Proofs of Propositions 4.29 and 4.30

We need the following lemma.

Lemma. *If* $\Gamma \vdash \varphi \to \psi$, *then* $\Gamma \vdash \exists v_i \varphi \to \exists v_i \psi$.

Proof. Assume that $\Gamma \vdash \varphi \to \psi$. By a tautology, $\Gamma \vdash \neg\psi \to \neg\varphi$. Generalizing on v_i and using (L2) we get $\Gamma \vdash \forall v_i \neg\psi \to \forall v_i \neg\varphi$. Then a tautology gives $\Gamma \vdash \neg\forall v_i \neg\varphi \to \neg\forall v_i \neg\psi$. This is the desired conclusion. □

4.29 The following is a sentential tautology:

(1) $$S_0 \to ([S_1 \leftrightarrow S_2 \wedge \neg S_0] \to \neg S_1)$$

We prove this by the method of Chapter 1:

2	1	5	3		6	7	8	2	3	4
S_0	$\to$	$([S_1$	$\leftrightarrow$	S_2	$\wedge$	$\neg$	$S_0]$	$\to$	$\neg$	$S_1)$
1	0	1	1		1	1	0	0	0	1

Two values have been assigned to S_0, so (1) is a tautology. Subsituting $v_0 = v_0$ for S_0, $v_0 \in v_2$ for S_1, and $v_0 \in v_2$ for S_2, we get the first-order tautology

$$\vdash v_0 = v_0 \to ([v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)] \to \neg(v_0 \in v_1)).$$

Hence by 3.4 we have

$$\vdash [v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)] \to \neg(v_0 \in v_1).$$

Hence by generalization and (L2) we get

$$\vdash \forall v_0([v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)]) \to \forall v_0(\neg(v_0 \in v_1)).$$

The lemma then gives

$$\vdash \exists v_1 \forall v_0([v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge \neg(v_0 = v_0)]) \to \exists v_1 \forall v_0(\neg(v_0 \in v_1)).$$

The hypothesis of the implication here is an instance of the comprehension axiom. Hence

(2) $$\mathrm{ZFC} \vdash \exists v_1 \forall v_0(\neg(v_0 \in v_1)).$$

Using the change of bound variable theorem 3.25 we get in succession

$$\mathrm{ZFC} \vdash \exists v_2 \forall v_0(\neg(v_0 \in v_2))$$
$$\mathrm{ZFC} \vdash \exists v_2 \forall v_1(\neg(v_1 \in v_2))$$

(3) $$\mathrm{ZFC} \vdash \exists v_0 \forall v_1(\neg(v_1 \in v_0))$$

By the extensionality axiom and change of bound variable theorem 3.25 we get in succession

$$\mathrm{ZFC} \vdash \forall v_0 \forall v_1 [\forall v_2 (v_2 \in v_0 \leftrightarrow v_2 \in v_1) \to v_0 = v_1];$$
$$\mathrm{ZFC} \vdash \forall v_0 \forall v_1 [\forall v_3 (v_3 \in v_0 \leftrightarrow v_3 \in v_1) \to v_0 = v_1];$$
$$\mathrm{ZFC} \vdash \forall v_0 \forall v_2 [\forall v_3 (v_3 \in v_0 \leftrightarrow v_3 \in v_2) \to v_0 = v_2];$$
$$(4) \qquad \mathrm{ZFC} \vdash \forall v_0 \forall v_2 [\forall v_1 (v_1 \in v_0 \leftrightarrow v_1 \in v_2) \to v_0 = v_2].$$

Now the following is a sentential tautology:

$$\neg S_0 \to (\neg S_1 \to (S_0 \leftrightarrow S_1)).$$

That this is a sentential tautology is seen as in chapter 1:

2	3	1	3	4	2	5	3	5
$\neg$	S_0	$\to$	$(\neg$	S_1	$\to$	$(S_0$	$\leftrightarrow$	$S_1))$
1	0	0	1	0	0	0	0	0

The assignments to $\leftrightarrow$ give a contradiction.

Now sustitute $v_1 \in v_0$ for S_0 and $v_1 \in v_2$ for S_1; we get the first-order tautology

$$\neg(v_1 \in v_0) \to [\neg(v_1 \in v_2) \to (v_1 \in v_0 \leftrightarrow v_1 \in v_2)].$$

Using generalization and (L2) we get

$$(5) \qquad \vdash \forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_2)) \to \forall v_1(v_1 \in v_0 \leftrightarrow v_1 \in v_2)].$$

By (4) and 3.28 we have

$$(6) \qquad \mathrm{ZFC} \vdash \forall v_1(v_1 \in v_0 \leftrightarrow v_1 \in v_2) \to v_0 = v_2)$$

Next, the following is a sentential tautology:

$$[S_0 \to (S_1 \to S_2)] \wedge (S_2 \to S_3) \to [S_0 \to (S_1 \to S_3)];$$

we check that this is a tautology:

5	4	8	6	9	2	10	4	11	1	3	2	7	3	7
$[S_0$	$\to$	$(S_1$	$\to$	$S_2)]$	$\wedge$	$(S_2$	$\to$	$S_3)$	$\to$	$[S_0$	$\to$	$(S_1$	$\to$	$S_3)]$
1	1	1	1	1	1	1	1	1	0	1	0	1	0	0

Different values have been assigned to S_3, so we have a tautology.

Now substitute $\forall v_1(\neg(v_1 \in v_0))$ for S_0, $\forall v_1(\neg(v_1 \in v_2))$ for S_1, $\forall v_1(v_1 \in v_0 \leftrightarrow v_1 \in v_2)$ for S_2, and $v_0 = v_2$ for S_3; we get the first-order tautology

$$(5) \wedge (6) \to [\forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]].$$

It follows that

$$\mathrm{ZFC} \vdash \forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2],$$

and hence by generalization and 3.38 we get

(7) $$\mathrm{ZFC} \vdash \forall v_0(\neg(v_1 \in v_0)) \to \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2].$$

Now $(S_0 \to S_1) \to (S_0 \to S_0 \wedge S_1)$ is clearly a sentential tautology, so by (7) we get

$$\mathrm{ZFC} \vdash \forall v_1(\neg(v_1 \in v_0)) \to [\forall v_1(\neg(v_1 \in v_0)) \wedge \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

and hence by the lemma we have

$$\mathrm{ZFC} \vdash \exists v_0 \forall v_1(\neg(v_1 \in v_0)) \to \exists v_0[\forall v_1(\neg(v_1 \in v_0)) \wedge \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

From (2) it then follows that

$$\mathrm{ZFC} \vdash \exists v_0[\forall v_1(\neg(v_1 \in v_0)) \wedge \forall v_2[\forall v_1(\neg(v_1 \in v_2)) \to v_0 = v_2]],$$

which is the desired conclusion. □

4.30 An instance of the comprehension axioms gives

$$\mathrm{ZFC} \vdash \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge v_0 \in v_3).$$

Hence by generalization we get

$$\mathrm{ZFC} \vdash \forall v_2 \forall v_3 \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_2 \wedge v_0 \in v_3).$$

Now the change of bound variables theorem 3.25 gives successively

$$\mathrm{ZFC} \vdash \forall v_4 \forall v_3 \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_4 \wedge v_0 \in v_3);$$
$$\mathrm{ZFC} \vdash \forall v_4 \forall v_5 \exists v_1 \forall v_0(v_0 \in v_1 \leftrightarrow v_0 \in v_4 \wedge v_0 \in v_5);$$
$$\mathrm{ZFC} \vdash \forall v_4 \forall v_5 \exists v_2 \forall v_0(v_0 \in v_2 \leftrightarrow v_0 \in v_4 \wedge v_0 \in v_5);$$
$$\mathrm{ZFC} \vdash \forall v_4 \forall v_5 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4 \wedge v_3 \in v_5);$$
$$\mathrm{ZFC} \vdash \forall v_0 \forall v_5 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_5);$$
$$\mathrm{ZFC} \vdash \forall v_0 \forall v_1 \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1).$$

Now two applications of Corollary 3.28 gives

(1) $$\mathrm{ZFC} \vdash \exists v_2 \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1).$$

By the extensionality axiom and successive applications of the change of bound variables theorem 3.25 we have

$$\mathrm{ZFC} \vdash \forall v_0 \forall v_1[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_1) \to v_0 = v_1];$$
$$\mathrm{ZFC} \vdash \forall v_0 \forall v_4[\forall v_2(v_2 \in v_0 \leftrightarrow v_2 \in v_4) \to v_0 = v_4];$$
$$\mathrm{ZFC} \vdash \forall v_0 \forall v_4[\forall v_3(v_3 \in v_0 \leftrightarrow v_3 \in v_4) \to v_0 = v_4];$$
$$\mathrm{ZFC} \vdash \forall v_2 \forall v_4[\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4) \to v_2 = v_4].$$

Then by Corollary 3.28 twice we get

$$\text{(2)} \qquad \text{ZFC} \vdash \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4) \to v_2 = v_4].$$

Now we claim that the following is a sentential tautology:

$$\text{(3)} \qquad (S_2 \leftrightarrow S_1) \to [(S_0 \leftrightarrow S_1) \to (S_0 \leftrightarrow S_2)].$$

Again we prove this by the method of Chapter 1; but this time the argument breaks into two cases, so we give a more formal proof.

Suppose that we have an assignment making (3) false (value 0). Then

(4) $S_2 \leftrightarrow S_1$ gets the value 1, and

$(S_0 \leftrightarrow S_1) \to (S_0 \leftrightarrow S_2)$ gets the value 0.

Hence

(5) $(S_0 \leftrightarrow S_1)$ gets the value 1 and

(6) $(S_0 \leftrightarrow S_2)$ gets the value 0.

By (6) we have two cases.

Case 1. S_0 gets the value 1 and S_2 gets the value 0. By (4) S_1 gets the value 0, and by (5) S_1 gets the value 1, contradiction.

Case 2. S_0 gets the value 0 and S_2 gets the value 1. By (4), S_1 gets the value 1, and by (5) S_1 gets the value 0, contradiction.

Thus, indeed, (3) is a sentential tautology.

Now in (3) we replace S_0 by $v_3 \in v_2$, S_1 by $v_3 \in v_0 \leftrightarrow v_3 \in v_1$, and S_2 by $v_3 \in v_4$. This gives the first-order tautology

$$(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to [(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to (v_3 \in v_2 \leftrightarrow v_3 \in v_4)].$$

Hence by generalization and (L2) we get

$$\text{(7)} \qquad \begin{aligned} &\vdash \forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to \\ &\quad [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \to \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4)]. \end{aligned}$$

Now in the sentential tautology given above in the proof of 4.29 following (6) we do the following replacements:

S_0 replaced by $\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1)$;
S_1 replaced by $\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1)$;
S_2 replaced by $\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4)$;
S_3 replaced by $v_2 = v_4$.

This gives the following first-order tautology:

$$
\begin{aligned}
&(\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4)]) \\
&\wedge [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_4) \rightarrow v_2 = v_4] \\
&\rightarrow ([(\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)]
\end{aligned}
$$

By (7) and (2) we then get

$$
\begin{aligned}
\text{ZFC} \vdash &(\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)]
\end{aligned}
\tag{8}
$$

Now we claim that the following is a sentential tautology:

$$[S_0 \rightarrow (S_1 \rightarrow S_2)] \rightarrow [S_1 \rightarrow (S_0 \rightarrow S_2)].$$

Again we prove this by the method of Chapter 1.

5	2	5	6	5	1	3	2	4	3	4
$[S_0$	$\rightarrow$	$(S_1$	$\rightarrow$	$S_2)]$	$\rightarrow$	$[S_1$	$\rightarrow$	$(S_0$	$\rightarrow$	$S_2)]$
1	1	1	0	0	0	1	0	1	0	0

The first implication is falsely assigned, contradiction. So the above is a sentential tautology. Hence the following is a first-order tautology:

$$
\begin{aligned}
&((\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad [\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)]) \\
&\rightarrow ((\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad [\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)])
\end{aligned}
$$

Hence by (8) we get

$$
\begin{aligned}
\text{ZFC} \vdash &((\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad [\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)])
\end{aligned}
$$

Generalizing on v_4 and using 3.38, we get

$$
\begin{aligned}
\text{ZFC} \vdash &((\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad \forall v_4([\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)]))
\end{aligned}
$$

By the sentential tautology given after (7) in the proof of 4.29 we then obtain

$$
\begin{aligned}
\text{ZFC} \vdash &(\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\
&\qquad (\forall v_3(v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\
&\qquad \forall v_4([\forall v_3(v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)]))
\end{aligned}
$$

Hence by the lemma we have

$$\begin{aligned}\mathrm{ZFC} \vdash \exists v_2 (\forall v_3 (v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow \\ \exists v_2 [(\forall v_3 (v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ \forall v_4 [\forall v_3 (v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)])])\end{aligned}$$

Hence by (1) we get

$$\begin{aligned}\mathrm{ZFC} \vdash \exists v_2 [(\forall v_3 (v_3 \in v_2 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \wedge \\ \forall v_4 [\forall v_3 (v_3 \in v_4 \leftrightarrow v_3 \in v_0 \wedge v_3 \in v_1) \rightarrow v_2 = v_4)])])\end{aligned}$$

This is as desired. □