New proof for Proposition 5.2

Again we argue model-theoretically, showing that $\models \mathrm{Subff}_\sigma^{v_0}\varphi \leftrightarrow \mathrm{Subf}_\sigma^{v_0}\varphi$, so that the proposition follows by the completeness theorem.

Suppose that $\overline{A}$ is a model for our language and $a : \omega \to A$.

First we note, by Proposition 4.6, that

(1) $\overline{A} \models \mathrm{Subf}_\sigma^{v_0}\varphi[a]$ iff $\overline{A} \models \varphi[a^0_{\sigma^{\overline{A}}(a)}]$.

In fact, in 4.6 replace ν by σ and v_i by v_0. Note that no free occurrence of v_0 in φ is within a subformula of the form $\forall v_k\mu$ with v_k occurring in σ, since only possibly v_0 occurs in σ. So (1) follows.

Now suppose that $\overline{A} \models \mathrm{Subff}_\sigma^{v_0}\varphi[a]$. Let $x = \sigma^{\overline{A}}(a)$. Since $gn(\varphi) > 0$ it follows that $v_{gn(\varphi)}$ does not occur in σ. Hence $\sigma^{\overline{A}}(a) = \sigma^{\overline{A}}(a^{gn(\varphi)}_x)$ by Proposition 2.4. Thus $\overline{A} \models (v_{gn(\varphi)} = \sigma)[a^{gn(\varphi)}_x]$. Now $\overline{A} \models (v_0 = v_{gn(\varphi)})[a^{0\ gn(\varphi)}_{x\ x}]$ so, since $\overline{A} \models \mathrm{Subff}_\sigma^{v_0}\varphi[a]$, we get $\overline{A} \models \varphi[a^{0\ gn(\varphi)}_{x\ x}]$. Since $v_{gn(\varphi)}$ does not occur in φ, it follows from Lemma 4.4 that $\overline{A} \models \varphi[a^0_x]$. Now by (1) it follows that $\overline{A} \models \mathrm{Subf}_\sigma^{v_0}\varphi[a]$.

Conversely, suppose that $\overline{A} \models \mathrm{Subf}_\sigma^{v_0}\varphi[a]$. Thus by (1) we have

(2) $\overline{A} \models \varphi[a^0_{\sigma^{\overline{A}(a)}}]$.

Now suppose that $y \in A$; we want to show that $\overline{A} \models (v_{gn(\varphi)} = \sigma \to \forall v_0[v_0 = v_{gn(\varphi)} \to \varphi])[a^{gn(\varphi)}_y]$. To this end, suppose that $\overline{A} \models (v_{gn(\varphi)} = \sigma)[a^{gn(\varphi)}_y]$; we want to show that $\overline{A} \models \forall v_0[v_0 = v_{gn(\varphi)} \to \varphi][a^{gn(\varphi)}_y]$, and to do this we take any $x \in A$, assume that $\overline{A} \models (v_0 = v_{gn(\varphi)})[a^{0\ gn(\varphi)}_{x\ y}]$, and prove that $\overline{A} \models \varphi[a^{0\ gn(\varphi)}_{x\ y}]$. Since $\overline{A} \models ((v_{gn(\varphi)} = \sigma)[a^{gn(\varphi)}_y]$, we have $y = \sigma^{\overline{A}}(a^{gn(\varphi)}_y)$, and so by Proposition 2.4 $y = \sigma^{\overline{A}}(a)$ since $v_{gn(\varphi)}$ does not occur in σ. Also, since $\overline{A} \models (v_0 = v_{gn(\varphi)})[a^{0\ gn(\varphi)}_{x\ y}]$, we have $x = y = \sigma^{\overline{A}}(a)$. By (2) we have $\overline{A} \models \varphi[a^{0\ gn(\varphi)}_{x\ y}]$, since $v_{\mathrm{gn}(\varphi)}$ does not occur in φ. □