6. Representability of recursive functions

In this chapter we prove Theorem A from chapter 5: all recursive functions and relations are representable. In order to do this, we need some lemmas about statements derivable from **P**. We will prove these statements model-theoretically, showing that they hold in any model $\overline{M}$ of **P**; hence by the completeness theorem they are derivable from **P**. For brevity we denote $+^{\overline{M}}$ by $+'$, $\mathbf{0}^{\overline{\mathbf{M}}}$ by $\mathbf{0}'$, $\mathbf{S}^{\overline{M}}$ by $\mathbf{S}'$, $\bullet^{\overline{M}}$ by $\bullet'$, and $\overline{m}^{\overline{M}}$ by $\overline{m}'$.

Lemma 6.1. $\mathbf{P} \vdash v_1 + (v_2 + v_0) = (v_1 + v_2) + v_0$.

Proof. We prove this by induction on v_0, applying the following instance of (P7):

$$\begin{aligned}&v_1 + (v_2 + \mathbf{0}) = (v_1 + v_2) + \mathbf{0}\\ &\wedge \forall v_0[v_1 + (v_2 + v_0) = (v_1 + v_2) + v_0 \rightarrow v_1 + (v_2 + \mathbf{S}v_0) = (v_1 + v_2) + \mathbf{S}v_0]\\ &\rightarrow \forall v_0[v_1 + (v_2 + v_0) = (v_1 + v_2) + v_0].\end{aligned}$$

So, assume $a, b \in M$. Then

$$\begin{aligned}a +' (b +' \mathbf{0}') &= a +' b \quad \text{by (P3)}\\ &= (a +' b) +' \mathbf{0}' \quad \text{by (P3)}\end{aligned}$$

Now assume that also $c \in M$ and $a +' (b +' c) = (a +' b) +' c$. Then

$$\begin{aligned}a +' (b +' \mathbf{S}'(c)) &= a +' \mathbf{S}'(b +' c) \quad \text{by (P4)}\\ &= \mathbf{S}'(a +' (b +' c)) \quad \text{by (P4)}\\ &= \mathbf{S}'((a +' b) +' c) \quad \text{by assumption}\\ &= (a +' b) +' \mathbf{S}'(c) \quad \text{by (P4)}\end{aligned}$$

It follows that for all c, $a +' (b +' c) = (a +' b) +' c$. □

Lemma 6.2. $\mathbf{P} \vdash v_2 \bullet (v_1 + v_0) = v_2 \bullet v_1 + v_2 \bullet v_0$.

Proof. Induction on v_0, the instance of (P7) being

$$\begin{aligned}&v_2 \bullet (v_1 + \mathbf{0}) = v_2 \bullet v_1 + v_2 \bullet \mathbf{0}\\ &\wedge \forall v_0[v_2 \bullet (v_1 + v_0) = v_2 \bullet v_1 + v_2 \bullet v_0 \rightarrow v_2 \bullet (v_1 + \mathbf{S}v_0) = v_2 \bullet v_1 + v_2 \bullet \mathbf{S}v_0]\\ &\rightarrow \forall v_0[v_2 \bullet (v_1 + v_0) = v_2 \bullet v_1 + v_2 \bullet v_0].\end{aligned}$$

So, let $a, b \in M$. Then

$$\begin{aligned}a \bullet' (b +' \mathbf{0}') &= a \bullet' b \quad \text{by (P3)}\\ &= a \bullet' b +' \mathbf{0}' \quad \text{by (P3)}\\ &= a \bullet' b + a \bullet \mathbf{0}'. \quad \text{by (P5)}\end{aligned}$$

Now suppose that also $c \in M$, and $a \bullet' (b +' c) = a \bullet' b +' a \bullet' c$. Then

$$\begin{aligned} a \bullet' (b +' \mathbf{S}'(c) &= a \bullet' \mathbf{S}'(b +' c) \quad \text{by (P4)} \\ &= a \bullet' (b +' c) + a \quad \text{by (P6)} \\ &= (a \bullet' b +' a \bullet' c) +' a \\ &= a \bullet' b +' ((a \bullet' c) +' a) \quad \text{by Lemma 6.1} \\ &= a \bullet' b +' a \bullet' \mathbf{S}'(c). \quad \text{by (P6)} \end{aligned}$$

This finishes the inductive proof. □

Lemma 6.3. $\mathbf{P} \vdash \mathbf{0} + v_0 = v_0$.

Proof. We prove this by induction on v_0. That is, we use the following instance of (P7), with φ the formula $\mathbf{0} + v_0 = v_0$:

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \wedge \forall v_0(\mathbf{0} + v_0 = v_0 \rightarrow \mathbf{0} + \mathbf{S}v_0 = \mathbf{S}v_0) \rightarrow \forall v_0(\mathbf{0} + v_0 = v_0).$$

Now $\mathbf{0}' +' \mathbf{0}' = \mathbf{0}'$ by (P3). Now suppose that $\mathbf{0}' +' a = a$. Then

$$\begin{aligned} \mathbf{0}' +' \mathbf{S}'(a) &= \mathbf{S}'(\mathbf{0}' +' a) \quad \text{by (P4)} \\ &= \mathbf{S}'(a). \quad \text{by supposition} \end{aligned}$$

It now follows that for all $a \in M$, $\mathbf{0}' +' a = a$. □

Lemma 6.4. $\mathbf{P} \vdash v_1 + v_0 = \mathbf{0} \rightarrow v_1 = \mathbf{0}$.

Proof. We prove this by induction on v_0. That is, we apply the following version of (P7), where φ is the formula $v_1 + v_0 = \mathbf{0} \rightarrow v_1 = \mathbf{0}$:

$$\begin{aligned} &(v_1 + \mathbf{0} = \mathbf{0} \rightarrow v_1 = \mathbf{0}) \wedge \forall v_0[(v_1 + v_0 = \mathbf{0} \rightarrow v_1 = \mathbf{0}) \rightarrow \\ &(v_1 + \mathbf{S}v_0 = \mathbf{0} \rightarrow v_1 = \mathbf{0})] \\ &\rightarrow \forall v_0(v_1 + v_0 = \mathbf{0} \rightarrow v_1 = \mathbf{0}). \end{aligned}$$

First suppose that $a +' \mathbf{0}' = \mathbf{0}'$. By (P3), $a +' \mathbf{0}' = a$. so $a = \mathbf{0}'$.

Second, suppose that $b \in M$ and ($a +' b = \mathbf{0}'$ implies that $a = \mathbf{0}'$). Also suppose that $a +' \mathbf{S}'(b) = \mathbf{0}'$. By (P4) we have $a +' \mathbf{S}'(b) = \mathbf{S}'(a +' b)$; so $\mathbf{S}'(a +' b) = \mathbf{0}'$. This contradicts (P2). Hence the supposition $a +' \mathbf{S}(b) = \mathbf{0}'$ is false, and so the implication ($a +' \mathbf{S}(b) = \mathbf{0}'$ implies that $a = 0'$) is true.

Hence the result of the lemma follows. □

Lemma 6.5. $\mathbf{P} \vdash \forall v_0[\neg(v_0 = \mathbf{0}) \rightarrow \exists v_1(\mathbf{S}v_1 = v_0)]$.

Proof. Induction on v_0. In (P7) we take φ to be the formula $\neg(v_0 = \mathbf{0}) \rightarrow \exists v_1(\mathbf{S}v_1 = v_0)$, giving the following instance of (P7):

$$\begin{aligned} &(\neg(\mathbf{0} = \mathbf{0}) \rightarrow \exists v_1(\mathbf{S}v_1 = \mathbf{0})) \wedge \forall v_0[(\neg(v_0 = \mathbf{0}) \rightarrow \exists v_1(\mathbf{S}v_1 = v_0)) \\ &\quad \rightarrow (\neg(\mathbf{S}v_0 = \mathbf{0}) \rightarrow \exists v_1(\mathbf{S}v_1 = \mathbf{S}v_0))] \rightarrow \forall v_0(\neg(v_0 = \mathbf{0}) \rightarrow \exists v_1(\mathbf{S}v_1 = v_0)). \end{aligned}$$

The implication "$\neg(\mathbf{0}' = \mathbf{0}')$ implies that there is an a such that $\mathbf{S}(a) = \mathbf{0}'$" is true since the hypothesis is false. Now assume that $a \neq \mathbf{0}'$ implies that there is a b such that $\mathbf{S}'(b) = a$, and assume that $\mathbf{S}'(a) \neq \mathbf{0}'$. Then there is a b such that $\mathbf{S}'(b) = \mathbf{S}'(a)$, namely a itself.

Hence the desired conclusion follows. □

Lemma 6.6. $\mathbf{P} \vdash \forall v_0 \forall v_1 [\mathbf{S}v_1 + v_0 = v_1 + \mathbf{S}v_0]$.

Proof. We prove this by induction on v_0, applying (P7) with the formula φ being $\mathbf{S}v_1 + v_0 = v_1 + \mathbf{S}v_0$; thus the instance of (P7) is

$$\begin{aligned}&\mathbf{S}v_1 + \mathbf{0} = v_1 + \mathbf{S0} \wedge (\forall v_0[\mathbf{S}v_1 + v_0 = v_1 + \mathbf{S}v_0]\\&\to \mathbf{S}v_1 + \mathbf{S}v_0 = v_1 + \mathbf{SS}v_0]) \to \forall v_0[\mathbf{S}v_1 + v_0 = v_1 + \mathbf{S}v_0].\end{aligned}$$

Take any $a \in M$. Then

$$\begin{aligned}\mathbf{S}'(a) + \mathbf{0}' &= \mathbf{S}'(a) \quad \text{by (P3)}\\&= \mathbf{S}'(a +' \mathbf{0}') \quad \text{by (P3)}\\&= a + \mathbf{S}'(\mathbf{0}'). \quad \text{by (P4)}\end{aligned}$$

Now assume that $\mathbf{S}'(a) +' b = a +' \mathbf{S}'(b)$. Hence

$$\begin{aligned}\mathbf{S}'(a) +' \mathbf{S}'(b) &= \mathbf{S}'(\mathbf{S}'(a) +' b) \quad \text{by (P4)}\\&= \mathbf{S}'(a +' \mathbf{S}'(b)) \quad \text{by assumption}\\&= a +' \mathbf{S}'(\mathbf{S}'(b)). \quad \text{by (P4)}\end{aligned}$$

□

Lemma 6.7. $\mathbf{P} \vdash v_0 + v_1 = v_0 + v_2 \to v_1 = v_2$.

Proof. We prove this by induction on v_0, the instance of (P7) being

$$\begin{aligned}&\forall v_1 \forall v_2(\mathbf{0} + v_1 = \mathbf{0} + v_2 \to v_1 = v_2)\\&\wedge \forall v_0[\forall v_1 \forall v_2(v_0 + v_1 = v_0 + v_2 \to v_1 = v_2) \to \forall v_1 \forall v_2(\mathbf{S}v_0 + v_1 = \mathbf{S}v_0 + v_2 \to v_1 = v_2)]\\&\to \forall v_0 \forall v_1 \forall v_2(v_0 + v_1 = v_0 + v_2 \to v_1 = v_2).\end{aligned}$$

So assume that $a, b \in M$. If $\mathbf{0}' +' a = \mathbf{0}' +' b$, then by Lemma 6.3 twice, $a = \mathbf{0}' +' a = \mathbf{0}' +' b = b$. Now assume that $c \in M$ and for all $a, b \in M$, $c +' a = c +' b$ implies that $a = b$. Suppose that $a, b \in M$ and $\mathbf{S}'(c) +' a = \mathbf{S}'(c) +' b$. By Lemma 6.6 twice, $c +' \mathbf{S}'(a) = c +' \mathbf{S}'(b)$. Hence by assumption $\mathbf{S}'(a) = \mathbf{S}'(b)$. Hence by (P1), $a = b$.

The desired conclusion follows. □

The next lemmas involve the terms $\overline{m}$. In a model $\overline{M}$ of $\mathbf{P}$ we then have elements $\overline{m}'$, defined by $\overline{0}' = \mathbf{0}'$ and $\overline{m+1}' = \mathbf{S}'\overline{m}'$.

Lemma 6.8. $\mathbf{P} \vdash \overline{m+n} = \overline{m} + \overline{n}$ *for any* $m, n \in \omega$.

Proof. Let $\overline{M}$ be any model of **P**; we want to show that $\overline{m+n}' = \overline{m}' +' \overline{n}'$ for any $m, n \in \omega$. We use (ordinary) induction on n, with m fixed. Note that $\overline{m+0}' = \overline{m}'$ and $\overline{0}' = \mathbf{0}'$. Hence $\overline{m+0}' = \overline{m}' +' \overline{0}'$ reduces to $\overline{m}' = \overline{m}' +' \mathbf{0}'$, which is true by (P3). Now for the induction step,

$$\begin{aligned}\overline{m+n+1}' &= \mathbf{S}'(\overline{m+n}')\\ &= \mathbf{S}'(\overline{m}' +' \overline{n}') \quad \text{inductive hypothesis}\\ &= \overline{m}' +' \mathbf{S}'(\overline{n}') \quad \text{(P4)}\\ &= \overline{m}' +' \overline{n+1}'. \end{aligned}$$

□

Lemma 6.9. $\mathbf{P} \vdash \overline{m \cdot n} = \overline{m} \bullet \overline{n}$ *for any* $m, n \in \omega$.

Proof. Again we work in a model $\overline{M}$ of **P**; we want to show that $\overline{m \cdot n}' = \overline{m}' \bullet' \overline{n}'$ for any $m, n \in \omega$. We go by induction on n, with m fixed. Note that $m \cdot 0 = 0$, and so $\overline{m \cdot 0}'$ is $\mathbf{0}'$. Hence the case $n = 0$ reduces to $\mathbf{0}' = \overline{m}' \bullet' \mathbf{0}'$, which holds by (P5). The inductive step:

$$\begin{aligned}\overline{m \cdot (n+1)} &= \overline{m \cdot n + m}'\\ &= \overline{m \cdot n}' +' \overline{m}' \quad \text{by Lemma 6.8}\\ &= \overline{m}' \bullet' \overline{n}' +' \overline{m}' \quad \text{inductive hypothesis}\\ &= \overline{m}' \bullet' \mathbf{S}'(\overline{n}') \quad \text{(P6)}\\ &= \overline{m}' \bullet' \overline{n+1}'. \end{aligned}$$

□

Lemma 6.10. *If* $m, n \in \omega$ *and* $m \neq n$*, then* $\mathbf{P} \vdash \neg(\overline{m} = \overline{n})$.

Proof. We use ordinary induction on n, proving that for all $m \neq n$, $\overline{m}' \neq \overline{n}'$. For $n = 0$ this follows from Lemma 6.5 and (P2). Now suppose that for all $m \neq n$ we have $\overline{m} \neq \overline{n}$, and $m \neq n+1$. If $m = 0$, then $\overline{m}' \neq \overline{n+1}'$ by (P2). Suppose that $m \neq 0$. Say $m = p + 1$. Then $p \neq n$, so $\overline{p}' \neq \overline{n}'$ by the inductive hypothesis. If $\overline{m}' = \overline{n+1}'$, then $\overline{p}' = \overline{n}'$ by (P1); hence $\overline{m}' \neq \overline{n+1}'$. □

Corollary 6.11. *If* $m, n \in \omega$*,* $\overline{M}$ *is a model of* **P***, and* $\overline{m}' = \overline{n}'$*, then* $m = n$.

Proof. If $m \neq n$, then $\mathbf{P} \vdash \neg(\overline{m} = \overline{n})$ by Lemma 6.10, and hence $\overline{m}' \neq \overline{n}'$. □

Next, let $\triangleleft$ be the formula $\exists v_2[v_0 + \mathbf{S}v_2 = v_1]$. Intuitively this says that $v_0 < v_1$. We use the symbol $\triangleleft$ to distinguish the formula from ordinary $<$. If σ and τ are terms, then $\sigma \triangleleft \tau$ is the formula $\exists v_2[\sigma + \mathbf{S}v_2 = \tau]$. So we need to avoid using terms which have occurrences of v_2 in them. We need several common properties of $<$.

Lemma 6.12. *If* $a, b \in \omega$ *and* $a < b$*, then* $\mathbf{P} \models \overline{a} \triangleleft \overline{b}$.

Proof. Assume that $a, b \in \omega$ and $a < b$. Choose $m \in \omega$ such that $a + m = b$. Then $m \neq 0$. By Lemma 6.8, $\overline{a} +' \mathbf{S}'(\overline{m-1}') = \overline{a}' +' \overline{m}' = \overline{b}'$. Hence $\overline{M} \models \overline{a} \triangleleft \overline{b}$. □

Lemma 6.13. $\mathbf{P} \vdash \neg(v_0 \lhd v_0)$.

Proof. Suppose that $a \in M$ and $a \lhd' a$. Choose $b \in M$ such that $a +' \mathbf{S}'(b) = a$. Then by (P3), $a +' \mathbf{S}'(b) = a +' \mathbf{0}'$, and hence by Lemma 6.7, $\mathbf{S}'(b) = \mathbf{0}'$, contradicting (P2). □

Lemma 6.14. $\mathbf{P} \vdash v_0 \lhd v_1 \wedge v_1 \lhd v_3 \to v_0 \lhd v_3$.

Proof. Suppose that $a, b, c \in M$ and $a \lhd' b \lhd' c$. Choose $d, e \in M$ such that $a +' \mathbf{S}'(d) = b$ and $b +' \mathbf{S}'(e) = c$. Then

$$\begin{aligned} a +' \mathbf{S}'(\mathbf{S}'(d +' e)) &= a +' (d +' \mathbf{S}'(\mathbf{S}'(e))) && \text{by (P4) twice} \\ &= a +' (\mathbf{S}'(d) +' \mathbf{S}'(e)) && \text{by Lemma 6.6} \\ &= (a +' \mathbf{S}'(d)) +' \mathbf{S}'(e) && \text{by Lemma 6.1} \\ &= b +' \mathbf{S}'(e) \\ &= c. \end{aligned}$$

Hence $a \lhd' c$. □

Lemma 6.15. $\mathbf{P} \vdash \forall v_0 \forall v_1 [v_0 \lhd v_1 \vee v_0 = v_1 \vee v_1 \lhd v_0]$.

Proof. We prove this by induction on v_0, using a version of (P7) in which φ is the formula $v_0 \lhd v_1 \vee v_0 = v_1 \vee v_1 \lhd v_0$. Thus the version of (P7) is

$$\begin{aligned} &(\mathbf{0} \lhd v_1 \vee \mathbf{0} = v_1 \vee v_1 \lhd \mathbf{0}) \\ &\wedge \forall v_0 [v_0 \lhd v_1 \vee v_0 = v_1 \vee v_1 \lhd v_0 \to \mathbf{S}v_0 \lhd v_1 \vee \mathbf{S}v_0 = v_1 \vee v_1 \lhd \mathbf{S}v_0] \\ &\to \forall v_0 [v_0 \lhd v_1 \vee v_0 = v_1 \vee v_1 \lhd v_0]. \end{aligned}$$

Let $a \in M$. We want to show, first, that

(1) $\mathbf{0}' \lhd' a$ or $\mathbf{0}' = a$ or $a \lhd' \mathbf{0}'$.

If $a = \mathbf{0}'$, then (1) holds. Suppose that $a \neq \mathbf{0}'$. By Lemma 6.5 choose $b \in M$ such that $\mathbf{S}'(b) = a$. Then $\mathbf{0}' +' \mathbf{S}'(b) = \mathbf{S}'(b) = a$ by Lemma 6.3, so $\mathbf{0}' \lhd' a$, and again (1) holds.

Now suppose that $b \in M$ and $a \lhd' b$ or $a = b$ or $b \lhd' a$. We want to show that

(2) $\mathbf{S}'(a) \lhd' b$ or $\mathbf{S}'(a) = b$ or $b \lhd' \mathbf{S}(a)$.

We consider three cases.

Case 1. $a \lhd' b$. Choose c such that $a +' \mathbf{S}'(c) = b$. If $c = \mathbf{0}'$, then

$$\begin{aligned} \mathbf{S}'(a) &= \mathbf{S}'(a) + \mathbf{0}' && \text{by (P3)} \\ &= a + \mathbf{S}'(\mathbf{0}') && \text{by Lemma 6.6} \\ &= b, \end{aligned}$$

and (2) holds.

If $c \neq \mathbf{0}'$, by Lemma 6.5 choose d such that $\mathbf{S}'(d) = c$. Then

$$\begin{aligned} \mathbf{S}'(a) +' \mathbf{S}'(d) &= a +' \mathbf{S}'(\mathbf{S}'(d)) \quad \text{by Lemma 6.6} \\ &= a +' \mathbf{S}'(c) \\ &= b; \end{aligned}$$

hence $\mathbf{S}'(a) \lhd' b$ and (2) holds.

Case 2. $a = b$. Then

$$\begin{aligned} b +' \mathbf{S}'(\mathbf{0}') &= \mathbf{S}'(b +' \mathbf{0}') \quad \text{by (P4)} \\ &= \mathbf{S}'(b) \quad \text{by (P3)} \\ &= \mathbf{S}'(a). \end{aligned}$$

Hence $b \lhd' \mathbf{S}'(a)$, and (2) holds.

Case 3. $b \lhd' a$. Choose c so that $b +' \mathbf{S}'(c) = a$. Then

$$\begin{aligned} b +' \mathbf{S}'(\mathbf{S}'(c)) &= \mathbf{S}'(b +' \mathbf{S}'(c)) \quad \text{by (P4)} \\ &= \mathbf{S}'(a). \end{aligned}$$

Hence $b \lhd' \mathbf{S}'(a)$, and (2) holds.

The lemma now follows. □

Lemma 6.16. $\mathbf{P} \vdash v_0 \lhd v_1 \rightarrow \mathbf{S}v_0 = v_1 \vee \mathbf{S}v_0 \lhd v_1$.

Proof. Suppose that $a, b \in M$ and $a \lhd' b$. Choose $c \in M$ such that $a +' \mathbf{S}'(c) = b$. Then $\mathbf{S}'(a) + c = b$ by Lemma 6.6.

Case 1. $c = \mathbf{0}$. Then $\mathbf{S}'(a) = b$ by (P3).

Case 2. $c \neq \mathbf{0}$. By Lemma 6.5 choose d so that $\mathbf{S}'(d) = c$. Thus $\mathbf{S}'(a) + \mathbf{S}'(d) = b$, so $\mathbf{S}'(a) \lhd' b$. □

Lemma 6.17. $\mathbf{P} \vdash v_0 \lhd v_1 \rightarrow v_3 + v_0 \lhd v_3 + v_1$.

Proof. Suppose that $a, b, c \in M$ and $a \lhd' b$. Choose d so that $a +' \mathbf{S}'(d) = b$. Then

$$\begin{aligned} (c +' a) +' \mathbf{S}'(d) &= c +' (a +' \mathbf{S}'(d)) \quad \text{by Lemma 6.1} \\ &= c +' b. \end{aligned}$$

Hence $c +' a \lhd' c +' b$. □

Lemma 6.18. $\mathbf{P} \vdash v_0 \lhd v_1 \rightarrow \mathbf{S}v_3 \bullet v_0 \lhd \mathbf{S}v_3 \bullet v_1$.

Proof. Suppose that $a, b, c \in M$ and $a \lhd' b$. Choose d so that $a +' \mathbf{S}'(d) = b$. Then

$$\begin{aligned} \mathbf{S}'(\mathbf{S}'(c) \bullet' d +' c) &= \mathbf{S}'(c) \bullet' d +' \mathbf{S}'(c) \quad \text{by (P4)} \\ &= \mathbf{S}'(c) \bullet' \mathbf{S}'(d). \quad \text{by (P6)} \end{aligned}$$

Hence

$$\begin{aligned}\mathbf{S}'(c) \bullet' a +' \mathbf{S}'(\mathbf{S}'(c) \bullet' d +' c) &= \mathbf{S}'(c) \bullet' a + \mathbf{S}'(c) \bullet \mathbf{S}'(d)\\ &= \mathbf{S}'(c) \bullet' (a +' \mathbf{S}'(d)) \quad \text{by Lemma 6.2}\\ &= \mathbf{S}'(c) \bullet' b;\end{aligned}$$

it follows that $\mathbf{S}'(c) \bullet' a \lhd' \mathbf{S}'(c) \bullet' b$. □

Lemma 6.19. *For any positive integer m we have*

$$\mathbf{P} \vdash \forall v_0 \left[v_0 \lhd \overline{m}) \leftrightarrow \bigvee_{i<m} v_0 = \overline{i} \right].$$

Proof. We prove this by (ordinary) induction on m. First suppose that $m = 1$. Suppose that $a \in M$. First suppose that $a \lhd' \overline{1}'$. Choose $b \in M$ such that $a +' \mathbf{S}'(b) = \overline{1}'$. Then by (P4), $\mathbf{S}'(a +' b) = a +' \mathbf{S}'(b) = \mathbf{S}'(\mathbf{0}')$. Hence by (P1), $a +' b = \mathbf{0}'$. Then $a = \mathbf{0}'$ by Lemma 6.4, so $\overline{M} \models \bigvee_{i<1} v_0 = \overline{i}[a]$.

Second suppose that $\overline{M} \models \bigvee_{i<1}(v_0 = \overline{i})[a]$. Thus $\overline{M} \models (v_0 = \overline{0})[a]$, so $a = \overline{0}'$. Hence $a +' \mathbf{S}'(\mathbf{0}') = \mathbf{0}' +' \mathbf{S}'(\mathbf{0}') = \mathbf{S}'(\mathbf{0}')$ by Lemma 6.3. Hence $a \lhd' \overline{1}'$. This finishes the case $m = 1$.

Now assume the statement for m. Let $a \in M$ be given. Suppose that $a \lhd' \overline{m+1}'$. Choose $b \in M$ such that $a +' \mathbf{S}'(b) = \overline{m+1}'$. By (P4) we have $\mathbf{S}'(a +' b) = a +' \mathbf{S}'(b) = \overline{m+1}'$, and then by (P1) we get $a +' b = \overline{m}'$. If $b = \mathbf{0}'$, then $a = a +' b = \overline{m}$ by (P3), and hence $\overline{M} \models \bigvee_{i<m+1} v_0 = \overline{i}[a]$. If $b \neq \mathbf{0}'$, by Lemma 6.5 choose $c \in M$ such that $\mathbf{S}'(c) = b$. Then $a +' \mathbf{S}'(c) = \overline{m}'$, hence $a \lhd' \overline{m}'$, so by the inductive hypothesis $\overline{M} \models \bigvee_{i<m} v_0 = \overline{i}[a]$, so $\overline{M} \models \bigvee_{i<m+1} v_0 = \overline{i}[a]$.

Conversely, suppose that $\overline{M} \models \bigvee_{i<m+1} v_0 = \overline{i}[a]$. Choose $i < m+1$ such that $a = \overline{i}'$. If $i < m$, then $\overline{M} \models \bigvee_{j<m} v_0 = \overline{j}[a]$, so by the inductive hypothesis $a \lhd' \overline{m}$. Say $a +' \mathbf{S}'(b) = \overline{m}$. Then $a +' \mathbf{S}'(\mathbf{S}'(b)) = \mathbf{S}'(a +' \mathbf{S}'(b)) = \overline{m+1}'$ by (P4), so $a \lhd' \overline{m+1}$. If $i = m$, then

$$\begin{aligned}a +' \mathbf{S}'(\mathbf{0}') &= \mathbf{S}'(a + \mathbf{0}') \quad \text{by (P4)}\\ &= \mathbf{S}'(a) \quad \text{by (P3)}\\ &= \overline{m+1}'.\end{aligned}$$

Hence $a \lhd' \overline{m+1}'$.

This finishes the proof. □

Lemma 6.20. *For any positive integer m,*

$$\mathbf{P} \vdash \forall v_0(v_0 \lhd \overline{m} \to \varphi) \leftrightarrow \bigwedge_{i<m} \varphi(\overline{i}).$$

Proof. Suppose that $a : \omega \to M$ is an assignment. First assume that

(1) $\overline{M} \models \forall v_0[v_0 \lhd \overline{m}) \to \varphi][a]$ and $i < m$;

we want to show that $\overline{M} \models \varphi(\bar{i})[a]$. Now by definition (see the beginning of Chapter 5), $\varphi(\bar{i})$ is $\mathrm{Subf}^{v_0}_{\bar{i}}\varphi$. Hence by Lemma 4.6 it suffices to prove that $\overline{M} \models \varphi\left[a^0_{\bar{i}^{\overline{M}}(a)}\right]$. Since $\bar{i}^{\overline{M}}(a)$ is simply $\bar{i}'$, it suffices to show that $\overline{M} \models \varphi\left[a^0_{\bar{i}'}\right]$. By Lemma 6.19, $\bar{i}' \lhd' \overline{m}'$. Hence by (1), $\overline{M} \models \varphi[^0_{\bar{i}'}]$.

Second, assume that

(2) $\overline{M} \models \left(\bigwedge_{i<m} \varphi(\bar{i})\right)[a]$;

we want to show that $\overline{M} \models \forall v_0[v_0 \lhd \overline{m} \to \varphi][a]$. To this end, suppose that $u \in M$ and $u \lhd' \overline{m}'$; we want to show that $\overline{M} \models \varphi[a^0_u]$. By Lemma 6.19 choose $i < m$ such that $u = \bar{i}'$. Now by (2), $\overline{M} \models \varphi(\bar{i})[a]$. By Lemma 4.6 we then have $\overline{M} \models \varphi\left[a^0_{\bar{i}'}\right]$. Since $u = \bar{i}'$, it follows that $\overline{M} \models \varphi[a^0_u]$. □

We also need a simpler way of representing finite sequences of natural numbers by a single number, or actually by two numbers. The representation via prime decompositions is too complicated at this stage. The new representation depends on the division algorithm: for any positive integers a, b there are unique nonnegative integers q, r such that $a = b \cdot q + r$ with $r < b$. We denote this unique integer r by $\mathrm{rm}(a, b)$. We also define $\mathrm{rm}(a, 0) = 0$ for any $a \in \omega$.

We also need a little elementary number theory. If $a, b > 1$, we say that they are *relatively prime* iff they have no common positive divisors except 1.

Lemma 6.21. *If $a, b > 1$, then they are relatively prime iff there are integers s, t (positive, negative, or zero) such that $1 = a \cdot s + b \cdot t$.*

Proof. $\Leftarrow$: Suppose that s and t are integers such that $1 = a \cdot s + b \cdot t$. Suppose that c is a positive divisor of both a and b. Say $a = c \cdot a'$ and $b = c \cdot b'$. Then

$$1 = a \cdot s + b \cdot t = c \cdot a' \cdot s + c \cdot b' \cdot t = c \cdot (a' \cdot s + b' \cdot t).$$

It follows that $c = 1$. Thus a and b are relatively prime.

$\Rightarrow$. There are integers s, t such that $a \cdot s + b \cdot t > 0$; for example, $s = 1$ and $t = 0$, giving $a \cdot s + b \cdot t = a > 0$. Let m be the smallest positive integer such that there are integers s, t such that $m = a \cdot s + b \cdot t$. Now write $a = m \cdot q + r$ with $0 \le r < m$. Then

$$r = a - m \cdot q = a - (a \cdot s + b \cdot t) \cdot q = a \cdot (1 - s) + b \cdot (-t).$$

By the choice of m we must have $r = 0$. Thus m divides a. Similarly, it divides b. So $m = 1$. □

Lemma 6.22. (Chinese Remainder Theorem) *Let $m_0, \ldots, m_r$ be natural numbers > 1, $r > 0$, with the m_i pairwise relatively prime, and let $a_0, \ldots, a_r$ be any natural numbers. Then there is a natural number x such that m_i divides $x - a_i$ for all $i \leq r$.*

Proof. We prove this by induction on r. For $r = 1$, since m_0 and m_1 are relatively prime, by Lemma 6.21 there are integers s, t such that $1 = m_0 \cdot s + m_1 \cdot t$. Then

$$a_0 - a_1 = (m_0 \cdot s + m_1 \cdot t)(a_0 - a_1) = m_0 \cdot s \cdot (a_0 - a_1) + m_1 \cdot t \cdot (a_0 - a_1).$$

Now since $m_0 \cdot m_1 > 0$, there is a natural number u such that $x \stackrel{\text{def}}{=} a_0 - m_0 \cdot s \cdot (a_0 - a_1) + u \cdot m_0 \cdot m_1 > 0$. Then $x - a_0$ is divisible by m_0, and

$$\begin{aligned} x - a_1 &= a_0 - a_1 - m_0 \cdot s \cdot (a_0 - a_1) + u \cdot m_0 \cdot m_1 \\ &= m_0 \cdot s \cdot (a_0 - a_1) + m_1 \cdot t \cdot (a_0 - a_1) - m_0 \cdot s \cdot (a_0 - a_1) + u \cdot m_0 \cdot m_1 \\ &= m_1 \cdot t \cdot (a_0 - a_1) + u \cdot m_0 \cdot m_1, \end{aligned}$$

so that $x - a_1$ is divisible by m_1. This takes care of the case $r = 1$.

Now assume the result for $r \geq 1$. Suppose now that $m_0, \ldots, m_{r+1}$ are natural numbers > 1, they are relatively prime, and $a_0, \ldots, a_{r+1}$ are natural numbers. By Lemma 6.21 choose integers s, t such that $1 = m_r \cdot s + m_{r+1} \cdot t$, and let u be a natural number such that $y \stackrel{\text{def}}{=} a_r - m_r \cdot s \cdot (a_r - a_{r+1}) + u \cdot m_r \cdot m_{r+1} > 0$. Now m_i and $m_r \cdot m_{r+1}$ are relatively prime when $i < m$. In fact, we have $1 = m_i \cdot s' + m_r \cdot t'$ for some integers s', t', and $1 = m_i \cdot s'' + m_{r+1} \cdot t''$ for some integers s'', t''. Hence

$$\begin{aligned} 1 &= (m_i \cdot s' + m_r \cdot t') \cdot (m_i \cdot s'' + m_{r+1} \cdot t'') \\ &= m_i \cdot (m_r \cdot s'' \cdot t' + m_i \cdot s' \cdot s'' + s' \cdot m_{r+1} \cdot t'') + m_r \cdot m_{r+1} \cdot t' \cdot t'', \end{aligned}$$

and so m_i and $m_r \cdot m_{r+1}$ are relatively prime by Lemma 6.21.

We now apply the inductive hypothesis to $m_0, \ldots, m_{r-1}, m_r \cdot m_{r+1}$ and $a_0, \ldots, a_{r-1}, y$ and obtain a natural number x such that $x - a_i$ is divisible by m_i for all $i < r$, and $x - y$ is divisible by $m_r \cdot m_{r+1}$. Say $x - y = m_r \cdot m_{r+1} \cdot c$. Thus

$$(1) \qquad x = y + m_r \cdot m_{r+1} \cdot c = a_r - m_r \cdot s \cdot (a_r - a_{r+1}) + u \cdot m_r \cdot m_{r+1} + m_r \cdot m_{r+1} \cdot c.$$

From this it is clear that $x - a_r$ is divisible by m_r. Also, in view of $1 = m_r \cdot s + m_{r+1} \cdot t$ we have

$$\begin{aligned} a_r - m_r \cdot s \cdot (a_r - a_{r+1}) &= a_r - (1 - m_{r+1} \cdot t)(a_r - a_{r+1}) \\ &= a_r - a_r + a_{r+1} + m_{r+1} \cdot t \cdot a_r - m_{r+1} \cdot t \cdot a_{r+1} \\ &= a_{r+1} + m_{r+1} \cdot t \cdot a_r - m_{r+1} \cdot t \cdot a_{r+1}; \end{aligned}$$

hence from (1) we get

$$x = a_{r+1} + m_{r+1} \cdot t \cdot a_r - m_{r+1} \cdot t \cdot a_{r+1} + u \cdot m_r \cdot m_{r+1} + m_r \cdot m_{r+1} \cdot c,$$

and hence $x - a_{r+1}$ is divisible by m_{r+1}. This finishes the inductive proof. □

We now define Gödel's β function; it is a function of three variables. For any $c, d, i \in \omega$ we define

$$\beta(c, d, i) = \mathrm{rm}(c, 1 + (i + 1) \cdot d).$$

The basic property of this function is as follows.

Lemma 6.23. *For any finite sequence $\langle a_0, \ldots, a_m \rangle$ of natural numbers there are natural numbers c, d such that $\beta(c, d, i) = a_i$ for all $i = 1, \ldots, m$.*

Proof. Let s be the maximum of $m, a_0, \ldots, a_m$, and let $d = s!$. For each $i \leq m$ let $d_i = 1 + (i + 1) \cdot d$. We claim that d_i and d_j are relatively prime for $i, j \leq m$ with $i < j$. In fact, suppose to the contrary that p is a prime dividing both d_i and d_j. Say $d_i = p \cdot s$ and $d_j = p \cdot t$. Then $d_j - d_i = p(t - s)$. Now $d_j - d_i = 1 + (j + 1) \cdot s! - (1 + (i + 1) \cdot s! = (j - i) \cdot s!$. Hence p divides $(j - i) \cdot s!$. Since $j - i \leq m \leq s$, it follows that p divides some $k \leq s$, hence it divides $(i + 1) \cdot s!$. But p also divides $d_i = 1 + (i + 1) \cdot s!$, so p divides 1, contradiction.

Since d_i and d_j are relatively prime for $i \neq j$, by the Chinese Remainder Theorem 6.22 choose c such that d_i divides $c - a_i$ for each $i \leq m$. Say $c - a_i = d_i \cdot q_i$, so $c = d_i \cdot q_i + a_i$. Now $a_i \leq s < d_i$, so $a_i = \mathrm{rm}(c, d_i) = \mathrm{rm}(c, 1 + (i + 1) \cdot d) = \beta(c, d, i)$ for each $i \leq m$. □

Lemma 6.24. *β is representable. In fact, it is representable by a formula φ in which v_0, v_1, v_2, v_3 occur free, with only v_4, v_5 bound, such that φ has the additional property that*

$$\mathbf{P} \vdash \forall v_0 \forall v_1 \forall v_2 \forall v_3 \forall v_6 [\varphi(v_0, v_1, v_2, v_3) \wedge \varphi(v_0, v_1, v_2, v_6) \rightarrow v_3 = v_6].$$

Proof. Let φ be the following formula:

$$\exists v_4 [v_0 = \mathbf{S}((\mathbf{S}v_2) \bullet v_1) \bullet v_4 + v_3 \wedge \exists v_5 [v_3 + \mathbf{S}v_5 = \mathbf{S}((\mathbf{S}v_2) \bullet v_1)]].$$

Note that $\overline{M} \models \varphi[c, d, i, a, a, \ldots]$ iff there is a $b \in \omega$ such that $c = (1 + (i + 1) \cdot d) \cdot b + a$ with $a < 1 + (i + 1) \cdot d$. This agrees with the definition of β. We claim that φ shows that β is representable.

To prove the claim, for the first condition in representablity, let $c, d, i \in \omega$, and set $\beta(c, d, i) = a$. We want to show that $\overline{M} \models \varphi(\overline{c}, \overline{d}, \overline{i}, \overline{a})$. By the definition of β, write $a = \mathrm{rm}(c, 1 + (i + 1) \cdot d)$, and by the definition of rm, let q be a natural number such that $c = (1 + (i + 1) \cdot d) \cdot q + a$ with $a < 1 + (i + 1) \cdot d$; and choose e so that $e + 1 + a = 1 + (i + 1) \cdot d$. By Lemmas 6.8 and 6.9 we have

$$\overline{a}' +' \mathbf{S}'(\overline{e}') = \mathbf{S}'(\mathbf{S}'(\overline{i}') \bullet' \overline{d}')$$

and hence

(1) $$\overline{M} \models \exists v_5 [\overline{a} + \mathbf{S}v_5 = \mathbf{S}((\mathbf{S}\overline{i}) \bullet \overline{d})].$$

Further, Lemmas 6.8 and 6.9 also give

$$\overline{c}' = \mathbf{S}'((\mathbf{S}'\overline{i}') \bullet' \overline{d}') \bullet' \overline{e}' +' \overline{a}'.$$

Together with (1) this gives $\overline{M} \models \varphi(\overline{c}, \overline{d}, \overline{i}, \overline{a})$.

The second property for representability is

$$\mathbf{P} \vdash \forall v_3[\varphi(\overline{c}, \overline{d}, \overline{i}, v_3) \rightarrow v_3 = \overline{a}]. \tag{2}$$

We claim that this follows from the additional condition of the lemma. In fact, assume that additional condition. Applications of Theorem 3.27 then give $\mathbf{P} \vdash \varphi(\overline{c}, \overline{d}, \overline{i}, \overline{a}) \wedge \varphi(\overline{c}, \overline{d}, \overline{i}, v_3) \rightarrow \overline{a} = v_3$. Then $\mathbf{P} \vdash \varphi(\overline{c}, \overline{d}, \overline{i}, v_3) \rightarrow \overline{a} = v_3$ by the first condition on representability, and (2) follows easily.

It remains to check the additional property in the lemma. So, suppose that $a, b, c, d, e \in M$ and $\overline{M} \models (\varphi(v_0, v_1, v_2, v_3) \wedge \varphi(v_0, v_1, v_2, v_6))[a, b, c, d, d, d, e]$. We want to show that $d = e$. We can choose additional elements $f, g \in M$ so that the following conditions hold:

(3) $a = \mathbf{S}'(\mathbf{S}'(c) \bullet' b) \bullet' f +' d$.

(4) $d \lhd' \mathbf{S}'(\mathbf{S}'(c) \bullet' b)$.

(5) $a = \mathbf{S}'(\mathbf{S}'(c) \bullet' b) \bullet' g +' e$.

(6) $e \lhd' \mathbf{S}'(\mathbf{S}'(c) \bullet' b)$.

For brevity, let $h = \mathbf{S}'(\mathbf{S}'(c) \bullet' b)$. Then (3)–(6) become

(7) $a = h \bullet' f +' d$.

(8) $d \lhd' h$.

(9) $a = h \bullet' g +' e$.

(10) $e \lhd' g$.

We claim that $f = g$. If not, then by Lemma 6.15 we have $f \lhd' g$ or $g \lhd' f$. Say by symmetry $f \lhd' g$. Hence

$$\begin{aligned} a &= h \bullet' f +' d \\ &\lhd' h \bullet' f + h \quad \text{by (8) and Lemma 6.17} \\ &= h \bullet' \mathbf{S}'(f). \quad \text{by (P6)} \end{aligned}$$

By Lemma 6.16 we have the following cases.

Case 1. $\mathbf{S}'(f) = g$ and $e = \mathbf{0}'$. Then by (8) and Lemma 6.17,

$$a = h \bullet' f +' d \lhd' h \bullet' f + h = h \bullet' \mathbf{S}'(f) = h \bullet' g = h \bullet' f +' e = a,$$

contradicting Lemma 6.13.

Case 2. $\mathbf{S}'(f) = g$ and $e \neq \mathbf{0}'$. By Lemma 6.5 there is a k such that $\mathbf{S}'(k) = e$. Thus $\mathbf{0}' + \mathbf{S}'(k) = e$ by Lemma 6.3, so $\mathbf{0}' \lhd e$. Hence

$$\begin{aligned} a &\lhd' h \bullet' \mathbf{S}'(f) \\ &= h \bullet' \mathbf{S}'(f) +' \mathbf{0}' \quad \text{by (P3)} \\ &\lhd' h \bullet' \mathbf{S}'(f) +' e \quad \text{by Lemma 6.17} \\ &= a, \end{aligned}$$

contradicting Lemmas 6.13 and 6.14.

Case 3. $\mathbf{S}'(f) \lhd' g$. Note that $h = \mathbf{S}'(k)$ for some k. Hence

$$\begin{aligned} a &\lhd' h \bullet' \mathbf{S}'(f) \quad \text{(see before Case 1)} \\ &\lhd' h \bullet' g. \quad \text{by Lemma 6.18} \end{aligned}$$

If $e = \mathbf{0}'$, then $a = h \bullet' g + e$ by (P3), contradicting Lemmas 6.13 and 6.14. If $e \neq \mathbf{0}'$, by Lemma 6.5 choose $s \in M$ such that $\mathbf{S}'(s) = e$. Then $\mathbf{0}' +' \mathbf{S}'(s) = e$ by Lemma 6.3, and so $\mathbf{0}' \lhd' e$. So $h \bullet' g = h \bullet' g + \mathbf{0}' \lhd h \bullet' g +' e$ using (P3) and Lemma 6.17. Since $a = h \bullet' g +' e$ by (8), this again contradicts Lemmas 6.13 and 6.14. □

Theorem 6.25. *Every recursive function is representable.*

Proof. Let $\langle f_0, \ldots, f_m \rangle$ be a recursive function construction sequence. We prove by complete induction that for every $i \leq m$, f_i is representable. So, assume that $i \leq m$ and we know that every f_j with $j < i$ is representable. By the definition of recursive function construction sequence we have the following cases.

Case 1. $f_i = \mathbf{s}$. We claim that the formula $\mathbf{S}v_0 = v_1$ represents $\mathbf{s}$. To prove this, suppose that $a \in \omega$. Then $\mathbf{S}\overline{a}$ is the same term as $\overline{a+1}$, and so $\overline{M} \models \mathbf{S}\overline{a} = \overline{a+1}$. Also, clearly for any $u \in M$ we have $\overline{M} \models (\mathbf{S}\overline{a} = v_1 \to v_1 = \overline{a+1})[u, u]$, and hence $\overline{M} \models \forall v_1(\mathbf{S}\overline{a} = v_1 \to v_1 = \overline{a+1})$.

Case 2. There exist j, m with $j < m$ such that $f_i = \mathbf{I}^m_j$. We claim that the formula $v_m = v_j$ represents $\mathbf{I}^m_j$. Suppose that $a_0, \ldots, a_{m-1} \in \omega$. Then $\mathbf{I}^m_j(a_0, \ldots, a_{m-1}) = a_j$. Obviously $\overline{M} \models \overline{a_j} = \overline{a_j}$. Also, for any $u \in M$, $\overline{M} \models (v_m = \overline{a_j} \to v_m = \overline{a_j})[u \ldots u]$, and hence $\overline{M} \models \forall v_m(v_m = \overline{a_j} \to v_m = \overline{a_j})$.

Case 3. There exist positive integers m, n and $j, k_0, \ldots, k_{m-1} < i$ such that each f_{k_s} is an n-ary operation on ω, f_j is an m-ary operation on ω, and $f_i = \mathbf{C}^m_n(f_j, f_{k_0}, \ldots, f_{k_{m-1}})$. For each $s < m$ let φ_s be a formula with free variables among $v_0, \ldots v_n$ which represents f_{k_s}, and let ψ be a formula with free variables among $v_0, \ldots, v_m$ which represents f_j. Let u be an integer such that v_u does not occur in ψ, and let ψ' be obtained from ψ by replacing all bound occurrences of v_n in ψ by v_u. Let t be an integer greater than m, n and all l such that v_l occurs in some φ_u or in ψ. We claim that the following formula represents f_i:

$$\exists v_t \ldots \exists v_{t+m-1} \left[\left(\bigwedge_{s<m} \varphi_s(v_0, \ldots, v_{n-1}, v_{t+s}) \right) \wedge \psi'(v_t, \ldots, v_{t+m-1}, v_n) \right].$$

To prove this, let $a_0,\ldots,a_{n-1}\in\omega$, let $b_s = f_{k_s}(a_0,\ldots,a_{n-1})$ for each $s<m$, and let $f_j(b_0,\ldots,b_{m-1}) = c$. Then because φ_s represents f_{k_s} we have

$$\mathbf{P}\vdash\varphi_s(\overline{a_0},\ldots,\overline{a_{n-1}},\overline{b_s})\quad\text{for each } s<m \tag{1}$$

And because ψ represents f_j we have

$$\mathbf{P}\vdash\psi'(\overline{b_0},\ldots,\overline{b_{m-1}},\overline{c}). \tag{2}$$

Putting (1) and (2) together with a tautology we then get

$$\mathbf{P}\vdash\left(\bigwedge_{s<m}\varphi_s(\overline{a_0},\ldots,\overline{a_{n-1}},\overline{b_s})\right)\wedge\psi'(\overline{b_0},\ldots,\overline{b_{m-1}},\overline{c}). \tag{3}$$

Now with $\overline{M}$ any model of $\mathbf{P}$ and $d:\omega\to M$, by Theorem 3.2 we get

$$\begin{aligned}\overline{M}\models&\mathrm{Subf}^{v_t}_{\overline{b_0}^{\overline{M}}(d)}\cdots\mathrm{Subf}^{v_{t+m-1}}_{\overline{b_{m-1}}^{\overline{M}}(d)}\\ &\left[\left(\bigwedge_{s<m}\varphi_s(\overline{a_0},\ldots,\overline{a_{n-1}},v_s)\right)\wedge\psi'(v_t,\ldots,v_{t+m-1},\overline{c})\right][d]\end{aligned}$$

It follows that

$$\overline{M}\models\exists v_t\ldots\exists v_{t+m-1}\left[\left(\bigwedge_{s<m}\varphi_s(\overline{a_0},\ldots,\overline{a_{n-1}},v_{t+s})\right)\wedge\psi'(v_t,\ldots,v_{t+m-1},\overline{c})\right][d].$$

This gives the first condition for representability.

For the second condition, suppose that $e\in M$ and

$$\overline{M}\models\exists v_t\ldots\exists v_{t+m-1}\left[\left(\bigwedge_{s<m}\varphi_s(\overline{a_0},\ldots,\overline{a_{n-1}},v_{t+s})\right)\wedge\psi'(v_t,\ldots,v_{t+m-1},v_n)\right][d^n_e].$$

We want to show that $e=c$. Choose $f_0,\ldots,f_{m-1}$ so that

$$\overline{M}\models\left[\left(\bigwedge_{s<m}\varphi_s(\overline{a_0},\ldots,\overline{a_{n-1}},v_{t+s})\right)\wedge\psi'(v_t,\ldots,v_{t+m-1},v_n)\right][d^{n\ t\ \ldots\ t+m-1}_{e\ f_0\ \ldots\ f_{m-1}}]. \tag{4}$$

By the second condition for representability of f_{k_s} by φ_s we get $f_s = a_s$ for each $s<m$. Hence from (4) we get $\overline{M}\models\psi'(\overline{a_0},\ldots,\overline{a_{m-1}},v_n)[d^n_e]$. Then the second condition for representability of f_j by ψ gives $e=c$, as desired.

Case 4. There exist $j<i$ and $a\in\omega$ such that f_i is $\mathbf{Q}_0(a,f_j)$. Let φ represent β with the additional property given in Lemma 6.24; the free variables of φ are among v_0,v_1,v_2,v_3. Let ψ represent f_j; the free variables of ψ are among v_0,v_1,v_2. Choose $t>3$

and greater than u for each variable v_u occurring in φ or ψ. Then we claim that the following formula χ represents f_i:

$$\exists v_t \exists v_{t+1}[\varphi(v_t, v_{t+1}, \mathbf{0}, \overline{a}) \wedge \varphi(v_t, v_{t+1}, v_0, v_1) \wedge \forall v_{t+2}[v_{t+2} \lhd v_0 \rightarrow \exists v_{t+3}\exists v_{t+4}$$
$$[\psi(v_{t+2}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, v_{t+2}, v_{t+3}) \wedge \varphi(v_t, v_{t+1}, \mathbf{S}v_{t+2}, v_{t+4})]].$$

The idea of this formula is to code, using the β-function, the whole finite sequence of values of f_i starting with the argument 0 and ending with v_0. To prove the claim, suppose that $m \in \omega$. Choose c, d so that $\beta(c, d, s) = f_i(s)$ for all $s \leq m$. In particular, $\beta(c, d, 0) = f_i(0) = a$, so

$$\mathbf{P} \vdash \varphi(\overline{c}, \overline{d}, \mathbf{0}, \overline{a}).$$

Hence for our model, for any $e : \omega \rightarrow M$ we have

(5) $$\overline{M} \models \varphi(\overline{c}, \overline{d}, \mathbf{0}, \overline{a})[e].$$

Also, $\beta(c, d, m) = f_i(m)$, so

$$\mathbf{P} \vdash \varphi(\overline{c}, \overline{d}, \overline{m}, \overline{f_i(m)}).$$

Hence

(6) $$\overline{M} \models \varphi(\overline{c}, \overline{d}, \overline{m}, \overline{f_i(m)})[e].$$

Now suppose that $s < m$. Then $f_j(s, f_i(s)) = f_i(s{+}1)$, $\beta(c, d, s) = f_i(s)$, and $\beta(c, d, s{+}1) = f_i(s + 1)$, so

$$\mathbf{P} \vdash \psi(\overline{s}, \overline{f_i(s)}, \overline{f_{i+1}(s)}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s}, \overline{f_i(s)}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, \overline{f_i(s+1)})$$

and hence

$$\overline{M} \models [\psi(\overline{s}, \overline{f_i(s)}, \overline{f_{i+1}(s)}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s}, \overline{f_i(s)}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, \overline{f_i(s+1)})][e].$$

It follows that

$$\overline{M} \models \exists v_{t+3}\exists v_{t+4}[\psi(\overline{s}, v_{t+3}, v_{t+4}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, v_{t+4})][e].$$

This being true for all $s < m$, it follows that

$$\overline{M} \models \Bigg[\bigvee_{s<m} (v_{t+2} = \overline{s}) \rightarrow \exists v_{t+3}\exists v_{t+4}[\psi(v_{t+2}, v_{t+3}, v_{t+4}) \wedge \varphi(\overline{c}, \overline{d}, v_{t+2}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \mathbf{S}v_{t+2}, v_{t+4})] \Bigg] [e].$$

Hence by Lemma 6.19 we get

$$\overline{M} \models \forall v_{t+2}[v_{t+2} \lhd \overline{m} \rightarrow \exists v_{t+3}\exists v_{t+4}[\psi(v_{t+2}, v_{t+3}, v_{t+4})$$
$$\wedge \varphi(\overline{c}, \overline{d}, v_{t+2}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \mathbf{S}v_{t+2}, v_{t+4})]][e].$$

Now together with (5) and (6) this gives $\overline{M} \models \chi(\overline{m}, \overline{f_i(m)})$, giving the first condition for representability for f_i.

For the other condition, assume that $m \in \omega$, $b \in M$, and $\overline{M} \models \chi[e^{0\ 1}_{m\ b}]$; we want to show that $b = f_i(m)$. Now choose $c, d \in M$ so that

(7) $\overline{M} \models [\varphi(v_t, v_{t+1}, \mathbf{0}, \overline{a}) \wedge \varphi(v_t, v_{t+1}, v_0, v_1) \wedge \forall v_{t+2}[v_{t+2} \lhd v_0 \rightarrow \exists v_{t+3}\exists v_{t+4}$
$[\psi(v_{t+2}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, v_{t+2}, v_{t+3}) \wedge \varphi(v_t, v_{t+1}, \mathbf{S}v_{t+2}, v_{t+4})]]]][e^{0\ 1\ t\ t+1}_{m\ b\ c\ d}]$.

Now we claim

(8) For all $k \leq m$, $\overline{M} \models \varphi(v_t, v_{t+1}, \overline{k}, \overline{f_i(k)})[e^{t\ t+1}_{c\ d}]$.

We prove this by (ordinary) induction on k. For $k = 0$ it is given by (7), since $f_i(0) = a$. Now assume that $k < m$ and

(9) $$\overline{M} \models \varphi(v_t, v_{t+1}, \overline{k}, \overline{f_i(k)})[e^{t\ t+1}_{c\ d}].$$

Now $\overline{k}' \lhd' \overline{m}'$ by Lemma 6.12, so by (7) we have

$$\overline{M} \models \exists v_{t+3}\exists v_{t+4}[\psi(\overline{k}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, \overline{k}, v_{t+3}) \wedge \varphi(v_t, v_{t+1}, \mathbf{S}\overline{k}, v_{t+4})]]]][e^{t\ t+1}_{c\ d}].$$

Hence we can choose $h, g \in M$ such that

(10) $\overline{M} \models \psi(\overline{k}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, \overline{k}, v_{t+3}) \wedge \varphi(v_t, v_{t+1}, \mathbf{S}(\overline{k}), v_{t+4})[e^{t\ t+1\ t+3\ t+4}_{c\ d\ h\ g}]$.

Now by (9) and (10) using the additional property of φ, we get $h = f_i(k)$. By (10) we have $\overline{M} \models \psi(\overline{k}, v_{t+3}, v_{t+4})[e^{t+3\ t+4}_{h\ g}]$, so by the second condition for ψ representing f_j we have $g = \overline{f_j(k, f_i(k))}' = \overline{f_i(k+1)}'$. Hence $\overline{M} \models \varphi(v_t, v_{t+1}, \mathbf{S}(\overline{k}), \overline{f_i(k+1)})[e^{t\ t+1}_{c\ d}]$ by (10). This gives $k + 1$ in (8).

So (8) holds by induction. The case $k = m$ is $\overline{M} \models \varphi(v_t, v_{t+1}, \overline{m}, \overline{f_i(m)})[e^{t\ t+1}_{c\ d}]$. Also from (7) we have $\overline{M} \models \varphi(v_t, v_{t+1}, \overline{m}, v_1)[e^{1\ t\ t+1}_{b\ c\ d}]$. Hence by the extra condition on φ it follows that $b = f_i(m)$, as desired.

This finishes the argument for $\mathbf{Q}_0$.

Case 5. There exist a positive integer n and $j, k < i$ such that f_j is an n-ary operation on ω, f_k is an $(n+2)$-ary operation on ω, and $f_i = \mathbf{Q}_n(f_j, f_k)$. The proof that f_i is representable is very similar to the above Case 4, but is somewhat more complicated.

Let φ represent β with the additional property in Lemma 6.24, ψ represent f_j, and χ represent f_k. Thus φ has variables among v_0, v_1, v_2, v_3, ψ has variables among $v_0, \ldots, v_n$, and χ has variables among $v_0, \ldots, v_{n+2}$. Choose $t > n + 2$ and greater than u for each variable v_u occurring in φ, ψ, or χ. Then we claim that the following formula θ represents f_i:

$$\begin{aligned}&\exists v_t\exists v_{t+1}\exists v_{t+5}[\varphi(v_t, v_{t+1}, \mathbf{0}, v_{t+5}) \wedge \psi(v_0, \ldots, v_{n-1}, v_{t+5})\\ &\wedge \varphi(v_t, v_{t+1}, v_n, v_{n+1}) \wedge \forall v_{t+2}[v_{t+2} \lhd v_n \rightarrow \exists v_{t+3}\exists v_{t+4}\\ &[\chi(v_0, \ldots, v_{n-1}, v_{t+2}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, v_{t+2}, v_{t+3})\\ &\wedge \varphi(v_t, v_{t+1}, \mathbf{S}v_{t+2}, v_{t+4})]]].\end{aligned}$$

To prove the claim, suppose that $a_0, \ldots, a_{n-1}, m \in \omega$. Choose c, d so that $\beta(c, d, s) = f_i(a_0, \ldots, a_{n-1}, s)$ for all $s \leq m$. Then $\beta(c, d, 0) = f_i(a_0, \ldots, a_{n-1}, 0) = f_j(a_0, \ldots, a_{n-1})$, so

$$\mathbf{P} \vdash \varphi(\overline{c}, \overline{d}, \mathbf{0}, \overline{f_j(a_0, \ldots, a_{n-1})}) \wedge \psi(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{f_j(a_0, \ldots, a_{n-1})}). \tag{11}$$

Also, $\beta(c, d, m) = f_i(a_0, \ldots, a_{n-1}, m)$, so

$$\mathbf{P} \vdash \varphi(\overline{c}, \overline{d}, \overline{m}, \overline{f_i(a_0, \ldots, a_{n-1}, m)}). \tag{12}$$

Now suppose that $s < m$. Then

$$f_j(a_0, \ldots, a_{n-1}, s, f_i(a_0, \ldots, a_{n-1}, s)) = f_i(a_0, \ldots, a_{n-1}, s+1)$$

and also $\beta(c, d, s) = f_i(a_0, \ldots, a_{n-1}, s)$ and $\beta(c, d, s+1) = f_i(a_0, \ldots, a_{n-1}, s+1)$. Hence

$$\begin{aligned}\mathbf{P} \vdash &\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{s}, \overline{f_i(a_0, \ldots, a_{n-1}, s)}, \overline{f_{i+1}(a_0, \ldots, a_{n-1}, s+1)}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{s}, \overline{f_i(a_0, \ldots, a_{n-1}, s)}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, \overline{f_i(a_0, \ldots, a_{n-1}, s+1)}).\end{aligned}$$

Thus in our model $\overline{M}$, with $e : \omega \to M$, we have

$$\begin{aligned}\overline{M} \models &[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{s}, \overline{f_i(a_0, \ldots, a_{n-1}, s)}, \overline{f_{i+1}(a_0, \ldots, a_{n-1}, s+1)}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{s}, \overline{f_i(a_0, \ldots, a_{n-1}, s)}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, \overline{f_i(a_0, \ldots, a_{n-1}, s+1)})][e].\end{aligned}$$

From Lemma 4.6 we then get

$$\begin{aligned}\overline{M} \models &[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{s}, v_{t+3}, v_{t+4}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{s}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, v_{t+4})][e^{t+3}_{w}{}^{t+4}_{u}]\end{aligned}$$

where $w = f_i(a_0, \ldots, a_{n-1}, s)$ and $u = f_i(a_0, \ldots, a_{n-1}, s+1)$. Hence

$$\begin{aligned}\overline{M} \models &\exists v_{t+3} \exists v_{t+4}[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{s}, v_{t+3}, v_{t+4}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{s}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \overline{s+1}, v_{t+4})][e].\end{aligned} \tag{13}$$

Note that (13) is true for all $s < m$. Now we claim

$$\begin{aligned}\overline{M} \models &\forall v_{t+2}[v_{t+2} \lhd \overline{m} \to \exists v_{t+3} \exists v_{t+4}[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, v_{t+2}, v_{t+3}, v_{t+4}) \\ &\wedge \varphi(\overline{c}, \overline{d}, v_{t+2}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \mathbf{S}v_{t+2}, v_{t+4})]][e].\end{aligned} \tag{14}$$

In fact, take any $w \in M$, and assume that $\overline{M} \models (v_{t+2} \lhd \overline{m})[e^{t+2}_w]$. Then by Lemma 6.19 there is an $s < m$ such that $w = s$. Hence by (13) and Lemma 4.6 we get

$$\begin{aligned}\overline{M} \models &\exists v_{t+3} \exists v_{t+4}[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, v_{t+2}, v_{t+3}, v_{t+4}) \\ &\wedge \varphi(\overline{c}, \overline{d}, v_{t+2}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, v_{t+2}, v_{t+4})][e^{t+2}_w],\end{aligned}$$

as desired, proving (14). Putting this together with (11) and (12) we have

$$\begin{aligned}\overline{M} \models &[\varphi(\overline{c}, \overline{d}, \mathbf{0}, \overline{f_j(a_0, \ldots, a_{n-1})}) \wedge \psi(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{f_j(a_0, \ldots, a_{n-1})}) \\ &\wedge \varphi(\overline{c}, \overline{d}, \overline{m}, \overline{f_i(a_0, \ldots, a_{n-1}, m)}) \wedge \forall v_{t+2}[v_{t+2} \lhd \overline{m} \rightarrow \\ &\exists v_{t+3}\exists v_{t+4}[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, v_{t+2}, v_{t+3}, v_{t+4}) \\ &\wedge \varphi(\overline{c}, \overline{d}, v_{t+2}, v_{t+3}) \wedge \varphi(\overline{c}, \overline{d}, \mathbf{S}v_{t+2}, v_{t+4})]]][e].\end{aligned}$$

Hence an easy argument using Lemma 4.6 gives

$$\begin{aligned}\overline{M} \models &[\exists v_t \exists v_{t+1} \exists v_{t+5}[\varphi(v_t, v_{t+1}, \mathbf{0}, v_{t+5}) \wedge \psi(\overline{a_0}, \ldots, \overline{a_{n-1}}, v_{t+5}) \\ &\wedge \varphi(v_t, v_{t+1}, \overline{m}, \overline{f_i(a_0, \ldots, a_{n-1}, m)}) \wedge \forall v_{t+2}[\rho(v_{t+2}, \overline{m}) \rightarrow \exists v_{t+3}\exists v_{t+4} \\ &[\chi(\overline{a_0}, \ldots, \overline{a_{n-1}}, v_{t+2}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, v_{t+2}, v_{t+3}) \\ &\wedge \varphi(v_t, v_{t+1}, \mathbf{S}v_{t+2}, v_{t+4})]]]]][e].\end{aligned}$$

That is, $\overline{M} \models \theta(\overline{a_0}, \ldots, \overline{a_{n-1}}, \overline{m}, \overline{f_i(a_0, \ldots, a_{n-1}, m)})[e]$. This gives the first condition for representability for f_i.

For the second condition, suppose that $a_0, \ldots, a_{n-1}, m \in \omega$, $b \in M$, and $\overline{M} \models \theta(\overline{a}_0, \ldots, \overline{a}_{n-1}, \overline{m}, v_{n+1})[e^{n+1}_b]$. We want to show that $b = f_i(a_0, \ldots, a_{n-1}, m)$. Choose $c, d, g \in M$ such that

$$\begin{aligned}(15)\qquad \overline{M} \models &[\varphi(v_t, v_{t+1}, \mathbf{0}, v_{t+5}) \wedge \psi(\overline{a}_0, \ldots, \overline{a}_{n-1}, v_{t+5}) \\ &\wedge \varphi(v_t, v_{t+1}, \overline{m}, v_{n+1}) \wedge \forall v_{t+2}[v_{t+2} \lhd \overline{m} \rightarrow \exists v_{t+3}\exists v_{t+4} \\ &[\chi(\overline{a}_0, \ldots, \overline{a}_{n-1}, v_{t+2}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, v_{t+2}, v_{t+3}) \\ &\wedge \varphi(v_t, v_{t+1}, \mathbf{S}v_{t+2}, v_{t+4})]]][e^{n+1}_b{}^{t}_{c}{}^{t+1}_{d}{}^{t+5}_{g}].\end{aligned}$$

Now we claim

(16) For all $s \leq m$, $\overline{M} \models \varphi(v_t, v_{t+1}, \overline{s}, \overline{f_i(a_0, \ldots, a_{n-1}, s)})[e^{t}_{c}{}^{t+1}_{d}]$.

We prove (16) by induction on s. From (15), $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{n-1}, v_{t+5})[e^{t+5}_g]$, so by the second condition for ψ representing f_j, $g = f_j(a_0, \ldots, a_{n-1})$. Now by (15) again, $\overline{M} \models \varphi(v_t, v_{t+1}, \mathbf{0}, v_{t+5})[e^{t}_{c}{}^{t+1}_{d}{}^{t+5}_{g}]$. Since $f_j(a_0, \ldots, a_{n-1}) = f_i(a_0, \ldots, a_{n-1}, 0)$, this proves (16) for $s = 0$, using Lemma 4.6.

Now assume that $s < m$ and

$$(17)\qquad \overline{M} \models \varphi(v_t.v_{t+1}, \overline{s}, \overline{f_i(a_0, \ldots, a_{n-1}, s)})[e^{t}_{c}{}^{t+1}_{d}].$$

Now $\overline{s}' \lhd' \overline{m}'$ by Lemma 6.12, so by (15) we can choose $h, g \in M$ such that

$$\begin{aligned}(18)\qquad \overline{M} \models &\chi(\overline{a}_0, \ldots, \overline{a}_{n-1}, \overline{s}, v_{t+3}, v_{t+4}) \wedge \varphi(v_t, v_{t+1}, \overline{s}, v_{t+3}) \\ &\wedge \varphi(v_t, v_{t+1}, \mathbf{S}\overline{s}, v_{t+4})[e^{t}_{c}{}^{t+1}_{d}{}^{t+3}_{h}{}^{t+4}_{g}].\end{aligned}$$

By (17), (18), and the additional property of φ we have $h = f_i(a_0, \ldots, a_{n-1}, s)$, using Lemma 4.6. Now by (18), $\overline{M} \models \chi(\overline{a}_0, \ldots, \overline{a}_{n-1}, \overline{s}, v_{t+3}.v_{t+4})[e^{t+3}_{h}{}^{t+4}_{g}]$, so by the second property of χ representing f_k we get

$$g = f_k(a_0, \ldots, a_{n-1}, s, f_i(a_0, \ldots, a_{n-1})) = f_i(a_0, \ldots, a_{n-1}, s+1).$$

Now by (18) again, $\overline{M} \models \varphi(v_t, v_{t+1}, \mathbf{S}(\overline{s}), v_{t+4})[e^{t\ t+1\ t+4}_{c\ d\ g}]$. This gives (16) for $s+1$. So by induction, (16) holds.

The case $s = m$ in (16) is $\overline{M} \models \varphi(v_t, v_{t+1}, \overline{m}, \overline{f_i(a_0, \ldots, a_{n-1}, m)})[e^{t\ t+1}_{c\ d}]$. From (15) we have $\overline{M} \models \varphi(v_t, v_{t+1}, \overline{m}, v_{n+1})[e^{t\ t+1\ n+1}_{c\ d\ b}]$. Hence by the additional property of φ it follows that $b = f_i(a_0, \ldots, a_{n-1}, m)$.

This finishes the treatment of $\mathbf{Q}_n$ for $n > 0$.

Case 6. There exist a positive integer m and a $j < i$ such that f_j is a special $(m+1)$-ary operation on ω and $f_i = \mathbf{M}_m(f_j)$. Let φ represent f_j. Choose s such that $s > 2$ and v_s does not occur in φ. Then we claim that the following formula ψ represents f_i:

$$\varphi(v_0, \ldots, v_m, \mathbf{0}) \wedge \forall v_s[v_s \lhd v_m \to \neg\varphi(v_0, \ldots, v_{m-1}, v_s, \mathbf{0})].$$

To prove this, suppose that $a_0, \ldots, a_{m-1} \in \omega$, and let $b = f_i(a_0, \ldots, a_{m-1})$. Thus $f_j(a_0, \ldots, a_{m-1}, b) = 0$, and so

$$\overline{M} \models \varphi(\overline{a_0}, \ldots, \overline{a}_{m-1}, \overline{b}, \mathbf{0})[e]. \tag{19}$$

Assume that $c \in M$ and $c \lhd' \overline{b}'$. By Lemma 6.19 there is an $s < b$ such that $c = \overline{s}'$. By the second condition for φ representing f_j we have

$$\overline{M} \models (\varphi(\overline{a}_0, \ldots, \overline{a}_{m-1}, \overline{s}, \mathbf{0}) \to \overline{f_j(a_0, \ldots, a_{m-1}, c)} = \mathbf{0})[e]. \tag{20}$$

Now $f_j(a_0, \ldots, a_{m-1}, c) \neq 0$, and so by Lemma 6.10, $\overline{f_j(a_0, \ldots, a_{m-1}, c))}' \neq \mathbf{0}'$. Hence from (20) we get

$$\overline{M} \models \neg\varphi(\overline{a}_0, \ldots, \overline{a}_{m-1}, \overline{c}, \mathbf{0})[e].$$

It now follows from Lemma 6.19 that

$$\overline{M} \models \forall v_s[v_s \lhd \overline{b} \to \neg\varphi(\overline{a}_0, \ldots, \overline{a}_{m-1}, v_s, \mathbf{0})][e].$$

Together with (21) this gives $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{m-1}, \overline{b})[e]$.

Now for the second representability condition, suppose $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{m-1}, v_m)[e^m_c]$ with $c \in M$; we want to show that $c = b$. Assume not. By Lemma 6.15 this gives two possibilities.

Case 1. $c \lhd \overline{b}$. Since $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{m-1}, \overline{b})[e]$, we get

$$\overline{M} \models \neg\varphi(\overline{a}_0, \ldots, \overline{a}_{m-1}, v_m, \mathbf{0})[e^m_c],$$

contradicting $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{m-1}, v_m)[e^m_c]$.

Case 2. $\overline{b}' \lhd c$. Since $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{m-1}, v_m)[e^m_c]$, we get

$$\overline{M} \models \neg\varphi(\overline{a}_0, \ldots, \overline{a}_{m-1}, \overline{b}, \mathbf{0})[e],$$

contradicting $\overline{M} \models \psi(\overline{a}_0, \ldots, \overline{a}_{m-1}, \overline{b})[e]$.

This finishes Case 6, and so the proof of Theorem 6.25. ☐

Theorem 6.26. *Every recursive relation is representable.*

Proof. Let R be any recursive relation. Say R is m-ary. Thus χ_R is recursive. By Theorem 6.25, let ψ represent χ_R. Thus ψ has free variables among $v_0, \ldots, v_m$. Let φ be the formula $\psi(v_0, \ldots, v_{m-1}, \mathbf{S0})$. Let $a_0, \ldots, a_{m-1} \in \omega$. If $\langle a_0, \ldots, a_{m-1}\rangle \in R$, then $\chi_R(a_0, \ldots, a_{m-1}) = 1$, and hence $\mathbf{P} \vdash \psi(\overline{a_0}, \ldots, \overline{a_{m-1}}, \overline{1})$; hence $\mathbf{P} \vdash \varphi(\overline{a_0}, \ldots, \overline{a_{m-1}})$. If $\langle a_0, \ldots, a_{m-1}\rangle \notin R$, then $\chi_R(a_0, \ldots, a_{m-1}) = 0$, and hence $\mathbf{P} \vdash \psi(\overline{a_0}, \ldots, \overline{a_{m-1}}, \overline{1}) \to \overline{1} = \mathbf{0}$ by the second condition in the definition of χ_R being representable. Since $\mathbf{P} \vdash \neg(\overline{1} = \mathbf{0})$ by (P2), it follows that $\mathbf{P} \vdash \neg\psi(\overline{a_0}, \ldots, \overline{a_{m-1}}, \overline{1})$, i.e., $\mathbf{P} \vdash \neg\varphi(\overline{a_0}, \ldots, \overline{a_{m-1}})$. □

EXERCISES

E6.1. The exponential function is defined as follows. $a^0 = 1$ and $a^{s(b)} = a^b \cdot a$. Show that the exponential function is representable.

E6.2. Prove that $\mathbf{P} \vdash \forall v_0 \forall v_1 [v_0 + v_1 = v_1 + v_0]$.

E6.3. Prove that $\mathbf{P} \vdash \forall v_1 \forall v_0 [v_1 \bullet v_0 + v_0 = \mathbf{S} v_1 \bullet v_0]$.

E6.4. Prove that $\mathbf{P} \vdash \forall v_0 [\mathbf{0} \bullet v_0 = \mathbf{0}]$.

E6.5. Prove that $\mathbf{P} \vdash \forall v_0 \forall v_1 [v_0 \bullet v_1 = v_1 \bullet v_0]$.

E6.6. Let φ be the formula defined in the proof of Lemma 6.24. Show that $\mathbf{P} \vdash \forall v_0 \forall v_1 \forall v_2 \exists v_3 \varphi$.