Errata 2

This material is supposed to replace Lemmas 4.21 and 4.22 in the notes.

Lemma 4.21. *Suppose that* $\mathbf{R}$ *is an* m*-ary relation symbol and* $\langle i(0),\ldots,i(m-1)\rangle$ *is a sequence of distinct natural numbers such that* $m\le i(j)$ *for all* $j<m$. *Then*

$$\vdash \mathbf{R}v_{i(0)}\ldots v_{i(m-1)} \leftrightarrow \exists v_0\ldots\exists v_{m-1}\left[\bigwedge_{j<m}(v_j=v_{i(j)})\wedge \mathbf{R}v_0\ldots,v_{m-1}\right].$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a:\omega\to A$. First suppose that $\overline{A}\models \mathbf{R}v_{i(0)}\ldots v_{i(m-1)}[a]$. Thus $\langle a_{i(0)},\ldots,a_{i(m-1)}\rangle\in\mathbf{R}^{\overline{A}}$. Let

$$b=a^{0\;\;\;\;1\;\;\;\;\ldots\;\;\;\;m-1}_{a_{i(0)}\;a_{i(1)}\;\ldots\;a_{i(m-1)}}.$$

Then for any $j<m$ we have $v_j^{\overline{A}}(b)=b_j=a_{i(j)}=b_{i(j)}=v_{i(j)}^{\overline{A}}(b)$. It follows that $\overline{A}\models\bigwedge_{j<m}(v_j=v_{i(j)})[b]$. Also, $\langle b_0,\ldots,b_{m-1}\rangle=\langle a_{i(0)},\ldots,a_{i(m-1)}\rangle\in\mathbf{R}^{\overline{A}}$. Hence $\overline{A}\models\mathbf{R}v_0\ldots v_{m-1}[b]$. Thus

$$\overline{A}\models\left[\bigwedge_{j<m}(v_j=v_{i(j)})\wedge \mathbf{R}v_0\ldots,v_{m-1}\right][b]$$

and hence

$$\overline{A}\models\exists v_0\ldots\exists v_{m-1}\left[\bigwedge_{j<m}(v_j=v_{i(j)})\wedge \mathbf{R}v_0\ldots,v_{m-1}\right][a] \tag{1}$$

Hence we have shown that $\overline{A}\models \mathbf{R}v_{i(0)}\ldots v_{i(m-1)}[a]$ implies (1).

Now suppose conversely that (1) holds. Choose $x(0),\ldots,x(m-1)\in A$ such that

$$\overline{A}\models\left[\bigwedge_{j<m}(v_j=v_{i(j)})\wedge \mathbf{R}v_0\ldots,v_{m-1}\right][b],$$

where $b=a^{0\;\;\;\;1\;\;\;\;\ldots\;\;\;\;m-1}_{x(0)\;x(1)\;\ldots\;x(m-1)}$. For any $j<m$ we have $b_j=x(j)=v_j^{\overline{A}}(b)=v_{i(j)}^{\overline{A}}(b)=v_{i(j)}^{\overline{A}}(a)=a_{i(j)}$. We also have $\langle b_0,\ldots,b_{m-1}\rangle\in\mathbf{R}^{\overline{A}}$. Hence $\langle a_{i(0)},\ldots,a_{i(m-1)}\rangle\in\mathbf{R}^{\overline{A}}$, and it follows that $\overline{A}\models \mathbf{R}v_{i(0)}\ldots v_{i(m-1)}[a]$.

So we have shown that $\overline{A}\models \mathbf{R}v_{i(0)}\ldots v_{i(m-1)}[a]$ iff (1). Therefore

$$\models \mathbf{R}v_{i(0)}\ldots v_{i(m-1)} \leftrightarrow \exists v_0\ldots\exists v_{m-1}\left[\bigwedge_{j<m}(v_j=v_{i(j)})\wedge \mathbf{R}v_0\ldots,v_{m-1}\right].$$

and it follows by the completeness theorem that

$$\vdash \mathbf{R}v_{i(0)}\ldots v_{i(m-1)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{j<m}(v_j = v_{i(j)}) \wedge \mathbf{R}v_0 \ldots, v_{m-1}\right]. \qquad \square$$

The proof of the following lemma is very similar to the proof of Lemma 4.21.

Lemma 4.22. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol and* $\langle i(0), \ldots, i(m)\rangle$ *is a sequence of distinct natural numbers such that* $m+1 \leq i(j)$ *for all* $j \leq m$*. Then*

$$\vdash \mathbf{F}v_{i(0)}\ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \exists v_0 \ldots \exists v_m\left[\bigwedge_{j\leq m}(v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m\right].$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that $\overline{A} \models \mathbf{F}v_{i(0)}\ldots v_{i(m-1)} = v_{i(m)}[a]$. Thus $\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(m-1)}) = a_{i(m)}$. Let

$$b = a^{0\quad 1\quad \ldots\quad m}_{a_{i(0)}\ a_{i(1)}\ \ldots\ a_{i(m)}}.$$

Then for any $j \leq m$ we have $v_j^{\overline{A}}(b) = b_j = a_{i(j)} = b_{i(j)} = v_{i(j)}^{\overline{A}}(b)$. It follows that $\overline{A} \models \bigwedge_{j\leq m}(v_j = v_{i(j)})[b]$. Also,

$$\begin{aligned}\mathbf{F}(b_0, \ldots, b_{m-1}) &= \mathbf{F}(a_{i(0)}, \ldots, a_{i(m-1)})\\ &= a_{i(m)}\\ &= b_m.\end{aligned}$$

Hence $\overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m)[b]$. Thus

$$\overline{A} \models \left[\bigwedge_{j\leq m}(v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m\right][b]$$

and hence

(1) $$\overline{A} \models \exists v_0 \ldots \exists v_m\left[\bigwedge_{j\leq m}(v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m\right][a]$$

Hence we have shown that $\overline{A} \models \mathbf{R}v_{i(0)}\ldots v_{i(m-1)}[a]$ implies (1).

Now suppose conversely that (1) holds. Choose $x(0), \ldots, x(m) \in A$ such that

$$\overline{A} \models \left[\bigwedge_{j\leq m}(v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m\right][b],$$

where $b = a^{0}_{x(0)}\,{}^{1}_{x(1)}\,{}^{\cdots}_{\cdots}\,{}^{m}_{x(m)}$. For any $j \leq m$ we have $b_j = x(j) = v_j^{\overline{A}}(b) = v_{i(j)}^{\overline{A}}(b) = v_{i(j)}^{\overline{A}}(a) = a_{i(j)}$. We also have $\langle \mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}) = b_m$. Hence $\mathbf{F}^{A}(a_{i(0)}, \ldots, a_{i(m-1)}) = a_{i(m)}$, and it follows that $\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(m-1)}) = v_{i(m)})[a]$.

So we have shown that $\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(m-1)}) = v_{i(m)})[a]$ iff (1). Therefore

$$\models \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1} \left[\bigwedge_{j \leq m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right].$$

and it follows by the completeness theorem that

$$\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \exists v_0 \ldots \exists v_{m-1} \left[\bigwedge_{j \leq m} (v_j = v_{i(j)}) \wedge \mathbf{F}v_0 \ldots, v_{m-1} = v_m \right]. \quad \square$$