The Halpern-Läuchli Theorem

We expand the origonal proof. We deal with trees of height ω, finitely branching, with a unique root, and with no maximal nodes. A set S of nodes is (h,k)-*dense* iff there is a node x of height h such that S dominates the nodes of height $h+k$ which are above x. k-*dense* means $(0,k)$-dense, and ∞-*dense* means k-dense for all k.

Proposition 1. *S is k-dense iff S dominates the nodes of height k.* □

Proposition 2. *S is ∞-dense iff S dominates all nodes of T.* □

We define $T\uparrow t=\{s:t\le s\}$. For each $n\in\omega$, $n(T)=\{T\uparrow x:|x|=n\}$. For $B\subseteq T$, $n(T,B)=\{(T\uparrow t)\cap B:|t|=n\}$. If $\mathbf{T}=(T_1,\ldots,T_d)$ is a system of trees, then an (h,k)-*matrix* for $\mathbf{T}$ is a product $\prod_{i=1}^d A_i$ with each A_i (h,k)-dense in T_i. A k-*matrix* is a $(0,k)$-matrix.

Theorem 3. (Halpern-Läuchli) *Let $\mathbf{T}=(T_1,\ldots,T_d)$ be a system of trees, each finitely branching, with a single root, and of height ω. Suppose that $Q\subseteq\prod_{i=1}^d T_i$. Then one of the following conditions holds:*

(i) For all $k\in\omega$ there is a k-matrix contained in Q.

(ii) There is an $h\in\omega$ such that for each k there is an (h,k)-matrix contained in $(\prod_{i=1}^d T_i)\backslash Q$.

Proof. We first introduce a certain algebra of symbols. *Atomic symbols* are

$$\exists A_i, \forall x_i, \forall a_i, \exists x_i \quad \text{for each positive integer } i.$$

For each positive integer d we define

$L_d=\{\sigma:\sigma$ is a function with domain $\{1,\ldots,2d\}$, and for each $i\in\{1,\ldots,d\}$ exactly one of the following holds:

(i) Each of $\exists A_i$ and $\forall x_i$ occurs exactly once in σ, with $\exists A_i$ before $\forall x_i$.

(ii) Each of $\forall a_i$ and $\exists x_i$ occurs exactly once in σ, with $\forall a_i$ before $\exists x_i$.

Examples:

$L_1=\{\langle\exists A_1,\forall x_1\rangle,\langle\forall a_1\exists x_1\rangle\}$.

$L_2=\{\langle\exists A_1,\forall x_1,\exists A_2,\forall x_2\rangle,\langle\exists A_1,\exists A_2,\forall x_1,\forall x_2\rangle,\ldots\}$.

Now we define a relation $\vdash_d$ on L_d. α,β stand for A_i,a_i,x_i and U and V are strings of atomic symbols each of length $d-1$.

Rules 1.

$U\exists\alpha\exists\beta V\vdash_d U\exists\beta\exists\alpha V$, if $U\exists\alpha\exists\beta V, U\exists\beta\exists\alpha V\in L_d$.

$U\forall\alpha\forall\beta V\vdash_d U\forall\beta\forall\alpha V$, if $U\forall\alpha\forall\beta V, U\forall\beta\forall\alpha V\in L_d$.

$U\exists\alpha\forall\beta V\vdash_d U\forall\beta\exists\alpha V$, if $U\exists\alpha\forall\beta V, U\forall\beta\exists\alpha V\in L_d$,

Rules 2.

$U\forall a_i\exists x_i V\vdash_d U\exists A_i\forall x_i V$ for all $i=1,\ldots,d$ such that $U\forall a_i\exists x_i V, U\exists A_i\forall x_i V\in L_d$.

$$U\exists A_i\forall x_i V \vdash U\forall a_i\exists x_i V \text{ for all } i=1,\ldots,d \text{ such that } U\forall a_i\exists x_i V, U\exists A_i\forall x_i V \in L_d.$$

To state rules 3, we first define, if $\langle V_i : r \leq i \leq k\rangle$ is a sequence of strings of atomis symbols, then $(V_i)_r^k$ is the concatenation $V_r \cdots V_k$.

Rules 3.

If σ is a permutation of $\{1,\ldots,d\}$, then
$(\forall a_{\sigma(i)})_1^r(\exists A_{\sigma(i)})_{r+1}^d V \vdash_d (\exists A_{\sigma(i)})_{r+1}^d(\forall a_{\sigma(i)})_1^r V$ for $r = 1\ldots d-1$.

Example

$$d=4, \quad r=2, \quad V=\exists x_3\forall x_1\forall x_4\exists x_2, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

gives

$$\forall a_2\forall a_3\exists A_1\exists A_4\exists x_3\forall x_1\forall x_4\exists x_2 \vdash_d \exists A_1\exists A_4\forall a_2\forall a_3\exists x_3\forall x_1\forall x_4\exists x_9.$$

$\models_d$ is the transitive closure of $\vdash_d$.

(1) $\forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d \models_d \exists A_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\forall x_d$.

Proof of (1): Let $\sigma(1)=d$, $\sigma(i+1)=i$ for $i=2,\ldots,d-1$, $r=1$. Then an instance of rules 3 is

$$(\forall a_{\sigma(i)})_1^1(\exists A_{\sigma(i)})_2^d(\forall x_i)_1^{d-1}\exists x_d \models_d (\exists A_{\alpha(i)})_2^d(\forall a_{\sigma(i)})_1^1(\forall x_i)_1^{d-1}\exists x_d,$$

or

$$(1a) \qquad \forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d \models_d (\exists A_i)_1^{d-1}\forall a_d(\forall x_i)_1^{d-1}\exists x_d$$

By Rules 1,

$$(1b) \qquad (\exists A_i)_1^{d-1}\forall a_d(\forall x_i)_1^{d-1}\exists x_d \models_d (\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall a_d\exists x_d$$

Again using rules 1,

$$\begin{aligned}(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall a_d\exists x_d &\models_d \exists A_1\forall x_1(\exists A_i)_2^{d-1}(\forall x_i)_2^{d-1}\forall a_d\exists x_d \\ &\cdots \\ &\models_d (\exists A_i\forall x_i)_1^{d-1}\forall a_d\exists x_d \qquad (1c)\end{aligned}$$

By rules 2,

$$(1d) \qquad (\exists A_i\forall x_i)_1^{d-1}\forall a_d\exists x_d \models_d (\forall a_i\exists x_i)_1^{d-1}\exists A_d\forall x_d$$

By Rules 1,

$$\begin{aligned}(\forall a_i\exists x_i)_1^{d-1}\exists A_d\forall x_d &\models_d (\forall a_i\exists x_i)_1^{d-2}\forall a_{d-1}\exists x_{d-1}\exists A_d\forall x_d \\ &\models_d (\forall a_i\exists x_i)_1^{d-3}\forall a_{d-2}\exists x_{d-2}\forall a_{d-1}\exists x_{d-1}\exists A_d\forall x_d \\ &\models_d (\forall a_i\exists x_i)_1^{d-3}\forall a_{d-2}\exists x_{d-2}\forall a_{d-1}\exists A_d\exists x_{d-1}\forall x_d \\ &\models_d (\forall a_i\exists x_i)_1^{d-3}\forall a_{d-2}\forall a_{d-1}\exists x_{d-2}\exists A_d\exists x_{d-1}\forall x_d \\ &\models_d (\forall a_i\exists x_i)_1^{d-3}\forall a_{d-2}\forall a_{d-1}\exists A_d\exists x_{d-2}\exists x_{d-1}\forall x_d \\ &\cdots \\ &\models_d (\forall a_i)_1^{d-1}\exists A_d(\exists x_i)_1^{d-1}\forall x_d \qquad (1e)\end{aligned}$$

Now with σ the identity and $r = d-1$, rules 3 give

$$(1f) \qquad (\forall a_i)_1^{d-1} \exists A_d (\exists x_i)_1^{d-1} \forall x_d \models_d \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d$$

Now (1a)–(1f) give (1).

(2) Suppose that $UV \in L_d$, U has length d, no atoms of the forms $\forall x_i, \exists x_i$ occur in U, and $\bar{U}$ is any rearrangement of U. Then $UV \models_d \bar{U}V$.

In fact, assume the hypotheses. So only $\exists A_i$ and $\forall a_i$ occur in U. If no $\forall a_i$ occurs, or no $\exists A_i$ occurs, the conclusion is clear by rules 1. So suppose some $\forall a_i$ occurs and some $\exists A_i$ occurs. By rules 1 there is a permutation σ of $\{1, \ldots, d\}$ such that

$$UV \models_d (\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V$$

By rules 3,

$$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \models_d (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$$

Then by rules 1,

$$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \models_d \bar{U}V.$$

Thus (2) holds.

(3) If $W \models_{d-1} \bar{W}$, then $\forall a_d W \exists x_d \models_d \forall a_d \bar{W} \exists x_d$.

For, assume that $W \models_{d-1} \bar{W}$. Say

$$W = S_0 \vdash_{d-1} S_1 \vdash_{d-1} S_2 \cdots \vdash_{d-1} S_n = \bar{W}.$$

We claim that

$$\forall a_d W \exists x_d = \forall a_d S_0 \exists x_d \vdash_d \forall a_d S_1 \exists x_d \cdots \vdash_d \forall a_d S_n \exists x_d = \forall a_d \bar{W} \exists x_d.$$

Consider the step from S_i to S_{i+1}. If rules (1) or rules (2) are used in going from S_i to S_{i+1}, clearly the same rules go from $\forall a_d S_i \exists x_d$ to $\forall a_d S_{i+1} \exists x_d$. Suppose that rules (3) are used. Say S_i is $(a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^{d-1} V$ and S_{i+1} is $(\exists A_{\sigma(i)})_{r+1}^{d-1} (\forall a_{\sigma(i)})_1^r V$. Then $\forall a_d S_i \exists x_d$ is $\forall a_d (a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^{d-1} V \exists x_d$ and $\forall a_d S_{i+1} \exists x_d$ is $\forall a_d (\exists A_{\sigma(i)})_{r+1}^{d-1} (\forall a_{\sigma(i)})_1^r V \exists x_d$. Hence $\forall a_d S_i \exists x_d \models_d \forall a_d S_{i+1} \exists x_d$ by (2). Hence (3) holds.

(4) $(\forall a_i)_1^d (\exists x_i)_1^d \models_d (\exists A_i)_1^d (\forall x_i)_1^d$.

In fact, we prove this by induction on d. For $d = 1$ the assertion is that $\forall a_1 \exists x_1 \models_d \exists A_1 \forall x_1$, which is an instance of rules 2. Now assume (4) for $d - 1 \geq 1$. Then rules 1 give

$$(4a) \qquad (\forall a_i)_1^d (\exists x_i)_1^d \models_d \forall a_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \exists x_d$$

By the inductive hypothesis and (3) we have

$$(4b) \qquad \forall a_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \exists x_d \models_d \forall a_d (\exists A_i)_1^{d-1} \forall x_i)_1^{d-1} \exists x_d$$

By (1) we have

$$(4c) \qquad \forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d \models_d \exists A_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\forall x_d$$

By the inductive hypothesis and (3) we have

$$(4d) \qquad \exists A_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\forall x_d \models_d \exists A_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall x_d$$

Now by rules (1) we get

$$(4e) \qquad \exists A_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall x_d \models_d (\exists A_i)_1^d(\forall x_i)_1^d$$

Now (4a)–(4e) give (4).

Now suppose that $\mathbf{T} = \langle T_i : 1 \leq i \leq d\rangle$ is a vector tree and $Q \subseteq \prod_{i=1}^d T_i$. We define a $(d+1)$-sorted language $\mathscr{L}$. The sorts are $S_1, \ldots, S_{d+1}$. Additional constants are as follows.

A d-ary function symbol Seq acting on d-tuples from $S_1 \times \cdots \times S_d$ with values in S_{d+1}.
For each $i = 1, \ldots, d$, a binary relation symbol $<_i$ acting on S_i.
$x_1, \ldots, x_d$ are variables ranging over $S_1, \ldots, S_d$ respectively.
$B_1, \ldots, B_d$ are constants for subsets of $S_1, \ldots, S_d$ respectively.
$A_1, \ldots, A_d$ are variables ranging over subsets of $S_1, \ldots, S_d$ respectively.
v_{ik} for $i = 1, \ldots, d$ and $k \in \omega$ are variables ranging over S_i,
$a_1, \ldots, a_d$ are variables ranging over subsets of $S_1, \ldots, S_d$ respectively.
Q, a constant for a subset of S_{d+1}

A structure for this language assigns T_i to S_i for $i = 1, \ldots, d$, the product $\prod_{i=1}^d T_i$ to S_{d+1}, and subsets B_i of T_1 for $i = 1, \ldots, d$, with Q assigned to Q.

Now with each sequence $\mathbf{n} = (n_1, \ldots, n_d)$ of positive integers and each sequence W of atomic symbols we associate a formula $\varphi = \varphi_{W\mathbf{n}}$. This is done by induction on the length of W

If W is empty, we let $\varphi_{W\mathbf{n}}$ be the formula $Seq(x_1, \ldots, x_d) \in Q$.

If $W = \exists A_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\exists A_i[A_i \subseteq B_i \wedge A_i$ is n_i-dense in $S_i \wedge \varphi_{W'\mathbf{n}}]$. Here "$A_i$ is n_i-dense in S_i" is the formula

$$\forall t \in S_i[|t| = n_i \to \exists s \in A_i[t \leq_i s]].$$

We use the variables v_{ik} to express this.

If $W = \forall x_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\forall x_i[x_i \in A_i \to \varphi_{W'\mathbf{n}}]$.

If $W = \forall a_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{W'\mathbf{n}}]$ Here $a_i \in n_i(S_i, B_i)$ is the formula

$$\exists t \in S_i[|t| = n_i \wedge \forall s[s \in a_i \leftrightarrow [t \leq s \wedge s \in B_i]]].$$

If $W = \exists x_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\exists x_i[x_i \in a_i \wedge \varphi_{W'\mathbf{n}}]$.

Now we let $\psi(W, n, p)$ be the statement "$\forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d[[\forall i = 1, \ldots, d[B_i$ is p-dense in $S_i] \to \varphi_{W\mathbf{n}}]]$"

(5) Suppose that W, W', ρ are sequences of atomic symbols. Suppose that under every assignment of values to the variables, φ_{Wn} implies $\varphi_{W'n}$. Then $\varphi_{\rho Wn}$ under any assignment implies $\varphi_{\rho W'n}$ under that assignment.

We prove this by induction on ρ. If ρ is empty, it is obvious. The induction step is clear upon looking at what $\varphi_{\rho Wn}$ is:

Case 1. $\rho = \exists A_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\rho' Wn}]$$

Case 2. $\rho = \forall x_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\forall x_i[x_i \in A_i \rightarrow \varphi_{\rho' Wn}].$$

Case 3. $\rho = \forall a_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\forall a_i[a_i \in n_i(S_i, B_i) \rightarrow \varphi_{\rho' Wn}]$$

Case 4. $\rho = \exists x_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{\rho' Wn}]$$

This proves (5).

(6) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists A_i \exists A_j Wn}$ implies that $\varphi_{\exists A_j \exists A_i W'n}$.

In fact, $\varphi_{\exists A_i \exists A_j Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{A_j W\mathbf{n}}];$$

expanding we get

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \exists A_j[A_j \subseteq B_j \wedge A_j \text{ is } n_j\text{-dense in } S_j \wedge \varphi_{W\mathbf{n}}]].$$

This is logically equivalent to

$$\exists A_i \exists A_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge A_j \subseteq B_j \wedge A_j \text{ is } n_j\text{-dense in } S_j \wedge \varphi_{W\mathbf{n}}].$$

Hence (6) holds.

(7) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists A_i \exists x_j Wn}$ implies that $\varphi_{\exists x_j \exists A_i W'n}$.

In fact, $\varphi_{\exists A_i \exists x_j Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{x_j W\mathbf{n}}];$$

expanding we get

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \exists x_j[x_j \in a_j \wedge \varphi_{W\mathbf{n}}]].$$

This is logically equivalent to

$$\exists A_i \exists x_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge x_j \in a_j \wedge \varphi_{W\mathbf{n}}].$$

Hence (7) holds.

(8) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists x_i \exists A_j Wn}$ implies that $\varphi_{\exists A_j \exists x_i W'n}$.

This is proved as for (7).

(9) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists x_i \exists x_j Wn}$ implies that $\varphi_{\exists x_j \exists x_i W'n}$.

In fact, $\varphi_{\exists x_i \exists x_j Wn}$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{x_j W\mathbf{n}}];$$

expanding we get

$$\exists x_i[x_i \in a_i \wedge \exists x_j[x_j \in a_j \wedge \varphi_{W\mathbf{n}}]].$$

This is logically equivalent to

$$\exists x_i \exists x_j[x_i \in a_i \wedge x_j \in a_j \wedge \varphi_{W\mathbf{n}}].$$

Hence (9) holds.

(10) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall x_i \forall x_j Wn}$ implies that $\varphi_{\forall x_j \forall x_i Wn}$.

In fact, $\varphi_{\forall x_i \forall x_j Wn}$ is

$$\forall x_i[x_i \in A_i \to \varphi_{\forall x_j Wn}];$$

expanding we get

$$\forall x_i[x_i \in A_i \to \forall x_j[x_j \in A_j \to \varphi_{Wn}]]$$

This is logically equivalent to

$$\forall x_i \forall x_j[x_i \in A_i \to [x_j \in A_j \to \varphi_{Wn}]]$$

Hence (10) follows.

(11) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall a_i \forall x_j Wn}$ implies that $\varphi_{\forall x_j \forall a_i Wn}$.

In fact, $\varphi_{\forall a_i \forall x_j Wn}$ is

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{\forall x_j Wn}];$$

expanding we get

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \forall x_j[x_j \in A_j \to \varphi_{Wn}]]$$

This is logically equivalent to

$$\forall a_i \forall x_j[x_i \in n_i(S_i, B_i) \to [x_j \in A_j \to \varphi_{Wn}]]$$

Hence (11) follows.

(12) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall x_i \forall a_j Wn}$ implies that $\varphi_{\forall a_j \forall x_i Wn}$.

This is similar to (11).

(13) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall a_i \forall a_j Wn}$ implies that $\varphi_{\forall a_j \forall a_i Wn}$.

This is similar to (11).

(14) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists A_i \forall a_j Wn}$ implies that $\varphi_{\forall a_j \exists A_i Wn}$.

In fact, $\varphi_{\exists A_i \forall a_j Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\forall a_j Wn}]$$

Expanding, we get

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \forall a_j[a_j \in n_j(S_j, B_j) \to \varphi_{Wn}]].$$

This is logically equivalent to

$$\exists A_i \forall a_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [a_j \in n_j(S_j, B_j) \to \varphi_{Wn}]].$$

This implies

$$\forall a_j \exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [a_j \in n_j(S_j, B_j) \to \varphi_{Wn}]].$$

Hence (14) holds.

One similarly treats other sequences of the form $\exists\alpha\forall\beta$.

(15) $A_i \subseteq B_i$ is n_i-dense in T_i iff $A_i \subseteq B_i$ and $\forall a_i \in n_i(T_i, B_i)[a_i \cap A_i \neq \emptyset]$.

In fact, for $\Rightarrow$, suppose that $A_i \subseteq B_i$ is n_i-dense in T_i and $a_i \in n_i(T_i, B_i$. Say $a_i = (T_i \uparrow t) \cap B_i$ with $|t| = n_i$. There is an $s \in A_i$ such that $t \leq s$. Thus $s \in a_i \cap A_i$.

For $\Leftarrow$, suppose that $A_i \subseteq B_i$ and $\forall a_i \in n_i(T_i, B_i)[a_i \cap A_i \neq \emptyset]$ and $|t| = n_i$. Set $a_i = (T_i \uparrow t) \cap B_i$. Choose $u \in a_i \cap A_i$. Then $t \leq u$, as desired.

(16) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall a_i \exists x_i Wn}$ implies that $\varphi_{\exists A_i \forall x_i Wn}$.

In fact, $\varphi_{\forall a_i \exists x_i W n}$ is

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{\exists x_i W n}];$$

expanding, we get

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \exists x_i[x_i \in a_i \wedge \varphi_{W n}]].$$

Now assume $\varphi_{\forall a_i \exists x_i W n}$. For each $a_i \in n_i(T_i, B_i)$ choose $x_i(a_i) \in a_i$ such that $\varphi_{W n}$. Let $A_i = \{x_i(a_i) : a_i \in n_i(T_i, B_i)\}$. Note that $\forall a_i \in n_i(T_i, B_i)[a_i \subseteq B_i]$. Hence $A_i \subseteq B_i$. Hence by (15), A_i is n_i-dense in T_i. Now $\forall x_i \in A_i \varphi_{W n}$. So (16) holds.

(17) If $W \vdash W'$ using rules 1 or 2, then $\forall n \exists p \psi(W.n.p)$ implies that $\forall n \exists p \psi(W'.n.p)$.

In fact, $\forall n \exists p \psi(W.n.p)$ is

$$\forall n \exists p \forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d[[\forall i = 1, \dots, d[B_i \text{ is } p\text{-dense in } S_i] \to \varphi_{W\mathbf{n}}]],$$

and similarly for $\forall n \exists p \psi(W'.n.p)$. Hence (17) follows from (5)–(16).

Now suppose that σ is a permutation of $\{1, \dots, d\}$. Let $W = (\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V$ and $\overline{W} = (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$ with $r \in \{1 \dots d-1\}$. Now for simplicity we assume that σ is the identity. Note that V is a string of length d whose entries are $\forall x_i$ for $r+1 \le i \le d$ and $\exists x_j$ for $1 \le j \le r$; moreover, only A_i for $i = r+1, \dots, d$ and a_i for $i = 1 \dots, r$ are free. If V is such a string, $\mathbf{a}$ is an assignment of values to the a_i's, $A_{r+1}, \dots, A_d$ an assignment of values to the A_i's, then the assertion $\varphi_{V n}[\mathbf{a}, A_{r+1}, \dots, A_d]$ has the natural meaning.

(18) If V is such a string, $\mathbf{a}$ assigns values to the a_i for $i = 1, \dots, r$, $A_{r+1}, \dots, A_d$ an assignment of values to the A_i's, $A'_{r+1} \subseteq A_{r+1}, \dots A'_d \subseteq A_d$, and $\varphi_{V n}[\mathbf{a}, A_{r+1}, \dots, A_d]$, then $\varphi_{V n}[\mathbf{a}, A'_{r+1}, \dots, A'_d]$.

We prove (18) by induction on the length of V. It is trivial for the empty string. Now suppose that the string is $\forall x_i V'$. Then $\varphi_{\forall x_i V' n}[\mathbf{a}, A_{r+1}, \dots, A_d]$ is

$$\forall x_i[x_i \in A_i \to \varphi_{V' n}[\mathbf{a}, A_{r+1}, \dots, A_d]],$$

so

$$\forall x_i[x_i \in A'_i \to \varphi_{V' n}[\mathbf{a}, A'_{r+1}, \dots, A'_d]],$$

If the string is $\exists x_i V'$, then $\varphi_{\exists x_i V' n}[\mathbf{a}, A_{r+1}, \dots, A_d]$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{V' n}[\mathbf{a}, A_{r+1}, \dots, A_d]],$$

and the conclusion is obvious. So (18) holds.

To prove the implication in (17) for rules 3, suppose that $\forall \mathbf{n} \exists p \psi(W, \mathbf{n}, p)$. Let F be such that $\forall \mathbf{n} \psi(W, \mathbf{n}, F(\mathbf{n}))$. Thus

(19) $$\forall \mathbf{n}[\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d \forall i = 1, \dots, d[B_i \text{ is } F(\mathbf{n})\text{-dense in } T_i] \to \varphi_{W\mathbf{n}}]]].$$

Since p'-density implies p-density for $p < p'$, we may assume that for all $\mathbf{n}$ and all $i = 1, \dots, d$, $F(\mathbf{n}) > n_i$.

Now fix a sequence $\mathbf{n} = (n_1, \ldots, n_d)$ of positive integers. We want to find p such that $\psi(\overline{W}, \mathbf{n}, p)$. Define G by induction, as follows.

$$G(0) = \max\{n_i : r < i \leq d\};$$
$$G(j+1) = F(\mathbf{k}), \text{ where } k_i = \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ G(j) & \text{if } r < i \leq d. \end{cases}$$

Now for each $i = 1, \ldots, r$ let z_i be the number of elements of T_i of height n_i, and let $m = \prod_{i=1}^r z_i$. For each $j \leq m$ let $p_j = G(m - j)$.

(20) If $j < m$, then $p_{j+1} \leq p_j$.

For,

$$p_j = G(m-j) = G(m-j-1+1) = F(\mathbf{k}^j), \text{ where } k_i^j = \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ G(m-j-1) & \text{if } r < i \leq d \end{cases}$$
$$= \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ p_{j+1} & \text{if } r < i \leq d \end{cases}$$

Since p_{j+1} is an entry of $\mathbf{k}^j$, (20) holds.

It follows that

(21) If a set is p_j-dense in T_i, then it is also p_{j+1}-dense in T_i.

We claim that $\psi(\overline{W}, \mathbf{n}, p_0)$. Now $\psi(\overline{W}, \mathbf{n}, p_0)$ is

$$\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d [\forall i = 1, \ldots, d [B_i \text{ is } p_0\text{-dense in } T_i \rightarrow \varphi_{\overline{W}\mathbf{n}}]].$$

So, assume that $B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d$ and $\forall i = 1, \ldots, d[B_i$ is p_0-dense in $T_i]$.

(22) If $a_1 \in n_1(T_1, B_1) \wedge \ldots \wedge a_r \in n_r(T_r, B_r)$, $0 \leq j < m$, $A_{r+1} \subseteq B_{r+1}, \ldots, A_d \subseteq B_d$, and $A_{r+1}, \ldots, A_d$ are p_j-dense in $T_{r+1}, \ldots, T_d$ respectively, then there exist $A'_{r+1} \subseteq A_{r+1}, \ldots, A'_d \subseteq A_d$ which are p_{j+1}-dense such that $\varphi_{V\mathbf{k}^j}[\vec{a}, A'_{r+1}, \ldots, A'_d]$.

By (19), $\varphi_{W\mathbf{k}^j}[A_{r+1}, \ldots, A_d]$, and hence by the form of W, there are $A'_{r+1} \subseteq A_{r+1}, \ldots, A'_d \subseteq A_d$ such that $A'_{r+1}, \ldots, A'_d$ are k^j_{r+1}-,$\ldots$, k^j_d-dense and $\varphi_{V\mathbf{k}^j}[\vec{a}, A'_{r+1}, \ldots, A'_d]$. Now $k^j_{r+1} = \cdots = k^j_d = p_{j+1}$, as desired.

Now clearly $|\prod_{i=1}^r n_i(T_i, B_i)| \leq |\prod_{i=1}^r z_i| = m$.

(23) For any $J \subseteq \prod_{i=1}^r n_i(T_i, B_i)$ with $|J| = j \leq m$, there are $A_{r+1} \subseteq B_{r+1}$, $\ldots$, $A_d \subseteq B_d$ such that each A_i is p_j-dense in T_i, and for every $\mathbf{a} \in J$, $\varphi_{V\mathbf{n}}[\mathbf{a}, A_{r+1}, \ldots, A_d]$.

We prove this by induction on j. It is obvious for $j = 0$. Now assume that $\mathbf{b} \notin J$ and $J \cup \{\mathbf{b}\} \subseteq \prod_{i=1}^r n_i(T_i, B_i)$ and the assertion is true for J. So $j < m$ and there are $A_{r+1} \subseteq B_{r+1}$, $\ldots$, $A_d \subseteq B_d$ such that each A_i is p_j-dense in T_i, and for every $\mathbf{a} \in J$, $\varphi_{V\mathbf{n}}[\mathbf{a}, A_{r+1}, \ldots, A_d]$. Now by (22) there exist $A'_{r+1} \subseteq A_{r+1}$, $\ldots$, $A'_d \subseteq A_d$ such that $A'_{r+1}, \ldots, A'_d$ are p_{j+1}-dense in $T_{r+1}, \ldots, T_d$ respectively, and $\varphi_{V\mathbf{k}^j}[\mathbf{b}, A'_{r+1}, \ldots, A'_d]$. By (18), $\varphi_{V\mathbf{n}}[\mathbf{c}, A'_{r+1}, \ldots, A'_d]$ for all $\mathbf{c} \in J \cup \{\mathbf{b}\}$. This proves (23).

This completes the proof of (17) for rules 3.

Now the proof of the theorem goes as follows. Let $W_0 = (\forall a_i)_1^d(\exists x_i)_1^d$, $W_1 = (\exists A_i)_1^d(\forall x_i)_1^d$. By (4), $W_0 \models_d W_1$. By (17) as extended, $\forall n \exists p \psi(W_0, n, p)$ implies $\forall n \exists p \psi(W_1, n, p)$.

Case 1. $\forall n \exists p \psi(W_0, n, p)$. Hence $\forall n \exists p \psi(W_1, n, p)$. For any $k \in \omega$ let $\mathbf{n}$ be constantly k. Then choose p so that $\psi(W_1, \mathbf{n}, p)$. Then there exist A_i for $i = 1, \ldots, d$ such that $A_i \subseteq B_I$ for all $i = 1, \ldots, d$, A_i is n_i-dense in T_i for all $i = 1, \ldots, d$, and for all $i = 1, \ldots, d$, $\forall x_i \in A_i$, $\varphi_{V\mathbf{n}}[A_1, \ldots, A_d, x_1, \ldots, x_d]$. Then $\forall i = 1, \ldots, d[A_i$ is k-dense in T_i and $(x_1, \ldots, x_d) \in Q$. Thus (i) in the theorem holds.

Case 2. There is an n such that for all p, $\neg\psi(W_0, n, p)$. Thus for every p there are $B_1, \ldots, B_d$ which are p-dense in their respective trees, and $a_i \in n_i(T_i, B_i)$ for $i = 1, \ldots d$ such that $\prod_{i=1}^d a_i \subseteq \prod_{i=1}^d T_i \backslash Q$. Let $h = \max\{n_i : 1 \leq i \leq d\}$. For any k, take $p = h + k$. Then $a_1, \ldots, a_d$ is an (m, k)-matrix contained in $\prod_{i=1}^d T_i \backslash Q$. □