A summary of some BA papers

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1. σ-algebras

Let A be a σ-algebra. We say that $X \subseteq A$ is an *irredundant σ-generating set for A* iff X is a σ-generating set for A but no proper subset is.

The following results are from Aniszczyk, Frankiewicz 1984.

Proposition 1.1. *Any family of size ω_1 of subsets of ω_1 is contained in a σ-field of subsets of ω_1 generated by a countable set.*

Proof. Let $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ with $|\mathscr{F}| \leq \omega_1$. Let A be the subfield of $\mathscr{P}(\omega_1)$ generated by $\mathscr{F} \cup \omega_1$. Let φ be an isomorphism of A into $\mathscr{P}(\omega)/fin$. For each $a \in A$ choose $\chi_a \subseteq \omega$ such that $\varphi(a) = [\chi_a]$. Let $\chi' = \chi \restriction \omega_1$.

For any $b \in \mathscr{F}$ let $K_b = \{x \subseteq \omega : |\chi_b \triangle x| < \omega\}$.

(1) K_b is a Borel subset of $\mathscr{P}(\omega)$.

In fact,

$$\begin{aligned}
K_b &= \{x \subseteq \omega : |\chi_b \triangle x| < \omega\} \\
&= \{x \subseteq \omega : |\chi_b \backslash x| < \omega \text{ and } |x \backslash \chi_b| < \omega\} \\
&= \bigcup \{x : \chi_b \backslash x = F \text{ and } x \backslash \chi_b = G : F \in [\chi_b]^{<\omega} \text{ and } G \in [\omega \backslash \chi_b]^{<\omega}\} \\
&= \bigcup \{x : \chi_b \backslash F \subseteq x, x \cap F = \emptyset, G \subseteq x, x \cap (\omega \backslash (G \cup \chi_b)) = \emptyset : \\
&\qquad F \in [\chi_b]^{<\omega} \text{ and } G \in [\omega \backslash \chi_b]^{<\omega}\} \\
&= \bigcup \left\{ \bigcap \{U_{HF} : H \in [\chi_b \backslash F]^{<\omega}\} \cap \bigcap \{U_{Gk} : K \in [\omega \backslash (G \cup \chi_b)]^{<\omega} : \right. \\
&\qquad \left. F \in [\chi_b]^{<\omega} \text{ and } G \in [\omega \backslash \chi_b]^{<\omega} \right\}
\end{aligned}$$

(2) $\forall b \in \mathscr{F}[\chi^{-1}[K_b] = \{b\}]$.

For if $c \in \mathscr{F}$ then $c \in \chi^{-1}[K_b]$ iff $\chi_c \in K_b$ iff $|\chi_b \triangle \chi_c| < \omega$ iff $[\chi_b] = [\chi_c]$ iff $\varphi(b) = \varphi(c)$ iff $b = c$.

It now follows that $\mathscr{F} \subseteq \{\chi^{-1}[A] : A \text{ borel}\}$. This last set is clearly a σ-field of subsets of ω_1, and it is countably generated. $\square$

Proposition 1.2. *If $\mathscr{F}$ is a family of ω_1 clubs on ω_1, then there is a club C on ω_1 such that $\forall D \in \mathscr{F}[C \backslash D$ is countable$]$.*

Proof. Let $\langle C_\xi : \xi < \omega_1 \rangle$ enumerate $\mathscr{F}$, possibly with repetitions. Then $\triangle_{\xi<\omega_1} C_\xi \backslash C_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$. $\square$

Proposition 1.3. *Let $A = \{X \subseteq \omega_1 : \exists$ club $C[C \subseteq X$ or $C \cap X = \emptyset]\}$. Then A is a σ-field of subsets of ω_1.*

Proof. A is clearly closed under complements. Now suppose that $X_i \in A$ for all $i \in \omega$.

Case 1. $\exists i \in \omega \exists$ club $C \subseteq X_i$. Then $C \subseteq \bigcup_{j\in\omega} X_j$, so $\bigcup_{j\in\omega} X_j \in A$.

Case 2. $\forall i \in \omega \exists$ club C_i such that $C_i \cap X_i = \emptyset$. Then $\bigcap_{i\in\omega} C_i$ is club, and $(\bigcap_{i\in\omega} C_i) \cap \bigcup_{i\in\omega} X_i = \emptyset$. So $\bigcup_{j\in\omega} X_j \in A$. □

Proposition 1.4. *The σ-BA A of Proposition 1.3 does not have an irredundant σ-generating set.*

Proposition 1.5. *Let X be the collection of all ultrafilters on ω and let $X' = \{\{F\} : F \in X\}$. Let A be the σ-field of subsets of X generated by X'. Then X' is an irredundant σ-generating set for A.*

Proposition 1.6. (CH) *$\mathscr{P}(\omega_1)$ does not have an irredundant σ-generating subset.*

Proposition 1.7. (CH) *The σ-field of Lebesgue measurable subsets of $\mathbb{R}$ does not have an irredundant σ-generating subset.*

The following is from Balcerzak 1988.

A *proper* σ-ideal I in $\mathscr{P}(X)$ is a σ-ideal such that $X \notin I$ and $\{x\} \in I$ for all $x \in X$. A proper ideal I in $\mathscr{P}(X)$ *satisfies ccc* iff there is no uncountable family $\mathscr{F} \subseteq \mathscr{P}(X)\backslash I$ of pairwise disjoint sets.

Proposition 1.8. (1.1) *If I and J are proper σ-ideals in $\mathscr{P}(X)$, $I \subseteq J$, and J satisfies ccc, then I satisfies ccc.* □

Proposition 1.9. (1.2) *If I and J are proper σ-ideals in $\mathscr{P}(X)$, then $I \cap J$ satisfies ccc iff both I and J satisfy ccc.* □

Suppose that X and Y are spaces and I, J are ideals in $\mathscr{P}(X), \mathscr{P}(Y)$ respectively. Then

$$\begin{aligned}
&\forall E \subseteq X \times Y \forall x \in X[E_x = \{y \in Y : (x,y) \in E\}];\\
&\forall E \subseteq X \times Y \forall y \in Y[E^y = \{x \in X : (x,y) \in E\}];\\
&V(I,J) = \{E \subseteq X \times Y : \{x \in X : E_x \notin J\} \in I\};\\
&H(I,J) = \{E \subseteq X \times Y : \{y \in Y : E^y \notin I\} \in J\}.
\end{aligned}$$

Theorem 1.10. *If $V(I,J)$ satisfies ccc, then both I and J satisfy ccc.*

The following is from Balcerzak 1990.

An *involution* is a function $f : \Xi \to X$ such that f^2 =identity. If I and J are ideals in $\mathscr{P}(\omega_1)$, then they are *isomorphic* iff there is a bijection $f : \omega_1 \to \omega_1$ such that $J = \{f[X] : X \in I\}$. They are *n-isomorphic* iff there are n involutions $f_i : \omega_1 \to \omega_1$ for $i < n$ such that $f_0 \circ \cdots \circ f_{n-1}$ is ann isomorphism from I to J.

Proposition 1.11. *For any infinite set, every bijection $f : X \to X$ is the composition of two involutions.*

The following is from Balcerzak 1992.

Ideals I and J are *orthogonal* iff there is an $a \in I$ such that $-a \in J$. If $\mathfrak{M}$ is a family of σ-ideals, then $I \in \mathfrak{M}$ is *orthogonalizable in* $\mathfrak{M}$ iff it has an orthogonal member in $\mathfrak{M}$. The set of all σ-ideals orthogonalizable in $\mathfrak{M}$ is denoted by ORT($\mathfrak{M}$). T is the set of all cardinalities of maximal almost disjoint families on ω_1. For each $\kappa \in T$, $A(\kappa)$ is the set of all σ-ideals on ω_1 which can be generated by mad families of size κ.

Proposition 1.12. *ORT(A(κ))=A(κ) for all $\kappa \in T$.*

The following is from Bell, J. 1976.

A BA A is κ-*universal* iff every BA of size $< \kappa$ can be isomorphically embedded in A.

Theorem 1.13. *Let κ be an infinite cardinal, and let A be a κ-complete BA. Then the following are equivalent:*

(i) A is κ-universal.

(ii) $\forall\kappa < \kappa[\mathrm{finco}(\lambda)$ can be isomorphically embedded in A.

(iii) $\forall\lambda < \kappa[A$ has an antichain of size $\lambda]$.

Proof. Obviously (i)⇒(ii)⇒(iii). Now assume (iii), and let B be a BA of size less than κ. If B is finite, clearly B can be isomorphically embedded in A. Say $|B| = \lambda < \kappa$. Let $\langle a_\xi : \xi < \lambda\rangle$ be an antichain in A. Since A is κ-complete, we may assume that $\sum_{\xi<\lambda} a_\xi = 1$. Let $\langle b_\xi : \xi < \lambda\rangle$ enumerate the nonzero elements of B, and for each $\xi < \lambda$ let F_ξ be an ultrafilter on B such that $b_\xi \in F_\xi$. Now define for any $x \in B$, $f(x) = \sum_{x\in F_\xi} a_\xi$. Then

$$f(x+y) = \sum_{x+y\in F_\xi} a_\xi = \sum_{x\in F_\xi} a_\xi + \sum_{y\in F_\xi} a_\xi = f(x) + f(y),$$

and

$$f(-x) = \sum_{-x\in F_\xi} a_\xi = -\sum_{x\in F_\xi} a_\xi = f(-x).$$

Moreover, $f(b_\xi) \geq a_\xi \neq 0$, so f is the desired isomorphic embedding. □

Corollary 1.14. *Every infinite σ-algebra is ω_1-universal.*

Corollary 1.15. *For any infinite cardinal κ, $\mathbb{P}(\kappa)$ is κ^+-universal.*

The following is from Bell, C. 1956.

If A is a collection of subsets of X and κ is an infinite cardinal, then $\mathscr{F}_\kappa(A)$ is the smallest κ-field of subsets of X containing A.

Proposition 1.16. (Theorem 2) *Let A be a collection of subsets of X and κ be a regular cardinal, Let $B \in \mathscr{F}_\kappa(A)$. Then there is an $A' \in [A]^{<\kappa}$ such that $B \in \mathscr{F}_\kappa(A')$.*

Proof. Clearly $\mathscr{F}_\kappa(A) \supseteq \bigcup\{\mathscr{F}_\kappa(A') : A' \in [A]^{<\kappa}\}$, so it suffices to show that this union is a κ-field of sets. Clearly it is closed under complements. Now suppose that $X \in$

$[\bigcup\{\mathscr{F}_\kappa(A') : A' \in [A]^{<\kappa}\}]^{<\kappa}$. For each $x \in X$ choose $A'_x \in [A]^{<\kappa}$ such that $x \in \mathscr{F}_\kappa(A'_x)$. Let $A'' = \bigcup_{x\in X} A'_x$. Then $|A''| < \kappa$ since κ is regular, and $\bigcup X \in \mathscr{F}_\kappa(A'')$. □

Proposition 1.17. (Theorem 4) *Let $\langle a_i : i \in I\rangle$ be a system of sets. Let $X = \bigcup_{i\in I} a_i$. Then the intersection A of all complete fields of subsets of X containing all elements a_i consists of all unions of sets of the form $\bigcap_{i\in I} a_i^{\varepsilon(i)}$ for $\varepsilon \in {}^I2$. Moreover, each element of A can be written uniquely as a union of nonempty such intersections.*

Proof. For each $j \in I$,

$$a_j = \bigcup\left\{\bigcap_{i\in I} a_i^{\varepsilon(i)} : \varepsilon(j) = 1\right\}.$$

Hence the proposition follows. □

Proposition 1.18. (Theorem 9) *Let K, L be collections of subsets of X, Y respectively. Suppose that κ is a successor cardinal λ^+. Let $f : K \to L$ be given. Then f can be extended to a κ-homomorphism of $\mathscr{F}_\kappa(K)$ into $\mathscr{F}_\kappa(L)$ iff for every $a \in {}^\lambda K$ and every $\varepsilon \in {}^\lambda 2$, the condition $\bigcap_{\alpha<\lambda} a_\alpha^{\varepsilon(\alpha)} = \emptyset$ implies that $\bigcap_{\alpha<\lambda}(f(a_\alpha))^{\varepsilon(\alpha)} = \emptyset$.*

Proof. $\Rightarrow$: clear. $\Leftarrow$: Assume the indicated condition. Suppose that $b \in \mathscr{F}_\kappa(K)$. By Proposition 1.16 let $K' \in [K]^{\leq\lambda}$ be such that $b \in \mathscr{F}_\kappa(K')$. Let $\langle a_\alpha : \alpha < \lambda\rangle$ enumerate K'. Then by Proposition 1.17 there is an $M \subseteq {}^\lambda 2$ such that $b = \bigcup_{\varepsilon\in M}\bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)}$, with each $\bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)} \neq 0$. We then define

$$f_a(b) = \bigcup_{\varepsilon\in M}\bigcap_{\alpha\in\lambda}(f(a_\alpha))^{\varepsilon(\alpha)}.$$

Now suppose that $K'' \in [K]^{\leq\lambda}$ be such that $b \in \mathscr{F}_\kappa(K'')$, and let $\langle c_\alpha : \alpha < \lambda\rangle$ enumerate K''. Further, let $\langle d_\alpha : \alpha < \lambda\rangle$ enumerate $K' \cup K''$. Say $a_\alpha = d_{\xi(\alpha)}$ for all $\alpha < \lambda$ and $c_\alpha = d_{\eta(\alpha)}$ for all $\alpha < \lambda$. Then

$$\begin{aligned} b &= \bigcup_{\varepsilon\in M}\bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)} \\ &= \bigcup_{\varepsilon\in M}\bigcap_{\alpha\in\lambda} d_{\xi(\alpha)}^{\varepsilon(\alpha)} \\ &= \bigcup_{\varepsilon\in M'}\bigcap_{\alpha\in\lambda} d_\alpha^{\varepsilon(\alpha)}, \end{aligned}$$

where M' is the set of all $\varepsilon \in {}^\lambda 2$ such that $\langle\varepsilon(\xi(\alpha)) : \alpha \in \lambda\rangle \in M$ and each $\bigcap_{\alpha\in\lambda} d_\alpha^{\varepsilon(\alpha)} \neq \emptyset$. Similarly we get

$$b = \bigcup_{\varepsilon\in M'}\bigcap_{\alpha\in\lambda} d_\alpha^{\varepsilon(\alpha)},$$

where the same M' is the set of all $\varepsilon \in {}^{\lambda}2$ such that $\langle \varepsilon(\eta(\alpha)) : \alpha \in \lambda \rangle \in N$ and each $\bigcap_{\alpha\in\lambda} d_\alpha^{\varepsilon(\alpha)} \neq \emptyset$. From this it follows that $f_a(b) = f_d(b) = f_c(b)$.

Now for any $b \in \mathscr{F}_\kappa(K)$ we define $f_*(b) = f_a(b)$ for any a as above.

If $x \in K$, we can apply the definition to $a = \langle x : \alpha < \lambda \rangle$, $M = \{\langle 1 : \alpha < \lambda \rangle\}$ to get $f_*(x) = f(x)$. Thus f_* extends f.

Now take any $b \in \mathscr{F}_\kappa(K)$. Then we obtain K' and a as above. Then

$$-b = \bigcup_{\varepsilon\in{}^{\lambda}2\backslash M} \bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)} = \bigcup_{\varepsilon\in M'} \bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)},$$

where $M' = \{\varepsilon \in {}^{\lambda}2\backslash M : \bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)} \neq \emptyset\}$. It follows that $f_*(b) \cap f_*(-b) = \emptyset$. If $\bigcap_{\alpha\in\lambda} a_\alpha^{\varepsilon(\alpha)} = \emptyset$, then by the condition of the Proposition, $\bigcap_{\alpha\in\lambda}(f(a_\alpha))^{\varepsilon(\alpha)} = \emptyset$. Hence $f_*(b) \cup f_*(-b) = X$. Thus f_* preserves complements.

Now suppose that $\langle b_\alpha : \alpha < \lambda \rangle$ is a system of elements of $\mathscr{F}_\kappa(K)$. Let $K' \in [K]^\lambda$ be such that each $b_\alpha \in \mathscr{F}_\kappa(K')$. Let $\langle a_\alpha : \alpha < \lambda \rangle$ enumerate K'. For each $\alpha < \lambda$ choose $M_\alpha \subseteq {}^{\lambda}2$ such that $b_\alpha = \bigcup_{\varepsilon\in M_\alpha} \bigcap_{\beta\in\lambda} a_\beta^{\varepsilon(\beta)}$, with each $\bigcap_{\beta\in\lambda} a_\beta^{\varepsilon(\beta)} \neq \emptyset$. Then $\bigcup_{\beta\in\lambda} b_\beta = \bigcup_{\varepsilon\in M'} \bigcap_{\beta\in\lambda} a_\beta^{\varepsilon(\beta)}$, where $M' = \bigcup_{\beta<\lambda} M_\beta$. $\square$

The following is from Bukovsky, Galavec 1972. Let $A_{\kappa\lambda}$ be the free κ^+-BA with λ κ^+-free generators.

Proposition 1.19. (Theorem 3) *(i) If $\lambda \leq \kappa$, then $A_{\kappa\lambda}$ has exactly 2^λ atoms.*

(ii) If $\kappa < \lambda$, then $A_{\kappa\lambda}$ is atomless.

Proposition 1.20. (Theorem 4) *If $\lambda \geq \omega$, then $|A_{\kappa\lambda}| = \lambda^\kappa$.*

2. $\mathscr{P}(\omega)/fin$

The following is from Aviles, Brech 2011.

If B is a BA and S and A are subalgebras of B, then B is the *internal push-out* of A and S provided the following conditions hold:

(1) $B\langle S\cup A\rangle$.

(2) $\forall a\in A\forall s\in S[a\cdot s=0\to\exists r\in A\cap S[a\leq r$ and $s\leq -r]]$.

A *push-out diagram* is a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

such that S and A are subalgebras of B, $R=A\cap S$, B is the internal push-out of A and S, and the arrows are inclusions.

A BA B is a *posex* of a subalgebra A if there exists a push-out diagram as above with S and R countable.

A BA A is *tightly σ-filtered* iff there exist and ordinal λ and a system $\langle B_\alpha:\alpha\leq\lambda\rangle$ such that the following conditions hold:

(3) $B_\alpha\subseteq B_\beta$ whenever$\alpha\leq\beta\leq\lambda$.

(4) $B_0=\{0,1\}$ and $B_\lambda=A$.

(5) $\forall\alpha<\lambda[B_{\alpha+1}$ is a posex of $B_\alpha]$.

(6) $B_\beta=\bigcup_{\alpha<\beta}B_\alpha$ for every limit ordinal $\beta\leq\lambda$.

Theorem 2.1. *Assume that 2^ω is regular. Then there is a BA B, unique up to isomorphism, such that the following conditions hold:*

(i) $|B|=2^\omega$.

(ii) B is tightly σ-filtered.

(iii) If A is a subalgebra of B and of another BA C, and C is a posex of A, then there is an isomorphic embedding $f:C\to B$ which makes the following diagram commute:

$$\begin{tikzcd} C \arrow[dr, "f"] & \\ A \arrow[u] \arrow[r] & B \end{tikzcd}$$

Theorem 2.2. *If $\mathscr{P}(\omega)/fin$ satisfies the conditions on B in Theorem 2.1, then $2^\omega\leq\omega_2$.*

The following is from Aviles, Todorcevic 2011.

$$
\begin{aligned}
I, J \text{ are orthogonal} \quad \text{iff} \quad & \forall a \in I \forall b \in J[a \cdot b = 0];\\
I \vee J &= \{a + b : a \in I,\ b \in J\};\\
I \wedge J &= \{a \cdot b : a \in I,\ b \in J\};\\
I^{\perp} &= \{a \in A : \forall b \in I[a \cdot b = 0]\};\\
I \restriction a &= \{b \in I : b \leq a\};\\
I \leq a \quad \text{iff} \quad & \forall b \in I[b \leq a].
\end{aligned}
$$

For a tree T, we denote by $[T]$ the set of all branches of T.

For $N \in [\omega]^{<\omega}\backslash 1$, an N-*gap* of a BA A is a family $\langle I_i : i \in N\rangle$ of pairwise orthogonal ideals of A such that $\forall c \in {}^N A[\forall i \in N[I_i \leq c(i)] \to \prod_{i\in N} c(i) \neq 0]$.

A multiple gap as above is *dense* iff $(\bigvee_{i\in N} I_i)^{\perp} = \{0\}$.

Let $B \subset N$. An N-gap $\langle I_i : i \in N\rangle$ is a B-*clover* iff $\neg\exists a \in A[\langle I_j \restriction a : j \in B\rangle$ is a B-gap and $\forall i \in N\backslash B[a \in I_i^{\perp}]]$. An N-gap is a *clover* iff it is a B-clover for every B such that $\emptyset \neq B \subset N$.

If $\langle I_i : i \in N\rangle$ is an N-gap, and $B \subset N$, then $\langle I_i : i \in N\rangle$ is a B-*jigsaw* iff $\forall A[B \subset A \subseteq N \forall a \in A[\langle I_i \restriction a : i \in A\rangle$ is an A-gap$\to \exists b < a[\langle I_i \restriction b : i \in B\rangle$ is a B-gap and $\forall i \in A\backslash B[b \in I_i^{\perp}]]]]$.

Two orthogonal ideals I_0 and I_1 are *countably separated* iff there is a sequence $\langle c_n : n \in \omega\rangle$ such that $\forall x \in I_0 \forall y \in I_1 \exists n[x \leq c_n$ and $y \cdot c_n = 0]$.

Theorem 2.3. (Theorem 4) *If I_0 and I_1 are orthogonal analytic ideals in $\mathscr{P}(\omega)/fin$, then one of the following holds:*

(i) I_0 and I_1 are countably separated in $\mathscr{P}(\omega)/fin$.

(ii) There exist $a \in {}^{2^\omega} I_0$ and $b \in {}^{2^\omega} I_1$ such that $\forall s[a_s \cap b_s = \emptyset]$ and $\forall s, t[s \neq t \to (a_s \cap b_t) \cup (a_t \cap b_s) \neq \emptyset$ and $\langle a_s : s \in 2^\omega\rangle$ and $\langle b_s : s \in 2^\omega\rangle$ are continuous.

Now let $\mathscr{X}$ be a family of subsets of n, and let $\langle I_i : i \in n\rangle$ be a system of ideals in a BA A. A multiple gap $\langle I_i : i \in n\rangle$ is $\mathscr{X}$-*countably separated* iff there exist elements c_i^k for $i \in n$ and $k \in \omega$ such that:

(1) $\prod_{i\in X} c_i^k = 0$ for every $k \in \omega$ and $X \in \mathscr{X}$.

(2) $\forall x \in \prod_{i<n} I_i \exists k \in \omega \forall i < n[x_i \leq c_i^k]$.

We say that $\langle I_i : i \in n\rangle$ is *weakly countably separated* iff it is $[n]^n$-countably separated. We say that $\langle I_i : i \in n\rangle$ is *strongly countably separated* iff it is $[n]^2$-countably separated.

Theorem 2.4. (Theorem 6) *Let $T = {}^{<\omega}n$. For each branch x of T and each $i < n$ let $a_x^i = \{s \in T : s^\frown\langle i\rangle \in x\}$. Let I_i be the ideal in $\mathscr{P}(T)/fin$ generated by $\{[a_x^i] : x$ a branch of $T\}$. Then $\langle I_i : i < n\rangle$ is an n-jigsaw which is not weakly countably separated.*

Theorem 2.5. (Theorem 7) *Let $\langle J_i : i \in n\rangle$ be analytic ideals in $\mathscr{P}(\omega)/fin$ which constitute a multiple gap. Then one of the following holds:*

(i) The ideals are weakly countably separated in $\mathscr{P}(\omega)/fin$.

(ii) There is a one-one function $u : {}^{<\omega}n$ such that $u(I_i) \subseteq J_i$ for all $i \in n$, where I_i is as in Theorem 2.4.

Proposition 2.6. (Proposition 8) *Let $T = {}^{<\omega}2$ and for $i \in$ and x a branch of T let $a^i_x = \{s \in T : s^\frown\langle i\rangle \in x\}$. Let I_i be the ideal of $\mathscr{P}(T)/fin$ generated by all $[a^i_x]$ for x a branch of T. Let J be the ideal of $\mathscr{P}(T)/fin$ generated by all $[b]$, b an antichain of T.*

Then I_0, J_0, J are analytic ideals, $\{I_0, I_1.J\}$ is a multiple gap, but it is not weakly countably separated and is neither a jigsaw nor a clover.

The following result is from Baayen, Paalman-de-Miranda 1963

Proposition 2.7. *For any infinite κ, $\mathscr{P}(\kappa)/fin$ has a disjoint subset of size λ iff $\lambda \leq \kappa^\omega$.*

Proof. First, for each $f \in {}^\omega\kappa$ let $D_f = \{f \restriction m : m \in \omega\}$. Then $D_f \cap D_g$ is finite for $f \neq g$. So this gives a disjoint subset of $\mathscr{P}(\kappa)/fin$ of size κ^ω.

Now suppose that $\mathscr{A} \subseteq \mathscr{P}(\kappa)/fin$ is pairwise disjoint. Say $\mathscr{A} = \{E/fin : E \in \mathscr{B}\}$ with $|\mathscr{A}| = |\mathscr{B}|$. For each $E \in \mathscr{B}$ let E' be a subset of E of size ω. Then $\langle E' : E \in \mathscr{B}\rangle$ is a system of pairwise almost disjoint subsets of κ, each of size ω. Thus $\{E' : E \in \mathscr{B}\} \subseteq [\kappa]^\omega$, so $|\mathscr{A}| = |\{E' : E \in \mathscr{B}\}| \leq \kappa^\omega$. $\square$

The following is from Baumgartner 1980.

Theorem 2.8. *The following is relatively consistent:*

(i) MA.

(ii) $2^\omega = \omega_2$.

(iii) Every uncountable subset of $\mathscr{P}(\omega)$ contains an uncountable chain or antichain with respect to $\subseteq$.

(iv) Every uncountable BA has an uncountable antichain.

(v) All ω_1-dense subsets of $\mathbb{R}$ are isomorphic.

The following is from Bashkirov 1978.

Let X be a maximal almost disjoint family of infinite subsets of ω. Let a be a one-one function with domain X and range disjoint from ω. For each $m \in \omega$ let $B(m) = \{m\}$, and for each $x \in X$ let $B(a_x) = \{\{a_x\} \cup M : M \subseteq x,\ x\backslash M$ is finite$\}$. Let $I_X = \omega \cup \mathrm{rng}(a)$.

Proposition 2.9. *$\langle B(x) : x \in I_X\rangle$ is a neighborhood system for I_X.*

Proof. Clearly $\forall x \in I_X[x \in B(x)]$. Suppose that $x \in U \in B(y)$ with $y \in I_X$. If $y \in \omega$, then $U = \{y\}$, hence $x = y$, so $\{x\} \in B(x)$ and $\{x\} \subseteq \{x\}$. If $y \in \mathrm{rng}(a)$, say $y = a_x$. Then there is a finite $M \subseteq x$ such that $U = \{a_x\} \cup M$ and $x\backslash M$ is finite.

Case 1. $x = a_x$. Then $U \in B(x)$ and $U \subseteq U$.

Case 2. $x \in M$. Then $\{x\} \in B(x)$ and $\{x\} \subseteq U$.

Finally, suppose that $U_1, U_2 \in B(x)$ with $x \in I_X$.

Case 1. $x \in \omega$. Then $U_1 = U_2 = \{x\}$ and the desired conclusion is clear.

Case 2. $x \in \operatorname{rng}(a)$; say $x = a_x$. Say $U_1 = \{a_x\} \cup M$ and $U_2 = \{a_x\} \cup N$, with $x \backslash M$ and $x \backslash N$ finite. Then $X \backslash (M \cap N) = (x \backslash M) \cup (x \backslash N)$ is finite, so $\{a_x\} \cup (M \cap N) \in B(x)$, and $\{a_x\} \cup (M \cap N) \subseteq U_1 \cap U_2$. □

Proposition 2.10. *I_X is locally compact but not compact.*

Proof. It suffices to show that each set $\{a_x\} \cup M$ with $M \subseteq x$ and $x \backslash M$ finite is compact. Suppose that $\mathscr{O}$ is a collection of open sets covering $\{a_x\} \cup M$. Choose $U \in \mathscr{O}$ such that $a_x \in U$. Say $U = a_x \cup N$ with $N \subseteq x$ and $x \backslash N$ finite. Then $M \backslash N$ is finite, since $M \backslash N \subseteq x \backslash N$. For each $y \in M \backslash N$ choose $V_y \in \mathscr{O}$ such that $y \in V_y$. Then $\{a_x\} \cup M \subseteq a_x \cup N \cup \bigcup_{y \in M \backslash N} V_y$.

Thus I_X is locally compact. $\{\omega\} \cup \{\{a_x\} \cup x : x \in X\}$ is a cover of I_X with no finite subcover. □

Proposition 2.11. *I_X is Hausdorff.*

Proof. Let $x, y \in I_X$, $x \neq y$. If one of x, y is in ω, clearly they have disjoint open neighborhoods. Suppose that $x = a_u$ and $y = a_v$. Then $u \cap v$ is finite, and so $\{a_u\} \cup (u \backslash v)$ and $\{a_v\} \cup (v \backslash u)$ are disjoint open neighborhoods. □

Proposition 2.12. *$Y \subseteq I_X$ is compact iff there exist a finite $F \subseteq X$ and a $G \subseteq \omega$ such that $\{m \in G : \forall x \in F[m \notin x]\}$ is finite, and $Y = \{a_x : x \in F\} \cup G$.*

Proof. $\Rightarrow$: Let $F = \{x : a_x \in Y\}$ and $G = \{m : m \in Y\}$. Thus $Y = \{a_x : x \in F\} \cup G$. If F is infinite, then $\{\{a_x \cup x : a_x \in F\} \cup \{\{m\} : m \in \omega\}$ covers Y, but there is no finite subcover. So F is finite. If $\{m \in G : \forall x \in F[m \notin x]\}$ is infinite, then the same set covers Y but has no finite subcover.

$\Leftarrow$: Assume the indicated conditions, and suppose that $\mathscr{U}$ is an open cover of Y. For each $x \in F$ let $\{a_x\} \cup M_x \in \mathscr{U}$ with $M_x \subseteq x$ and $x \backslash M_x$ finite. Let

$$G' = \{m \in G : \exists x \in F[m \in x \backslash M_x]\} \cup \{m \in G : \forall x \in F[m \notin x]\}.$$

Thus G' is finite. For each $m \in G'$ choose $V_x \in \mathscr{U}$ such that $m \in V_x$. Then $\{\{a_x\} \cup M_x : x \in F\} \cup \{V_x : x \in G'\}$ covers Y. □

Let I_X^* be the one-point compactification of I_X. Thus $I_X^* = I_X \cup \{\infty\}$, with open sets those of I_X plus all sets $\{\infty\} \cup (I_X \backslash F)$ with F compact in I_X.

Proposition 2.13. *I_X^* is Hausdorff.*

Proof. Let x, y be distinct points of I_X^*. If $x, y \neq \infty$, then the desired disjoint open neighborhoods exist by Proposition 2.11. Suppose wlog $x = \infty$. If $y \in \omega$, then $\{\infty\} \cup (I_X \backslash \{y\})$ and $\{y\}$ are disjoint open neighborhoods. If $y = a_u$, then $\{\infty\} \cup (I_X \backslash (\{a_u\} \cup u))$ and $\{a_u\} \cup u$ are disjoint open neighborhoods. □

Proposition 2.14. *The clopen sets form a base for the topology on I_X^*.*

Proof. By Proposition 2.12, each set $\{a_u\} \cup M$, with $M \subseteq u$ and $u \backslash M$ finite, is clopen. Hence it suffices to show that each set $\{\infty\} \cup (I_X \backslash Y)$, Y compact in I_X, is a union of clopen sets. Suppose that $y \in \{\infty\} \cup (I_X \backslash Y)$; we want to find a clopen set V such that $y \in V \subseteq \{\infty\} \cup (I_X \backslash Y)$.

Case 1. $y \in \omega$. Then take $V = \{y\}$.

Case 2. $y = a_x$. Let F, G be as in Proposition 2.12. Let $Y' = Y \cup \bigcup_{a_y \in F} y$. Then Y' is compact by Proposition 2.12, and it is clearly open; so it is clopen. Clearly $y \in \{\infty\} \cup (I_X \backslash Y') \subseteq \{\infty\} \cup (I_X \backslash Y)$.

Case 3. $y = \infty$. Let F, G be as in Proposition 2.12. Let $Y' = Y \cup \bigcup_{a_y \in F} y$. Then Y' is compact by Proposition 2.12, and it is clearly open; so it is clopen. Clearly $y \in \{\infty\} \cup (I_X \backslash Y') \subseteq \{\infty\} \cup (I_X \backslash Y)$. $\square$

Theorem 2.15. *Let A be the subalgebra of $\mathscr{P}(\omega)$ generated by $X \cup \{\{m\} : m \in \omega\}$. Then* $\mathrm{Ult}(A)$ *is homeomorphic to I_X^*.*

Proof. For each $m \in \omega$ let F_m be the ultrafilter on A generated by $\{m\}$.

(1) For each $x \in X$ the set $\{x\} \cup \{M : \omega \backslash M$ is finite$\}$ has fip.

In fact, otherwise we get $x \cap \bigcap_{M \in F} M = \emptyset$ for some finite set F of M's such that $\omega \backslash M$ is finite. Hence $x \subseteq \bigcup_{M \in F}(\omega \backslash M)$. Since x is infinite and $\bigcup_{M \in F}(\omega \backslash M)$ is finite, this is a contradiction.

For each $x \in X$ let G_x be the filter on A generated by $\{x\} \cup \{M : \omega \backslash M$ is finite$\}$. By (1), G_x is proper. We claim that it is an ultrafilter. For, let $A' = \{s \in A : s \in G_x$ or $(\omega \backslash s) \in G_x\}$. Clearly $\{m\} \in A'$ for all $m \in \omega$. Clearly $x \in G_x$. Suppose that $y \in X \backslash \{x\}$. Then $x \cap y$ is finite, so $(\omega \backslash (x \cap y)) \in G_x$. Now $(\omega \backslash (x \cap y)) = (\omega \backslash x) \cup (\omega \backslash y)$ and $x \in G_x$, so $(\omega \backslash y) \in G_x$. Thus $y \in A'$. Clearly A' is closed under $\backslash$ and $\cup$, so $A' = A$. This shows that G_x is an ultrafilter.

Let $Y = \{s \in A : (\omega \backslash s) \in X\} \cup \{\omega \backslash F : F \in [\omega]^{<\omega}\}$. We claim that Y has fip. For, suppose that $K \in [X]^{<\omega}$, $H \in [[\omega]^{<\omega}]^{<\omega}$, and $\bigcap_{s \in K}(\omega \backslash s) \cap \bigcap_{b \in H}(\omega \backslash b) = \emptyset$. Then $\bigcap_{s \in K}(\omega \backslash s) \subseteq \bigcup_{b \in H} b$, so $\bigcap_{s \in K}(\omega \backslash s)$ is finite. Take any $c \in X \backslash K$. Then $\forall s \in K[c \cap s$ is finite$]$, so $c \cap \bigcup_{s \in K} s$ is finite, and so $c \cap \bigcap_{s \in K}(\omega \backslash s)$ is infinite, contradiction.

Let L be the filter generated by Y. Clearly L is an ultrafilter.

(2) $\mathrm{Ult}(A) = \{F_m : m \in \omega\} \cup \{G_x : x \in X\} \cup \{L\}$.

To prove this, let M be any ultrafilter on A. If M is principal, say M is generated by $\{m\}$. Then $M = F_m$. Assume that M is not principal. If $x \in M$ for some $x \in X$, clearly $M = G_x$. If $x \notin M$ for all $x \in X$, clearly $M = L$. Thus (2) holds.

Define $f(F_m) = m$, $f(G_x) = a_x$, $f(L) = \infty$. Thus f is a bijection from $\mathrm{Ult}(A)$ to I_X^*, so it suffices to show that f is continuous. So let $N \in f^{-1}[U]$ with U a clopen set in I_X^*; we want to find an open subset V in $\mathrm{Ult}(A)$ such that $N \in V \subseteq f^{-1}[U]$.

Case 1. $U = \{m\}$ with $m \in \omega$. Then $f(N) = m$, hence $N = F_m$. So $N \in \{F_m\} \subseteq f^{-1}[U]$.

Case 2. $U = \{a_x\} \cup M$ with $x \in X$, $M \subseteq x$, $x \backslash M$ finite. Then $f(N) \in U$.

Subcase 2.1. $f(N) = a_x$. Then $N = G_x$. So $N \in \mathcal{S}(x) \subseteq f^{-1}[U]$.

Subcase 2.2. $f(N) = m \in M$. Then $N = F_m$ and $N \in \mathcal{S}(\{m\}) \subseteq f^{-1}[U]$.

Case 3. (See the proof of Proposition 2.14, Case 3.) $U = \{\infty\} \cup (I_X \backslash Y)$ with $Y = \bigcup_{x \in F}(\{a_x\} \cup x) \cup G$, F a finite subset of X, $G \subseteq \omega$ such that $\{m \in G : \forall x \in F[m \notin x]\}$ finite.

Subcase 3.1. $N = F_m$ for some $m \in \omega$. Then $f(F_m) = m \in U$. Hence $N \in \mathcal{S}(\{m\}) \subseteq f^{-1}[U]$.

Subsubcase 3.2. $N = G_x$ with $x \in X$. So $f(N) = a_x$, hence $x \notin F$. Suppose that $P \in \mathcal{S}(x)$.

Subsubcase 3.2.1. $\{m\} \in P$ for some $m \in \omega$. Then $x \cap \{m\} \in P$, hence $m \in x$, and $f(P) = m \in U$.

Subcase 3.2.2. $a_x \in P$ for some $x \in X$, and P is nonprincipal. Then $P = G_x$ and $x \notin F$. So $f(P) = a_x \in U$.

Subsubcase 3.2.3. P is nonprincipal, and $\omega\backslash s \in P$ for all $s \in X$. Then $P = L$ and $f(P) = \infty \in U$.

Subcase 3.3. $N = L$. So $f(N) = \infty$. Let $V = \bigcap_{x \in F}(\omega\backslash x) \cup (\omega\backslash\{m \in G : \forall x \in F[m \notin x]\}$. Thus $N \in f^{-1}[V]$. Suppose that $P \in f^{-1}[V]$.

Subsubcase 3.3.1. $\{m\} \in P$ for some $m \in \omega$. Then $f(P) = m \in V$.

Subsubcase 3.3.2. $P = G_x$ for some $x \in X$. Then $f(P) = x \in V$.

Subsubcase 3.3.3. $P = L$. Then $f(P) = \infty \in V$. □

The following is from Baumgartner, Frankiewicz, Zbierski 1990.

Theorem 2.16. *There is a model of ZFC with 2^ω arbitrarily large in which every BA A of size $\leq 2^\omega$ can be isomorphically embedded in $\mathscr{P}(\omega)/fin$. Moreover, each automorphism of A can be extended to an automorphism of $\mathscr{P}(\omega)/fin$.*

The following is from Baumgartner, Komjath 1981.

Theorem 2.17. ($\diamondsuit$) *There is an atomless field of subsets of ω such that every nonzero element is uncountable and every chain and antichain is countable.*

Theorem 2.18. ($\diamondsuit$) *There is an uncountable atomless field of subsets of ω such that the countable elements form a maximal ideal and every chain and antichain is countable.*

Theorem 2.19. *If every antichain in A is countable, the A has a countable dense subalgebra.*

The following is from Baumgartner, Weese 1982.

Let F be a mad family of subsets of ω. A set $a \subseteq \omega$ is a *partitioner* of F iff $\forall b \in F[b \cdot a$ or $b \cdot -a$ is finite]. The partitioners of F form a BA P_F. I_F is the ideal of P_F generated by F together with the finite subsets of ω. The algebra P_F/I is the *partition algebra* of F. A BA A is *partition-representable* iff there is a mad family F such that A is isomorphic to P_F/I.

Theorem 2.20. (Theorem 2.1) *Every countable BA is partition-representable.*

Theorem 2.21. (Theorem 2.2) *For all $\lambda \leq 2^\omega$ the algebra* $\mathrm{finco}(\lambda)$ *is partition-representable.*

Theorem 2.22. (Theorem 2.4) $\mathrm{intalg}(\mathbb{R})$ *is partition-representable.*

Theorem 2.23. (CH; Theorem 3.1) *Every BA of size $\le 2^\omega$ is partition-representable.*

Theorem 2.24. (Theorem 4.1) *Assume CH, and let $\kappa \ge \omega_2$ and let $\mathbb{P} = \mathrm{Fn}(\kappa, \omega, \omega)$. In a generic extension, if A has an independent subset of size ω_2 then A is not partition-representable. In particular, no infinite complete BA is partition-representable.*

The following is from Frankiewicz 1977.

Theorem 2.25. (Lemma 1) *If $\omega_1 \le \kappa$ and $\mathscr{P}(\omega)/fin \cong \mathscr{P}(\kappa)/fin$, then $\mathscr{P}(\omega)/fin \cong \mathscr{P}(\omega_1)/fin$.*

Proof. Let $A = \mathscr{P}(\omega)/fin$, $B = \mathscr{P}(\omega_1)/fin$, $C = \mathscr{P}(\kappa)/fin$. Note that if $X \in [\omega]^\omega$ then $A \restriction [X]_A \cong A$, and $C \restriction [\omega_1]_C \cong B$. Assume that f is an isomorphism from A onto C. Then there is an $X \in [\omega]^\omega$ such that $f^{-1}([\omega_1]_C) = [X]_A$. Hence $A \cong B$. $\square$

Theorem 2.26. (Lemma 2) *If κ is singular and $\forall \lambda < \kappa[\mathscr{P}(\lambda)/fin \not\cong \mathscr{P}(\kappa)/fin]$, then $\mathscr{P}(\kappa)/fin \not\cong \mathscr{P}(\kappa^+)/fin$.*

Proof. We consider the following property of a cardinal μ:

(1_μ) There is an isomorphism f of $\mathscr{P}(\mathrm{cf}(\kappa))$ into $\mathscr{P}(\mu)/fin$ such that:

(a) $\quad \forall \alpha < \mathrm{cf}(\kappa)[(\mathscr{P}(\mu)/fin) \restriction f(\{\alpha\}) \not\cong \mathscr{P}(\mu)/fin]$;

(b) $\quad \forall X \in \mathscr{P}(\mu)[(\mathscr{P}(\mu)/fin) \restriction [X] \cong \mathscr{P}(\mu)/fin \to \exists \alpha < \mathrm{cf}(\kappa)[f(\{\alpha\}) \cdot [X] \ne 0]$.

We claim that (1_κ) but not (1_{κ^+}). Once we prove this, it follows that $\mathscr{P}(\kappa)/fin \not\cong \mathscr{P}(\kappa^+)/fin$. In fact, suppose that (1_κ), not (1_{κ^+}), and g is an isomorphism of $\mathscr{P}(\kappa)/fin$ onto $\mathscr{P}(\kappa^+)/fin$. Let f be as in (1_κ). We claim that $g \circ f$ shows that (1_{κ^+}). (Contradiction.) For, if $\alpha < \mathrm{cf}(\kappa)$ then $(\mathscr{P}(\kappa^+)/fin) \restriction g(f(\{\alpha\})) \cong (\mathscr{P}(\kappa)/fin) \restriction f(\{\alpha\}) \not\cong \mathscr{P}(\kappa)/fin \cong \mathscr{P}(\kappa^+)/fin$, so that (1_{κ^+})(a) holds. If $X \in \mathscr{P}(\kappa^+)$ and $(\mathscr{P}(\kappa^+)/fin) \restriction [X] \cong \mathscr{P}(\kappa^+)/fin$, let Y be such that $g^{-1}([X]) = [Y]$. Thus $Y \in \mathscr{P}(\kappa)$ and $(\mathscr{P}(\kappa)/fin) \restriction [Y] \cong \mathscr{P}(\kappa)/fin$. By (1_κ)(b) choose $\alpha < \mathrm{cf}(\kappa)$ such that $f(\{\alpha\}) \cdot [Y] \ne 0$. Then $g(f(\{\alpha\})) \cdot [X] \ne 0$, as desired.

To prove (1_κ), let $\langle \gamma_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing sequence of cardinals, with $\gamma_0 = 0$, γ_1 infinite, $\gamma_\lambda = \bigcup_{\alpha<\lambda} \gamma_\alpha$ for λ limit, with $\sup_{\alpha<\mathrm{cf}(\alpha)} \gamma_\alpha = \kappa$. For each $X \subseteq \mathrm{cf}(\kappa)$ let $f(X) = \left[\bigcup_{\alpha \in X} (\gamma_{\alpha+1} \backslash \gamma_\alpha)\right]$. Clearly f is an isomorphism of $\mathscr{P}(\mathrm{cf}(\kappa))$ into $\mathscr{P}(\kappa)/fin$. For (1_κ)(a), suppose that $\alpha < \mathrm{cf}(\kappa)$. Then $f(\{\alpha\}) = [\gamma_{\alpha+1} \backslash \gamma_\alpha]$, and $|\gamma_{\alpha+1} \backslash \gamma_\alpha| = \gamma_{\alpha+1} < \kappa$. Thus $(\mathscr{P}(\kappa)/fin) \restriction f(\alpha) \cong \mathscr{P}(\gamma_{\alpha+1})/fin$, and by hypothesis this is not isomorphic to $\mathscr{P}(\kappa)/fin$. For (1_κ)(b), suppose that $X \in \mathscr{P}(\kappa)$ and $(\mathscr{P}(\kappa)/fin) \restriction [X] \cong \mathscr{P}(\kappa)/fin$. Then $|X| = \kappa$ by the hypothesis of the theorem. Choose $\alpha < \mathrm{cf}(\kappa)$ such that $X \cap (\gamma_{\alpha+1} \backslash \gamma_\alpha)$ is infinite. Then $f(\{\alpha\}) \cap [X] \ne 0$.

Now suppose that (1_{κ^+}); we want to get a contradiction. Let f be as in the definition of (1_{κ^+}). By (1_{κ^+})(a), for each $\alpha < cf(\kappa)$ there is an $X_\alpha \in [\kappa^+]^{\le\kappa}$ such that $f(\{\alpha\}) = [X_\alpha]$. Let $Y = \bigcup_{\alpha<\mathrm{cf}(\kappa)} X_\alpha$. Thus $|Y| \le \kappa$. So $|\kappa^+ \backslash Y| = \kappa^+$, hence $(\mathscr{P}(\kappa^+)/fin) \restriction [\kappa^+ \backslash Y] \cong \mathscr{P}(\kappa^+)/fin$. By (1_{κ^+})(b) it follows that there is an $\alpha < \mathrm{cf}(\kappa)$ such that $f(\{\alpha\}) \cdot [\kappa^+ \backslash Y] \ne 0$. But $f(\{\alpha\}) \cdot [\kappa^+ \backslash Y] = [X_\alpha] \cdot [\kappa^+ \backslash Y] = 0$, contradiction. $\square$

Lemma 2.27. *Suppose that μ is an infinite cardinal and $\mathscr{P}(\mu)/fin$ is not isomorphic to $\mathscr{P}(\mu^+)/fin$. Then for any κ,λ, if $\mu\le\kappa<\lambda$ then $\mathscr{P}(\kappa)/fin$ is not isomorphic to $\mathscr{P}(\lambda)/fin$.*

Proof. Assume that μ is an infinite cardinal and $\mathscr{P}(\mu)/fin$ is not isomorphic to $\mathscr{P}(\mu^+)/fin$. Suppose that there exist κ,λ with $\mu\le\kappa<\lambda$ and $\mathscr{P}(\kappa)/fin$ is isomorphic to $\mathscr{P}(\lambda)/fin$; we want to get a contradiction. Let such a κ be minimum. Let f be an isomorphism from $\mathscr{P}(\kappa)/fin$ to $\mathscr{P}(\lambda)/fin$. Choose $X\subseteq\kappa$ such that $f([X]_\kappa)=[\kappa^+]_\lambda$. Then

$$\mathscr{P}(|X|)/fin\cong(\mathscr{P}(\kappa)/fin)\restriction[X]_\kappa\cong(\mathscr{P}(\lambda)/fin)\restriction[\kappa^+]_\lambda\cong\mathscr{P}(\kappa^+)/fin.$$

Now $|X|\le\kappa$, so by the minimality of κ we have $|X|=\kappa$. Thus $\mathscr{P}(\kappa)/fin\cong\mathscr{P}(\kappa^+)/fin$. Let g be an isomorphism of $\mathscr{P}(\kappa)/fin$ to $\mathscr{P}(\kappa^+)/fin$. By Theorem 2.26, κ is regular. Let $\langle\theta_\alpha:\alpha<\kappa\rangle$ be a strictly increasing sequence of ordinals with supremum κ, with $\mu<\theta_0$.

$$\forall X\in[\kappa]^\kappa\exists\alpha<\kappa[[\theta_\alpha]\cdot[X]\ne0]. \tag{1}$$

In fact, suppose that $X\in[\kappa]^\kappa$. Choose $Y\subseteq X$ with $|Y|=\mu$. Say $Y\subseteq\theta_\alpha$ with $\alpha<\kappa$. Then $[\theta_\alpha]\cdot[X]\ge[Y]\ne0$. Say $g([\theta_\alpha])=[Y_\alpha]$. Note that $|\theta_\alpha|<\kappa$. Now $(\mathscr{P}(\kappa)/fin)\restriction[\theta_\alpha]\cong(\mathscr{P}(\kappa^+)/fin)\restriction[Y_\alpha]$, so by the minimality of κ we have $|Y_\alpha|<\kappa$. Choose $X\subseteq\kappa$ so that $g([X])=[\kappa^+\backslash\bigcup_{\alpha<\kappa}Y_\alpha]$. Now $|\kappa^+\backslash\bigcup_{\alpha<\kappa}Y_\alpha|=\kappa^+$, and

$$\mathscr{P}(\kappa)/fin)\restriction[X]\cong(\mathscr{P}(\kappa^+)/fin)\restriction g([X]),$$

so it follows by the minimality of κ that $|X|=\kappa$. By (1), choose $\alpha<\kappa$ so that $[\theta_\alpha]\cdot[X]\ne0$. Hence $0=[Y_\alpha]\cdot g([X])=g([\theta_\alpha])\cdot g([X])\ne0$, contradiction. □

The following is from Balcar, Frankiewicz 1978.

A *scale* is a system $\langle f_\alpha:\alpha<\mu\rangle$ of members of ${}^\omega\omega$ such that $\forall\alpha,\beta<\mu[\alpha<\beta\to f_\alpha<^*f_\beta]$ and $\forall g\in{}^\omega\omega\exists\alpha<\mu[g<^*f_\alpha]$.

Lemma 2.28. *There is at most one regular cardinal μ such that there is a scale $\langle f_\alpha:\alpha<\mu\rangle$.*

Proof. Suppose that $\mu<\nu$ and $\langle f_\alpha:\alpha<\mu\rangle$ and $\langle f_\alpha:\alpha<\nu\rangle$ are scales. For each $\alpha<\mu$ let $\beta(\alpha)<\nu$ be minimum such that $f_\alpha<^*g_{\beta(\alpha)}$. Let $\gamma=(\sup_{\alpha<\mu}\beta(\alpha))+1$. Then choose $\alpha<\mu$ such that $g_\gamma<^*f_\alpha$. Then $f_\alpha<^*g_{\beta(\alpha)}<^*g_\gamma<^*f_\alpha$, contradiction. □

Lemma 2.29. *If μ is an uncountable regular cardinal and $\mathscr{P}(\mu)/fin$ is isomorphic to $\mathscr{P}(\omega)/fin$, then there is a scale of length μ.*

Proof. Let f be an isomorphism of $\mathscr{P}(\mu)/fin$ onto $\mathscr{P}(\omega)/fin$. Let $\langle p_n:n\in\omega\rangle$ be a partition of μ into sets of size μ. For each $n\in\omega$ let $f([p_n])=[q_n]$. Define $r_0=q_0$ and $r_{n+1}=q_{n+1}\backslash\bigcup_{i\le n}q_i$. Then each r_n is infinite; this is obvious for $n=0$, and $q_{n+1}=$

$(q_{n+1} \cap \bigcup_{i \le n} q_n) \cup r_{n+1}$, so r_{n+1} is infinite. Clearly $r_n \cap r_m = \emptyset$ for $m \neq n$. Obviously $f([p_0]) = [r_0]$. Also,

$$\begin{aligned}
[r_{n+1}] &= \left[q_{n+1} \backslash \bigcup_{i \le n} q_i \right] \\
&= [q_{n+1}] \cdot -\sum_{i \le n} [q_i] \\
&= f([p_{n+1}]) \cdot -\sum_{i \le n} f([p_i]) \\
&= f\left(\left[p_{n+1} \backslash \bigcup_{i \le n} p_i \right] \right) \\
&= f([p_{n+1}]).
\end{aligned}$$

Now let $\langle a_n : n \in \omega \rangle$ be a system of pairwise disjoint finite sets such that $\bigcup_{n \in \omega} a_n = \omega \backslash \bigcup_{i \in \omega} q_i$. Define $s_i = r_i \cup a_i$ for all $i \in \omega$. Then $\langle s_i : i \in \omega \rangle$ is a partition of ω into infinite sets, and $f([p_i]) = [s_i]$ for all $i \in \omega$.

Now for each $\alpha < \mu$ let $t_\alpha = \mu \backslash \alpha$, and let $u_\alpha \subseteq \omega$ be such that $f([t_\alpha]) = [u_\alpha]$.

(1) $\forall \alpha < \mu \forall n \in \omega [u_\alpha \cap s_n$ is infinite].

In fact, suppose that $\alpha < \mu$ and $n \in \omega$. Then $|t_\alpha \cap p_n| = \mu$, hence $[t_\alpha] \cdot [p_n] \neq 0$, so $[u_\alpha] \cdot [s_n] \neq 0$, and (1) follows.

Now for each $\alpha < \mu$ and $n \in \omega$ let $h_\alpha(n)$ be the least element of $u_\alpha \cap s_n$.

(2) If $\alpha < \beta$, then $h_\alpha \le^* h_\beta$.

For, suppose that $\alpha < \beta$. Then $t_\beta \subseteq t_\alpha$, so $[t_\beta] \le [t_\alpha]$, and hence $[u_\beta] \le [u_\alpha]$. Let m be greater than each member of $\{n \in \omega : (u_\beta \backslash u_\alpha) \cap s_n \neq 0\}$. Suppose that $n \ge m$. Then $(u_\beta \backslash u_\alpha) \cap s_n = 0$, so $u_\beta \cap s_n \subseteq u_\alpha \cap s_n$. Hence $h_\alpha(n) \le h_\beta(n)$, proving (2).

(3) $\forall g \in {}^\omega\omega \exists \alpha < \mu [g \le^* h_\alpha]$.

In fact, suppose that $\forall \alpha < \mu [g \not\le^* h_\alpha]$. Thus each set $I_\alpha = \{n \in \omega : h_\alpha(n) < g(n)\}$ is infinite. For each $n \in \omega$ let $G_n = \{i \in \omega : i < g(n)\}$. Set $V = \bigcup_{n \in \omega} (s_n \cap G_n)$.

(4) $\forall \alpha < \mu [\{h_\alpha(n) : n \in I_\alpha\} \subseteq V]$.

For, let $n \in I_\alpha$. Then $h_\alpha(n) < g(n)$, so $h_\alpha(n) \in G_n$. Also by definition, $h_\alpha(n) \in s_n$. So $h_\alpha(n) \in V$.

(5) $\forall \alpha < \mu [\{h_\alpha(n) : n \in I_\alpha\}$ is infinite].

In fact, let $\alpha < \mu$. Now I_α is infinite, and for all $n \in I_\alpha$, $h_\alpha(n) \in s_n$, and the $s_n's$ are pairwise disjoint. So (5) holds.

(6) $\forall \alpha < \mu [V \cap u_\alpha$ is infinite].

For, take $\alpha < \mu$. By (5), $\{h_\alpha(n) : n \in I_\alpha\}$ is infinite, and by (4) it is a subset of V. Clearly it is a subset of u_α too.

(7) $\forall n \in \omega[V \cap s_n$ is finite].

For, let $n \in \omega$. Then $V \cap s_n = s_n \cap G_n \subseteq g(n)$, and (7) holds.

Now by (7), $[V] \cdot [s_n] = 0$. Say $f([U]) = [V]$. Then $[U] \cdot [p_n] = 0$, so $U \cap p_n$ is finite. It follows that U is countable. Choose $\alpha < \mu$ such that $U \cap t_\alpha = \emptyset$. Then $[U] \cdot [t_\alpha] = 0$, so $[V] \cdot [u_\alpha] = 0$. Hence $V \cap u_\alpha$ is finite, contradicting (6).

Hence (3) holds.

(8) $\forall \alpha < \mu \exists \beta \in (\alpha, \mu)[h_\alpha <^* h_\beta]$.

For, let $\alpha < \mu$. Define $g(n) = h_\alpha(n) + 1$ for all $n \in \omega$. By (3), choose β so that $g \leq^* h_\beta$. By (2) we may assume that $\alpha < \beta$. Clearly $h_\alpha <^* h_\beta$.

Now from (8) the existence of a scale of length μ is clear. $\square$

Theorem 2.30. *If κ and λ are uncountable and distinct, then $\mathscr{P}(\kappa)/fin \not\cong \mathscr{P}(\lambda)/fin$.*

Proof. By Lemma 2.27 with $\mu = \omega$ it suffices to show that $\mathscr{P}(\omega_1)/fin \not\cong \mathscr{P}(\omega_2)/fin$. Suppose that $\mathscr{P}(\omega_1)/fin \cong \mathscr{P}(\omega_2)/fin$. Again by Lemma 2.27 with $\mu = \omega$ we have $\mathscr{P}(\omega)/fin \cong \mathscr{P}(\omega_1)/fin$. Then by Lemma 2.29, there is a scale of length ω_1. Now $\mathscr{P}(\omega_1)/fin \cong \mathscr{P}(\omega_2)/fin$ and $\mathscr{P}(\omega)/fin \cong \mathscr{P}(\omega_1)/fin$, so $\mathscr{P}(\omega)/fin \cong \mathscr{P}(\omega_2)/fin$. Hence by Lemma 2.29 there is a scale of length ω_2. This contradicts Lemma 2.28. $\square$

The following is from Bell, M. 1980.

If A is a collection of sets, then $\bigwedge A = \{\bigcap A' : A' \in [A]^{<\omega}\}$ and $\bigvee A = \{\bigcup A' : A' \in [A]^{<\omega}\}$. Let $P = \{f \in {}^\omega\omega : \forall n \in \omega[f(n) \leq n + 1]\}$ and $N = \{f \restriction n : f \in P, \ n \in \omega\}$. Let $T = \{\pi \in {}^\omega N : \forall n \in \omega[\mathrm{dmn}(\pi(n)) = n + 1]\}$. For each $s \in N$ let $C_s = \{t \in N : s \subseteq t\}$, and for each $\pi \in T$ let $C_\pi = \bigcup_{n\in\omega} C_{\pi(n)}$.

Proposition 2.31. *(i) $N = \{g : \exists n \in \omega[g \in {}^{n+1}\omega$ and $\forall i \leq n[g(i) \leq i + 1]]\}$.*

(ii) $T = \{\pi : \mathrm{dmn}(\pi) = \omega$ and $\forall n \in \omega[\pi(n) \in {}^{n+1}\omega$ and $\forall i \leq n[(\pi(n))(i) \leq i + 1]]\}$.

(iii) $\forall \pi \in T[C_\pi = \{t \in N : \exists n \in \omega[\pi(n) \subseteq t]\}]$.

(iv) $\forall \pi \in T[N\backslash C_\pi$ is infinite].

Proof. (i)–(iii) are clear. For (iv), suppose that $\pi \in T$. Note that $Q \stackrel{\text{def}}{=} \prod_{n\in\omega}(n + 2)\backslash\{(\pi(n))(n)\}$ is uncountable. Clearly $Q \subseteq N\backslash C_\pi$. $\square$

Now we define $\mathscr{A} = \{C_\pi : \pi \in T\} \cup \{N\backslash C_\pi : \pi \in T\}$ and $\mathscr{B} = \bigvee(\bigwedge(\mathscr{A}))$.

Proposition 2.32. *$\mathscr{B}$ is a subalgebra of $\mathscr{P}(N)$, and $\{\{x\} : s \in N\} \cup \{C_s : s \in N\} \subseteq \mathscr{B}$.*

Clearly $\{s\} : s \in N\}$ is dense in $\mathscr{B}$. Let $\mathscr{C} = \mathscr{B}/[N]^{<\omega}$.

Proposition 2.33. *$\mathscr{C}$ does not have a countable dense subalgebra. It has a set of generators which is a countable union of linked subsets.*

The following is from Bell, M. 1982.

If G is a graph, then $C(G)$ is the set of all complete subgraphs of G. For $v \in G$ we set $v^+ = \{C \in C(G) : v \in C\}$ and $v^- = \{C \in C(G) : v \notin C\}$. Then $\bigcup_{v\in G}\{v^+, v^-\}$ is a subbase for a topology on $C(G)$.

Proposition 2.34. *$C(G)$ is a closed subspace of $\mathscr{P}(G)$.*

Proof. Clearly $C(G)$ is a subspace of $\mathscr{P}(G)$. Suppose that $X \in \mathscr{P}(G)\backslash C(G)$; we want to find an open subset U of $\mathscr{P}(G)$ such that $X \in U$ and $U \cap C(G) = \emptyset$. Since $X \subseteq G$ but is not a complete subgraph of G, there exist distinct $a, b \in X$ such that $\{a, b\}$ is not an edge of G. Then $X \in U_{\{a,b\}\emptyset}$ and $U_{\{a,b\}\emptyset} \cap C(G) = \emptyset$. □

Proposition 2.35. *$C(G)$ is supercompact.*

A space X is *Frechet-Urysohn* iff for all $A \subseteq X$ and all $a \in \overline{A}$ there is a sequence $\langle b_n : n \in \omega\rangle$ of elements of A which converges to a.

Proposition 2.36. *$C(G)$ is Frechet-Urysohn iff every complete subgraph of G is countable.*

The following is from Bell, M. 1983.

Let

$$\begin{aligned}\mathscr{A} &= \left\{\prod_{i\in\omega} A_i : \forall i < \omega[A_i \subseteq \omega \text{ and } |A_i\backslash A_{i+1}| < \omega]\right\};\\ H &= \text{ subalgebra of } \mathscr{P}({}^{\omega}\omega) \text{ generated by } \mathscr{A}.\end{aligned}$$

Proposition 2.37. *H can be isomorphically embedded in $\mathscr{P}(\omega)/fin$.*

Proposition 2.38. *For any $j \geq 2$ there is a subalgebra of H which is $\sigma - j$-linked but not $\sigma - (j+1)$-linked.*

Proposition 2.39. *There is a subalgebra of H which is ccc but not σ-2-linked.*

The following is from Bell, M. 1985

Proposition 2.40. *Let S be a collection of closed subsets of X, with S closed under finite intersection. Then S is a closed subbase for X iff $\forall$ closed $K\forall$ open $U[K \subseteq U \to \exists F \in [S]^{<\omega}[K \subseteq \bigcup F \subseteq U]]$.*

If $k < n$ and $s \in [\omega_1]^n$, then $s_{(k)}$ is the k-th element of S in the order of ω_1. Thus $s = \{s_{(0)}, \ldots, s_{(n-1)}\}$.

Suppose that n is a positive integer and $f : [\omega_1]^n \to [\omega_1]^{<\omega}$ is such that $\forall s \in [\omega_1]^n[s \cap f(s) = \emptyset]$. Suppose that $A \subseteq \omega_1$. Then

$$\begin{aligned} A \text{ is } \textit{free} \quad &\text{iff} \quad \forall s \in [A]^n[A \cap f(s) = \emptyset];\\ A \text{ is } \textit{almost free} \quad &\text{iff} \quad \forall s \in [A]^n[A \cap f(s) \subseteq \{\gamma : s_{(0)} < \gamma < s_{(n-1)}\}.\end{aligned}$$

If S is an infinite set and $0 < n \in \omega$, for each $s \in S$ let $s^+ = \{F \in [S]^{\leq n} : s \in F\}$ and $s^- = \{F \in [S]^{\leq n} : s \notin F\}$. Then $\{s^+ : s \in S\} \cup \{s^- : s \in S\}$ generates a subalgebra A_{Sn} of $\mathscr{P}([S]^{\leq n})$.

Proposition 2.41. *There is an $(n+1)-ary$ closed generating set for A_{Sn}.*

Proposition 2.42. *The compactness number of* $\mathrm{Ult}(A_{S(2n-1)})$ *and of* $\mathrm{Ult}(A_{S(2n)})$ *is* $n+1$.

The following is from Bell 1985a.

If S is a collection of sets, then $\mathrm{Cen}(S)$ is the set of all centered subsets of S. For $s \in S$ we let $s^+ = \{C \in \mathrm{Cen}(S) : s \in C\}$ and $s^- = \{C \in \mathrm{Cen}(S) : s \notin C\}$. $\mathfrak{C}(S)$ is the field of subsets of S generated by all sets s^+, s^-.

Proposition 2.43. *A is isomorphic to* $\mathrm{Cen}(S)$ *for some family S of sets iff A is generated by a set G such that for all finite $F, H \subseteq G$, $\prod F \cdot -\sum H \neq 0$ iff $\prod F \neq 0$ and $F \cap H = \emptyset$.*

The following is from van Douwen 1990.

Let

$$T_\kappa = \{\tau \subseteq \kappa \times \kappa : \tau \text{ is a one-one function and } |\kappa \backslash \mathrm{dmn}(\tau)| + |\kappa \backslash \mathrm{rng}(\tau)| < \kappa\}.$$

Proposition 2.44. *For each $\tau \in T_\kappa$ there is an automorphism τ^* of $\mathbb{P}(\kappa)/[\kappa]^{<\kappa}$ such that for any $X \subseteq \kappa$, $\tau^*([X]) = [\tau[X]]$.*

Proof. First we claim

(1) $\tau[X \triangle Y] = \tau[X] \triangle \tau[Y]$.

For, suppose that $x \in X \triangle Y$ and $x \in \mathrm{dmn}(\tau)$. Then $\tau(x) \in \tau[X]$. If $\tau(x) \in \tau[Y]$, say $y \in Y \cap \mathrm{dmn}(\tau)$ and $\tau(x) = \tau(y)$. Then $x = y$, so $x \in Y$, contradiction. Hence $\tau(x) \notin \tau[Y]$. By symmetry this proves $\subseteq$ in (1).

Suppose that $z \in \tau[X] \backslash \tau[Y]$. Say $x \in X \cap \mathrm{dmn}(\tau)$ and $z = \tau(x)$. If $x \in Y$, then $z \in \tau[Y]$, contradiction. Thus $x \notin Y$, so $z \in \tau[X \backslash Y]$. By symmetry this prove $\supseteq$ in (1).

Now by (1), $|X \triangle Y| < \kappa$ iff $|\tau[X \triangle Y]| < \kappa$ iff $|\tau[X] \triangle \tau[Y]| < \kappa$. Hence τ^* is well-defined and one-one.

(2) $\tau[\tau^{-1}[X]] \subseteq X$.

In fact, suppose that $m \in \tau[\tau^{-1}[X]]$. Then there is an $n \in \tau^{-1}[X]$ such that $\tau(n) = m$. Now $n \in \mathrm{dmn}(\tau$ and $\tau(n) \in X$; so $m \in X$.

(3) $X \backslash \tau[\tau^{-1}[X]] \subseteq ((\kappa \backslash \mathrm{dmn}(\tau)) \cup (\kappa \backslash \mathrm{rng}(\tau)))$.

For, suppose that $m \in X \cap \mathrm{dmn}(\tau \cap \mathrm{rng}(\tau)$. Say $\tau(n) = m$. Then $n \in \tau^{-1}[X]$, and $\tau(n) = m$. So $m \in \tau[\tau^{-1}[X]]$.

By (2) and (3), $\tau^*([\tau^{-1}[X]]) = [X]$. So τ^* maps onto $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$.

(4) $\tau[X] \subseteq \tau[X \cup Y]$.

In fact, let $m \in \tau[X]$. Then there is an $n \in X \cap \mathrm{dmn}(\tau)$ such that $m = \tau(n)$. Since $n \in X \cup Y$, this shows that $m \in \tau[X \cup Y]$.

(5) $\tau[X \cup Y]\backslash(\tau[X] \cup \tau[Y]) \subseteq ((\kappa\backslash\mathrm{dmn}(\tau)) \cup (\kappa\backslash\mathrm{rng}(\tau)))$.

For, suppose that $m \in \tau[X \cup Y] \cap \mathrm{dmn}(\tau) \cap \mathrm{rng}(\tau)$. and $m \notin \tau[Y]$. Say $n \in X \cup Y$, $n \in \mathrm{dmn}(\tau)$, and $m = \tau(n)$. If $n \in Y$, then $m \in \tau[Y]$, contradiction. So $n \notin Y$, hence $n \in X$. Thus $m \in \tau[X]$. This proves (5).

By (4) and (5), $\tau^*([X] + [Y]) = \tau^*([X \cup Y]) = [\tau[X \cup Y]] = [\tau[X]] + [\tau[Y]] = \tau^*([X]) + \tau^*([Y])$.

Next,

(6) $\tau^*(-[X]) = -\tau^*([X])$.

In fact, $\tau^*([X]) + \tau^*(-[X]) = \tau^*([X]) + \tau^*([\kappa\backslash X]) = \tau^*([\kappa]) = 1$. We also claim

(7) $\tau[X] \cap \tau[\kappa\backslash X] \cap \mathrm{dmn}(\tau \cap \mathrm{rng}(\tau) = \emptyset$.

In fact, suppose that $m \in \tau[X] \cap \tau[\kappa\backslash X] \cap \mathrm{dmn}(\tau \cap \mathrm{rng}(\tau) = \emptyset$. Choose $n \in X$ such that $\tau(n) = m$, and choose $p \in \kappa\backslash X$ such that $\tau(p) = m$. Then $n = p$, contradiction. So (7) holds.

From (7) it is clear that $\tau^*([X]) \cdot \tau^*(-[X]) = 0$. Hence (6) follows.

Now we have shown that τ^* is an automorphism of $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$. □

Let $T^*_\kappa = \{\tau^* : \tau \in T_\kappa\}$ and let $S^*_\kappa = \{\tau^* : \tau$ is a permutation of κ.

Proposition 2.45. (Theorem 6.1) *There is a homomorphism h from T^*_ω onto $\mathbb{Z}$ with kernel S^*_ω.*

Proposition 2.46. (Proposition 6.2) *If $\kappa > \omega$ then $T^*_\kappa = S^*_\kappa$.*

The following is from van Douwen 1991.

Theorem 2.47. $|\mathscr{P}(\kappa)/[\kappa]^{<\kappa}| = 2^\kappa$.

Proof. Write $\kappa = \bigcup_{\alpha<\kappa} X_\alpha$ with each $|X_\alpha| = \kappa$ and $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$. For each $S \subseteq \kappa$ let $a_S = \bigcup_{\alpha\in S} X_\alpha$. Then for $S, T \in \mathscr{P}(\kappa)$ and $S \neq T$ we have $[a_S] \neq [a_T]$. □

Corollary 2.48. (Fact 1.1) *If $\mathscr{P}(\kappa)/[\kappa]^{<\kappa} \cong \mathscr{P}(\lambda)/[\lambda]^{<\lambda}$, then $2^\kappa = 2^\lambda$.* □

Proposition 2.49. *$\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is $\mathrm{cf}(\kappa)$-complete but not $\mathrm{cf}(\kappa)^+$-complete.*

Proof. Suppose that $\langle X_\alpha : \alpha < \gamma\rangle$ is a system of elements of $[\kappa]^\kappa$ with $\gamma < \mathrm{cf}(\kappa)$. Then $\left[\bigcup_{\alpha<\gamma} X_\alpha\right]$ is an upper bound for $\{[X_\alpha] : \alpha < \gamma\}$. Suppose that $[Y]$ is any upper bound for $\{[X_\alpha] : \alpha < \gamma\}$. Then

$$\left[\bigcup_{\alpha<\gamma} X_\alpha\right]\backslash[Y] = \left[\bigcup_{\alpha<\gamma}(X_\alpha\backslash Y)\right],$$

and $\bigcup_{\alpha<\gamma}(X_\alpha\backslash Y)$ has size less than κ since $\forall\alpha<\gamma[|X_\alpha\backslash Y|<\kappa]$ and $\gamma<\mathrm{cf}\kappa$. So $\left[\bigcup_{\alpha<\gamma}X_\alpha\right]$ is the least upper bound for $\{[X_\alpha]:\alpha<\gamma\}$.

Now to show that $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is not $\mathrm{cf}(\kappa)^+$-complete, we take two cases.

Case 1. κ is regular. Let $\langle X_\alpha:\alpha<\kappa\rangle$ be a pairwise disjoint system of members of $[\kappa]^\kappa$ with union κ. We claim that $\sum_{\alpha<\kappa}[X_\alpha]$ does not exist. Suppose that $Y\in[\kappa]^\kappa$ and $[Y]$ is an upper bound for $\{[X_\alpha]:\alpha<\kappa\}$. For each $\alpha<\kappa$ let W_α be of size less than κ such that $X_\alpha\backslash Y\subset W_\alpha\subseteq X_\alpha$. Let $Z=\bigcup_{\alpha<\kappa}(X_\alpha\backslash W_\alpha)$. Then

$$Z\backslash Y=\bigcup_{\alpha<\kappa}(X_\alpha\backslash W_\alpha)\backslash Y)=\emptyset.$$

Moreover, if $x_\alpha\in W_\alpha\cap Y$ for each $\alpha<\kappa$, Then $\{x_\alpha:\alpha<\kappa\}$ is a subset of Y of size κ which is disjoint from Z; so $[Z]<[Y]$. If $\alpha<\kappa$, then $X_\alpha\backslash Z=W_\alpha$ has size less than κ, as desired.

Case 2. κ is singular. Let $\langle\lambda_\xi:\xi<\mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of nonzero cardinals with supremum κ. Let $\langle X_\xi:\xi<\mathrm{cf}(\kappa)\rangle$ be a pairwise disjoint system of members of $[\kappa]^\kappa$ with union κ. We claim that $\sum_{\xi<\mathrm{cf}(\kappa)}[X_\alpha]$ does not exist. Suppose that $Y\in[\kappa]^\kappa$ and $[Y]$ is an upper bound for $\{[X_\xi]:\xi<\mathrm{cf}(\kappa)\}$. For each $\xi<\mathrm{cf}(\kappa)$ let W_ξ be such that $X_\xi\backslash Y\subset W_\xi\subseteq X_\xi$, $|W_\xi|<\kappa$, and $|W_\xi\cap Y|=\lambda_\xi$. Let $Z=\bigcup_{\xi<\mathrm{cf}(\kappa)}(X_\xi\backslash W_\xi)$. Then

$$Z\backslash Y=\bigcup_{\xi<\mathrm{cf}(\kappa)}(X_\xi\backslash W_\xi)\backslash Y)=\emptyset.$$

Moreover, if $V_\xi\in[W_\xi\cap Y]^{\lambda_\xi}$ for each $\xi<\mathrm{cf}(\kappa)$, Then $\bigcup_{\xi<\mathrm{cf}(\kappa)}V_\xi$ is a subset of Y of size κ which is disjoint from Z; so $[Z]<[Y]$. If $\xi<\mathrm{cf}(\kappa)$, then $X_\xi\backslash Z=W_\xi$ has size less than κ, as desired.

Corollary 2.50. (Fact 1.2) *If* $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}\cong\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$, *then* $\mathrm{cf}(\kappa)=\mathrm{cf}(\lambda)$. □

Proposition 2.51. (Fact 3.1) *If* $2^{<\kappa}=\kappa$, *then* $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ *has a disjoint subset of size* 2^κ.

Proof. Let $T=\bigcup_{\alpha<\kappa}{}^\alpha 2$. Then $|T|=\kappa$ by assumption. For each $f\in{}^\kappa 2$ let $f'=\{f\restriction\alpha:\alpha<\kappa\}$. Then $|f'|=\kappa$. If $f,g\in{}^\kappa 2$ and $f\neq g$, then $|f'\cap g'|<\kappa$. Let k be a bijection from T to κ. For each $f\in{}^\kappa 2$ let $a_f=[k[f']]$. Then $\langle a_f:f\in{}^\kappa 2$ is a disjoint system. □

Let B be a BA. We say that $S\subseteq B$ *separates* $P\subseteq B$ iff $\forall p,q\in P[p\neq q\to\exists s\in S[p\leq s$ and $q\leq -s]$. Then B *has the* (σ,π)*-separated chain condition* ((σ,π)*-scc*) iff

$$\forall P\subseteq B\backslash\{0\}[\exists S\subseteq B[S\text{ separates }P\text{ and }|S|<\sigma]\to|P|<\pi].$$

Proposition 2.52. *If* $2^{<\kappa}=\kappa$, *then* $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ *does not have the* $(\kappa^+,2^\kappa)$*-scc.*

Proof. Let T be as in the proof of Proposition 2.51, and let $k:T\to\kappa$ be a bijection. Let $P=\{[k[f']]:f\in{}^\kappa 2\}$. Thus $P\subseteq\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$. For each $s\in T$ let

$s^* = \{l \in T : k \subseteq l\}$, and let $S = \{[k[s^*]] : s \in T\}$. Thus $S \subseteq \mathscr{P}(\kappa)/[\kappa]^{<\kappa}$. Suppose that f, g are distinct members of ${}^{\kappa}2$. Say $f(\alpha) \neq g(\alpha)$. Then $[k[f']] \leq [k[(f \restriction (\alpha+1))^*]]$ and $[k[g']] \leq -[k[(f \restriction (\alpha+1))^*]]$. So S separates P. Since $|S| = \kappa$ and $|P| = 2^\kappa$, this shows that $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ does not have the $(\kappa^+, 2^\kappa)$-scc. $\square$

Proposition 2.53. (Fact 4.1) *$\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ has the $(\mathrm{cf}(\kappa), \kappa^+)$-scc.*

Proof. Suppose that $P \subseteq \mathscr{P}(\kappa)/[\kappa]^{<\kappa}\backslash\{0\}$, $S \subseteq \mathscr{P}(\kappa)/[\kappa]^{<\kappa}$, S separates P, and $|S| < \mathrm{cf}(\kappa)$; we want to show that $|P| \leq \kappa$. We may assume that $0 \notin S$. Choose $P' \subseteq [\kappa]^\kappa$ and $S' \subseteq [\kappa]^\kappa$ so that $P = \{[p] : p \in P'\}$, $S = \{[s] : s \in S'\}$, $[p] \neq [p']$ for $p, p' \in P'$ and $p \neq p'$, and $[s] \neq [s']$ for $s, s' \in S'$ and $s \neq s'$. Let $S'' = \{s : \kappa\backslash s \in S'\}$, and for each $p \in P'$ let

$$(*) \qquad p^* = p\backslash\bigcup\{p \cap s : s \in S' \cup S'',\ |p \cap s| < \kappa\}.$$

Since $|S' \cup S''| < \mathrm{cf}(\kappa)$, it follows that $|p^*| = \kappa$. Suppose that $p, q \in P'$ and $p \neq q$. Choose $s \in S'$ such that $p \subseteq_\kappa s$ and $q \subseteq \kappa\backslash s$. Then $|p \cap (\kappa\backslash s)| < \kappa$, and $\kappa\backslash s \in S''$, so $p \cap (\kappa\backslash s)$ is in the big union of $(*)$ for p^*, so $p^* \subseteq s$. Also, $|q \cap s| < \kappa$, so $q \cap s$ is in the big union of $(*)$ for q, and so $q^* \subseteq \kappa\backslash s$. Hence $p^* \cap q^* = \emptyset$. It follows that $|P| \leq \kappa$. $\square$

Proposition 2.54. (Fact 3.4(b)) *If $\sigma > \pi$, then A has the π-cc iff A has the (σ, π)-scc.*

Proof. $\Rightarrow$: obvious. $\Leftarrow$: Suppose that A has the (σ, π)-scc but $X \subseteq A$ is a system of pairwise disjoint nonzero elements of A with $|X| = \pi$. Let $S = P$. Clearly S separates P. This is a contradiction. $\square$

Proposition 2.55. (Fact 4.4) *If there is a $\gamma < \sigma$ such that $2^\gamma \geq \pi$, then there is a BA A of size π which does not have the (σ, π)-scc.*

Proof. Assume that $\gamma < \sigma$ and $2^\gamma \geq \pi$.

Case 1. $\sigma > \pi$. Let A be the finite-cofinite algebra on π. Then $[\pi]^1$ separates $[\pi]^1$ and $|[\pi]^1| = \pi$, and this proves that A does not have the (σ, π)-scc.

Case 2. $\sigma \leq \pi$. Since $2^\gamma \geq \pi$, we can pick $P \subseteq \mathscr{P}(\gamma)$ such that $|P| = \pi$. Define

$$S = \{\{A \in \mathscr{P}(\gamma) : \alpha \in A\} : \alpha \in \gamma\}.$$

Let A be the subalgebra of $\mathscr{P}(\mathscr{P}(\gamma))$ generated by $[P]^1 \cup S$. Since $|P| = \pi$ and $|S| \leq \gamma, \sigma \leq \pi$, we have $|A| = \pi$. Note that $[P]^1$ is pairwise disjoint. Finally, $S \cup \{-s : s \in S\}$ separates $[P]^1$. For, suppose that $p, q \in P$ with $p \neq q$. Say $\alpha \in p\backslash q$. Then $\{p\} \subseteq \{A \in \mathscr{P}(\gamma) : \alpha \in A\} \in S$ and $\{q\} \subseteq \mathscr{P}(\gamma)\backslash\{A \in \mathscr{P}(\gamma) : \alpha \in A\}$. $\square$

Proposition 2.56. (Fact 4.6) *If $|A| \leq \lambda$, then A can be isomorphically embedded into $\mathscr{P}(\lambda)$.*

Proof. We may assume that A is a field of subsets of some set X. Let f be a surjection from λ onto $A\backslash\{0\}$. Then define, for any $a \in A$, $e(a) = f^{-1}[a]$. Then $e(a+b) =$

$f^{-1}[a+b] = f^{-1}[a] \cup f^{-1}[b]$ and $e(-a) = f^{-1}[-a] = X \backslash f^{-1}[a]$. So e is a homomorphism. It is clearly one-one. $\square$

Proposition 2.57. (Fact 4.6) *$\mathscr{P}(\lambda)$ can be isomorphically embedded into $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$.*

Proof. Let B be a partition of λ into λ sets each of size λ. It suffices to show that $\mathscr{P}(B)$ can be isomorphically embedded into $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$. For each $X \subseteq B$ let $e(X) = [\bigcup X]$. Clearly e is the desired embedding. $\square$

Corollary 2.58. (Fact 4.6) *If $|A| \leq \lambda$, then A can be isomorphically embedded into $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$.* $\square$

Proposition 2.59. *If B has the (σ, π)-scc and $A \leq B$, then A has the (σ, π)-scc.* $\square$

Proposition 2.60. (Fact 4.3) *If $2^{<\mathrm{cf}(\kappa)} > \kappa < \lambda$, then $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$ does not have the $(\mathrm{cf}(\kappa), \kappa^+)$-scc.*

Proof. Assume that $2^{<\mathrm{cf}(\kappa)} > \kappa < \lambda$. Then there is a $\gamma < \mathrm{cf}(\kappa)$ such that $2^\gamma > \kappa$. Hence by Proposition 2.55 there is a BA A of size κ^+ which does not have the $(\mathrm{cf}(\kappa), \kappa^+$-scc. By Corollary 2.58, A can be isomorphically embedded into $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$. Hence by Proposition 2.59, $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$ does not have the $(\mathrm{cf}(\kappa), \kappa^+)$-scc. $\square$

Proposition 2.61. (Main lemma) *If $\kappa < 2^{<\mathrm{cf}(\kappa)}$ and $\kappa < \lambda$, then $\mathscr{P}(\kappa)/[\kappa]^{<\kappa} \not\cong \mathscr{P}(\lambda)/[\lambda]^{<\lambda}]$.*

Proof. By Propositions 2.53 and 2.60. $\square$

Proposition 2.62. *If κ and λ are regular and $\mathscr{P}(\kappa)/[\kappa]^{<\kappa} \cong \mathscr{P}(\lambda)/[\lambda]^{<\lambda}$, then $\kappa = \lambda$.*

Proof. By Corollary 2.50. $\square$

Proposition 2.63. (Theorem 2.1) *If κ is regular and $\kappa < 2^{<\kappa}$, then $\forall\lambda \neq \kappa[\mathscr{P}(\kappa)/[\kappa]^{<\kappa} \not\cong \mathscr{P}(\lambda)/[\lambda]^{<\lambda}]$.*

Proof. Assume that κ is regular and $\kappa < 2^{<\kappa}$, and suppose that $\lambda \neq \kappa$.

Case 1. $\lambda < \kappa$. Then $\mathrm{cf}(\lambda) \neq \kappa = \mathrm{cf}(\kappa)$ and the desired result holds by Corollary 2.50.

Case 2. $\kappa < \lambda$. Apply Proposition 2.61. $\square$

Proposition 2.64. *If $m < n \in \omega$, κ is regular or $\forall\gamma < \kappa[2^\gamma < 2^\kappa]$, and $\exists\lambda \neq \kappa^{+m}[\mathscr{P}(\lambda)/[\lambda]^{<\lambda} \cong \mathscr{P}(\kappa^{+m})/[\kappa^{+m}]^{<\kappa^{+m}}]$, then $\forall\lambda \neq \kappa^{+n}[\mathscr{P}(\lambda)/[\lambda]^{<\lambda} \not\cong \mathscr{P}(\kappa^{+n})/[\kappa^{+n}]^{<\kappa^{+n}}]$.*

Proof. Assume that $m < n \in \omega$, κ is regular or $\forall\gamma < \kappa[2^\gamma < 2^\kappa]$, $\lambda \neq \kappa^{+m}$, and $\mathscr{P}(\lambda)/[\lambda]^{<\lambda} \cong \mathscr{P}(\kappa^{+m})/[\kappa^{+m}]^{<\kappa^{+m}}$. Then $n \geq 1$, so κ^{+n} is regular. If $\mu < \kappa^{+n}$ then $\mathrm{cf}(\mu) < \kappa^{+n} = \mathrm{cf}(\kappa^{+n})$ and so $\mathscr{P}(\mu)/[\mu]^{<\mu} \not\cong \mathscr{P}(\kappa^{+n})/[\kappa^{+n}]^{<\kappa^{+n}}$ by Corollary 2.50. So it remains to take care of the case $\mu > \kappa^{+n}$. By Proposition 2.63 it suffices to show that

$\kappa^{+n} < 2^{<\kappa^{+n}}$. To do this it suffices to show that $\kappa^{+\omega} \le 2^{\kappa^{+m}}$. Now by Corollary 2.48 we have $2^\lambda = 2^{\kappa^{+m}}$, so it suffices to show that $\kappa^{+\omega} \le \lambda$.

Now $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa^{+m})$ by Corollary 2.50. Hence we cannot have $\kappa^{+m} < \lambda < \kappa^{+\omega}$. We now consider several cases.

Case 1. $m \ne 0$. Then $\lambda \not< \kappa^{+m}$, since $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa^{+m})$ by Corollary 2.50.

Case 2. $m = 0$ and κ is regular. Similarly, $\lambda \not< \kappa^{+m}$.

Case 3. $m = 0$ and κ is singular. Suppose that $\lambda < \kappa$. Then by assumption, $2^\lambda < 2^\kappa$. This contradicts Corollary 2.48. $\square$

Theorem 2.65. *If κ is regular or $\forall\gamma < \kappa[2^\gamma < 2^\kappa]$, then there is at most one $m \in \omega$ such that $\exists\lambda \ne \kappa^{+m}[\mathscr{P}(\lambda)/[\lambda]^{<\lambda} \cong \mathscr{P}(\kappa^{+m})/[\kappa^{+m}]^{<\kappa^{+m}}]$.* $\square$

The following is from Farah 2001.

Proposition 2.66. *The following are equivalent;*

(i) There is a continuous function mapping ω^* onto ${}^2\omega^*$.

(ii) There is an $n \in \omega$ such that there is a continuous function mapping ${}^n\omega^*$ onto ${}^{n+1}\omega^*$.

(iii) There is a continuous function mapping ω^* onto ${}^\omega\omega^*$.

The following is from Hajnal, Juhasz, Soukup 1987.

If X is an infinite set and $\mathscr{A}$ is a collection of subsets of X, then $I_\mathscr{A}$ is the ideal generated by $\mathscr{A}$. If $\mathscr{A}$ is almost disjoint, then it is *saturated* iff for every $H \notin I_\mathscr{A}$ there is an $A \in \mathscr{A}$ such that $A \subseteq H$.

Theorem 2.67. (Theorem) *If $\mathbb{P}$ is the poset that adds ω_1 dominating reals to M, then in $M[G]$ there is a saturated almost disjoint family.*

The following is from Hrusak, Ferreira 2003.

If $\mathscr{A}$ is a MAD family on ω, then $I(\mathscr{A}) = \{X \subseteq \omega : \exists F \in [\mathscr{A}]^{<\omega}[X \subseteq^* \bigcup F]\}$. If $\mathscr{I}, \mathscr{J}$ are ideals on ω, let $\mathscr{I} \le_K \mathscr{J}$ iff $\exists f : \omega \to \omega \forall I \in \mathscr{I}[f^{-1}[I] \in \mathscr{J}]$. For MAD families $\mathscr{A}, \mathscr{B}$ we write $\mathscr{A} \le_K \mathscr{B}$ iff $I(\mathscr{A}) \le_K I(\mathscr{B})$.

Theorem 2.68. (Corollary 2.4) *For any MAD family $\mathscr{A}$ there is a strictly decreasing chain of length $(2^\omega)^+$ below $\mathscr{A}$ in the Katetov ordering.*

Theorem 2.69. (Proposition 2.5) *For any MAD family $\mathscr{A}$ there is a collection of 2^ω pairwise Katetov incomparable MAD families below $\mathscr{A}$.*

The following is from Kojman, Shelah 2001.

We define

$$\mathrm{Col}(\kappa, \lambda) = \{f : f \text{ is a function with domain in } \kappa \text{ and range contained in } \lambda\}.$$

Theorem 2.70. (Theorem 2.1) *Suppose that μ is singular with $\mathrm{cf}(\mu) = \omega$, and let $\lambda = \mu^\omega$. Then $\overline{\mathscr{P}(\mu)/[\mu]^{<\mu}}$ has a complete subalgebra isomorphic to $\overline{\mathrm{Col}(\omega_1, \lambda)}$.*

The following is from Koszmider 1998.

A set $\mathscr{A} \subseteq [\omega_1]^{\omega_1}$ is *strongly almost disjoint* iff $|\mathscr{A}| = \omega_2$ and all intersections of two members of $\mathscr{A}$ are finite. A sequence $X \in {}^{\omega_2}\mathscr{P}(\omega_1$ is a *strong chain* iff $\forall\alpha, \beta < \omega_2[\alpha < \beta \to [X_\alpha \backslash X_\beta$ is finite and $X_\beta \backslash X_\alpha$ is uncountable].

Theorem 2.71. *Chang's conjecture implies that there is no strong chain.*

Theorem 2.72. *It is relatively consistent that a strong chain exists.*

The following is from Mill 1983.

Proposition 2.73. (CH) *For any BA A the following are equivalent:*
(i) A satisfies CSP and $|A| \leq \omega_1$.
(ii) A is a homomorphic image of $\mathscr{P}(\omega)$.

Proposition 2.74. (CH) ω^* *has exactly* 2^{ω_1} *autohomeomorphisms.*

The following is from Mill 1983.

Proposition 2.75. (Dow, CH) *There is a BA A with a countably generated ideal I such that $\mathscr{P}(\omega)$ can be embedded in A/I but not in A itself.*

The following is from Rabus 1994.

A (κ, λ^*)-*pre-gap* is a pair (A, B) such that $A \in {}^{\kappa}\mathscr{P}(\omega)$ is $\subseteq^*$-increasing, $B \in {}^{\lambda}\mathscr{P}(\omega)$ is $\subseteq^*$-decreasing, and $\forall\alpha < \kappa\forall\beta < \lambda[A_\alpha \subseteq^* B_\beta]$. An infinite set $C \subseteq \omega$ is *beside* (A, B) iff $\forall\beta < \lambda[C \subseteq B_\beta]$ and $\forall\alpha < \kappa[C \backslash A_\alpha$ is infinite]. A *tight gap* is a pre-gap with no set beside it.

Theorem 2.76. *In M let $A \in {}^{\omega_1}\mathscr{P}(\omega)$ be $\subseteq^*$-increasing. Then there is a ccc poset $\mathbb{P}$ such that $\mathbb{P}$ forces the existence of $B \in {}^{\omega_2}\mathscr{P}(\omega)$ which is $\subseteq^*$-decreasing, with (A, B) a tight gap.*

3. Cardinal functions

Arhangelski 1972 proved:

Theorem 3.1. *If* $|\mathrm{Ult}(A)| \leq 2^\omega$, $\mathrm{c}(A) = \omega$, *and* $t(A) = \omega$, *then* $\mathrm{s}(A) = \omega$.

The following is from Balcar, Hrušák 2005.

If A is any BA, then $\mathrm{fin}_A = \{a \in {}^\omega A : \{i \in \omega : a_i \neq 0 \text{ is finite}\}$. Clearly fin_A is an ideal in ${}^\omega A$.

Proposition 3.2. *Let A be a non-trivial BA. For each $a \subseteq \omega$ define $f(a) \in {}^\omega A$ by*

$$(f(a))_i = \begin{cases} 1 & \text{if } i \in a, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is a function $g : \mathscr{P}(\omega)/fin \to {}^\omega A/fin_A$ such that the following conditions hold:

(i) $g([a]) = [f(a)]$ for all $a \subseteq \omega$.
(ii) g is an isomorphic embedding.
(iii) $\mathrm{rng}(g)$ is a regular subalgebra of ${}^\omega A/fin_A$.

Proof.

$$\begin{aligned}
[a] = [b] \quad & \text{iff} \quad a \triangle b \text{ is finite} \\
& \text{iff} \quad \{i \in \omega : i \in a \backslash b \text{ or } i \in b \backslash a\} \text{ is finite} \\
& \text{iff} \quad \{i \in \omega : [(f(a))_i = 1 \text{ and } (f(b))_i = 0] \text{ or} \\
& \qquad [(f(a))_i = 0 \text{ and } (f(b))_i = 1]\} \text{ is finite} \\
& \text{iff} \quad \{i \in \omega : (f(a) \triangle f(b))_i \neq 0\} \text{ is finite} \\
& \text{iff} \quad [f(a)] = [f(b)].
\end{aligned}$$

Hence (i) holds; and (ii) is then clear.

For (iii), suppose that $X \subseteq \mathscr{P}(\omega)$ and $\sum_{x \in X}[x]$ exists; say $\sum_{x \in X}[x] = [y]$. Then if $x \in X$ then $[x] \leq [y]$, and hence $[f(x)] \leq [f(y)]$. So $[f(y)]$ is an upper bound for $\{[f(x)] : x \in X\}$. Suppose that $[z]$ is any upper bound, but $[f(y)] \cdot -[z] \neq 0$. Thus $\{i \in \omega : (f(y))_i \cdot -z_i \neq 0\}$ is infinite. Now $w \stackrel{\text{def}}{=} \{i \in \omega : (f(y))_i \cdot -z_i \neq 0\} = \{i \in y : z_i \neq 1\}$. Thus $w \subseteq y$, so $[w] \leq [y]$ and $[w] \neq 0$. So there is an $x \in X$ such that $[w] \cdot [x] \neq 0$. Thus $w \cap x$ is infinite. Now $[f(x)] \leq [z]$, so $f(x) \cdot -z \in \mathrm{fin}_A$. Thus $\{i \in x : z_i \neq 1\}$ is finite. So $w \backslash \{i \in x : z_i \neq 1\}$ is infinite. But $w \subseteq \{i \in y : z_i \neq 1\}$, contradiction. $\square$

Theorem 3.3. (Theorem 2.2) $\mathfrak{h}({}^\omega\mathrm{Fr}(\omega)/fin) \leq \min(\mathfrak{h}, \mathrm{add}(\mathrm{meag}))$.

Theorem 3.4. (Theorem 2.3 (Dow)) $\mathfrak{h}({}^\omega\mathrm{Fr}(\omega)/fin) < \mathfrak{h}$ *in the iterated Mathias model.*

Theorem 3.5. (Theorem 2.4) $\mathfrak{t} = \mathfrak{t}({}^\omega\mathrm{Fr}(\omega)/fin)$.

Corollary 3.6. ($\mathfrak{t} = \mathfrak{h}$) *$\mathscr{P}(\omega)/fin$ and $\mathscr{P}(\mathbb{R})/fin$ have isomorphic completions.*

Theorem 3.7. *It is relatively consistent that $\mathfrak{t} < \mathfrak{h}((^{\omega}\mathrm{Fr}(\omega)/fin)$.*

The following is from Balcar, Frankiewicz 1979.

If ν is a cardinal and X is a topological space, then a point p of X is a *ν-point* iff there is a family of ν pairwise disjoint open sets of X each of which has p in its closure.

Theorem 3.8. (Theorem A) *Every ultrafilter on $\mathscr{P}(\omega)/fin$ is a $\mathfrak{b}$-point in the space* $\mathrm{Ult}(\mathscr{P}(\omega)/fin)$*.*

Theorem 3.9. (Theorem B) *If $\mathrm{cf}(2^\omega) \leq \mathfrak{b}$, then every ultrafilter on $\mathscr{P}(\omega)/fin$ is a 2^ω-point in* $\mathrm{Ult}(\mathscr{P}(\omega)/fin)$*.*

The following is from Balcar, Pelant, Simon 1980.

If P is a dense-in-itself topological space, then $n(P)$ is the least cardinality of a family of nowhere dense sets covering P.

Proposition 3.10. *For any BA A, $n(\mathrm{Ult}(A)) = \min\{\kappa :$there is a family $\mathscr{A}$ of partitions of unity of A such that for every $F \in \mathrm{Ult}(A)$ there is a $P \in \mathscr{A}$ such that $P \cap F = \emptyset\}$.*

Let κ be the least cardinal such that $\mathscr{P}(\omega)/fin$ is not $(\kappa, 2^\omega)$-distributive.

Theorem 3.11. (Lemma 2.5) $\omega_1 \leq \kappa \leq 2^\omega$*.*

Theorem 3.12. (Corollary 2.9) *κ is regular.*

Theorem 3.13. (Theorem 3.5(i)) *If $\kappa < 2^\omega$, then $\kappa \leq n(\mathscr{P}(\omega)/fin) \leq \kappa^+$.*

Theorem 3.14. (Theorem 3.5(ii)) *If $\kappa = 2^\omega$, then $\kappa \leq n(\mathscr{P}(\omega)/fin) \leq 2^{2^\omega}$.*

Theorem 3.15. (Theorem 4.2) $\kappa \leq \mathrm{cf}(2^\omega)$.

Theorem 3.16. (Theorem 4.5) $\kappa \leq \mathfrak{b}$.

The following is from Balcar, Simon 1988

Theorem 3.17. *If κ is regular and uncountable, then $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is $(\omega, ., \mathfrak{b}_\kappa)$-nowhere distributive.*

Theorem 3.18. *If κ is singular with $\mathrm{cf}(\kappa) > \omega$, then $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is $(\omega, ., \kappa^+)$-nowhere distributive.*

The following is from Bonnet, Shelah 1985

Theorem 3.19. *There is a subset L of the real line such that the following condition hold:*

(i) $|L| = \mathrm{cf}(2^\omega)$.

(ii) Each open interval of $\mathbb{R}$ *contains* $\mathrm{cf}(2^\omega)$ *elements of* L*.*
(iii) $\mathrm{Inc}(\mathrm{Intalg}(L)) < \mathrm{cf}(2^\omega)$.

The following is from Brendle, Shelah 1999.

For U a nonprincipal ultrafilter on ω,

$$\pi\mathfrak{p}(U) = \min\{|X| : X \subseteq U \text{ and } \neg\exists B \in U\forall A \in X[B \subseteq^* A]\};$$
$$\pi\mathfrak{p}(U) = \min\{|X| : X \subseteq U \text{ and } \neg\exists B \in [\omega]^\omega\forall A \in X[B \subseteq^* A]\}$$

Proposition 3.20. $\mathfrak{p} = \min\{\pi\mathfrak{p}(U) : U$ *a nonprincipal ultrafilter on* $\omega\}$*.*

The following is from Campero-Arena, Cancino, Hrusak, Miranda-Perea 2016.

Theorem 3.21. (Corollary 2.4) $\mathrm{Inc}_{\mathrm{mm}}(\mathscr{P}(\omega)/fin) = 2^\omega$*.*

Theorem 3.22. (Corollary 3.3) *It is consistent with* $\neg CH$ *that there is a maximal tree in* $\mathscr{P}(\omega)/fin$ *of size* ω_1*.*

The following is from Cichon 1984.

For κ an infinite cardinal and $p \in [\kappa]^{<\omega}$ the set $I_\kappa \stackrel{\mathrm{def}}{=} \{q \in [\kappa]^{<\omega} : p \cap q = \emptyset\}$ is an ideal in $\mathscr{P}([\kappa]^{<\omega})$. Let $B_\kappa = \mathscr{P}([\kappa]^{<\omega})/I_\kappa$.

Theorem 3.23. (Theorem 1.1) *If* $\mathrm{cf}(\kappa) = \omega$*, then* $\mathfrak{p}(B_\kappa) \geq \omega_1$*.*

Theorem 3.24. (Theorem 1.3) *If* $\kappa < \mathfrak{d}$*, then* $\mathfrak{p}(B_\kappa) \geq \omega_1$*.*

Theorem 3.25. (Theorem 2.1) *If* $\kappa^\omega = \kappa$*, then* $\mathfrak{p}(B_\kappa) = \omega$*.*

The following is from Cichon 1989.

Let I be an ideal on a set X. A subset $A \subseteq X$ is a (κ, λ)-*Luzin set for* I iff $|A| = \kappa$ and $\forall B \in I[|A \cap B| < \lambda]$. The article investigates this and similar notions, in particular with respect to Cichon's diagram.

The following is from Cichon, Kraszewski 1998.

Define
$$Pif(2) = \{f : \varphi \text{ is a function and } \mathrm{dmn}(\varphi) \in [\omega]^\omega \text{ and } \mathrm{rng}(\varphi) \in 2\}.$$

For $\varphi \in Pif(2)$ let $[\varphi]^*_2 = \{x \in {}^\omega 2 : \varphi \subseteq x\}$. Let I^*_2 be the ideal on ${}^\omega 2$ generated by all sets $[\varphi]^*_2$ for $\varphi \in Pif(2)$.

Theorem 3.26. (Lemma 4.1) $\mathrm{cov}(I^*_2) = \mathfrak{r}$*.*

Theorem 3.27. (Lemma 4.2) $\mathrm{non}(I^*_2) = \mathfrak{s}$*.*

The following is from Day 1970.

Theorem 3.28. *A linearly ordered set L is isomorphic to a maximal chain in a κ-complete atomic BA iff L is κ-complete, has a maximum and minimum element, and does not have a complete dense interval.*

The following is from Dordal 1989.

An *α-tower* in $[\omega]^\omega$ is a sequence $\langle x_\xi : \xi < \alpha\rangle$ of members of $[\omega]^\omega$ such that $\forall \xi, \eta < \alpha[\xi < \eta \to x_\eta \subseteq^* x_\xi$, while there is no $y \in [\omega]^\omega$ such that $\forall \xi < \alpha[y \subseteq^* x_\xi]$.

Theorem 3.29. (Lemma 1.1) *Forcing with* $([\omega]^\omega . \subseteq^*)$ *collapses* 2^ω *to* $\mathfrak{h}$.

An *α-tower* in ${}^\omega\omega$ is a sequence $\langle f_\xi : \xi < \alpha\rangle$ of members of ${}^\omega\omega$ such that $\forall \xi, \eta < \alpha[\xi < \eta \to f_\xi <^* f_\eta$, while there is no $g \in {}^\omega\omega$ such that $\forall \xi < \alpha[x_\xi <^* g]$.

Theorem 3.30. (Lemma 1.2) *If κ is regular and there is a κ-tower in ${}^\omega\omega$, then $\mathfrak{b} \le \kappa \le \mathfrak{d}$.*

The following is from Hernandez-Hernandez 2009.

Recall the definition of fin_A from the beginning of this chapter.

Theorem 3.31. (Proposition 2.3) $\mathfrak{h}({}^\omega\mathscr{P}(\omega)/\text{fin}) = \mathfrak{h}$.

Theorem 3.32. *It is relatively consistent that* $\mathfrak{h}({}^\omega(\mathscr{P}(\omega)/fin)/fin) \ne \mathfrak{h}$.

The following is from Jech 1977.

For any BA A, the game G consists of players choosing $a_0 \ge a_1 \ge \cdots$. I wins if $\prod_{n\in\omega} a_n = 0$.

Proposition 3.33. *If A has a σ-closed dense subset, then II has a winning strategy.*

Proposition 3.34. *I fails to have a winning strategy iff A is (ω, ∞)-distributive.*

Proposition 3.35. *There is a BA such that the game is undetermined.*

The following is from Jech 1984.

The cut and choose game G_{cc} runs as follows. Let A be a BA. I chooses $a \in A$, Then I cuts a into two disjoint pieces a_1^0 and a_1^1. Then II chooses one of these; etc. Let $a_n^{f(n)}$ be the element chosen by II on the n-th move. Then I wins iff $\prod_{n\in\omega} a_n^{f(n)} = 0$.

Proposition 3.36. *I fails to have a winning strategy in the game G_{cc} over A iff A is $(\omega, 2)$-distributive.*

A *Suslin algebra* is an atomless ccc $(\omega, 2)$-distributive BA.

Proposition 3.37. *G_{cc} is undetermined in a Suslin algebra.*

The following is from Jech, Prikry 1984.

We consider the cofinality of ${}^{\omega_1}\omega$ under eventual domination.

Theorem 3.38. *If 2^ω is real-valued measurable, then* $\mathrm{cf}({}^{\omega_1}\omega) = 2^{\aleph_1}$.

Theorem 3.39. *If $2^\omega < \aleph_{\omega_1}$ and $2^\omega < 2^{\omega_1}$, then* $\mathrm{cf}({}^{\omega_1}\omega) = 2^{\aleph_1}$.

The following is from Juhasz 1993.

Theorem 3.40. *Let κ be uncountable and regular, and let X be compact Hausdorff with $w(X) \geq \kappa$. Then there is a closed subspace F of X such that*

$$\begin{aligned} w(F) &\in [\kappa, 2^{<\kappa}] \text{ and} \\ |F| &\leq \sum\{2^{2^\lambda} : \lambda < \kappa\} \end{aligned}$$

The following is from Just 1988.

Theorem 3.41. *It is consistent with $\neg CH$ that every BA has altitude at most ω_1.*

The following is from Just, Koszmider 1991.

Theorem 3.42. *If in M κ is a cardinal of uncountable cofinality, then there is a generic extension $M[G]$ such that in $M[G]$ the following hold:*

(i) $2^\omega = \kappa$.

(ii) There is a BA A such that $\mathrm{cf}(A) = |A| = \omega_1$.

(iii) For every cardinal $\lambda \leq \kappa$ of uncountable cofinality there is a BA B such that $h(B) = \lambda$.

The following is from Koppelberg, Shelah 1995 (Shelah 415).

Theorem 3.43. *It is relatively consistent to have a cardinal κ, a system $\langle A_\alpha : \alpha < \kappa\rangle$ of BAs, and an ultrafilter D on κ such that*

$$\mathrm{d}(\prod_{\alpha<\kappa} A_\alpha/D) \leq \pi(\prod_{\alpha<\kappa} A_\alpha/D) < \left|\prod_{\alpha<\kappa} \pi(A_\alpha)\right| = \left|\prod_{\alpha<\kappa} \mathrm{d}(A_\alpha)\right|.$$

Theorem 3.44. *If μ is a strong limit cardinal,* $\mathrm{cf}(\mu) = \omega$, *and $2^\mu = \mu^+$, then there is a BA A such that* $|A| = |\mathrm{End}(A)| = \mu^+$ *and* $|\mathrm{Id}(A)| = 2^{\mu^+}$.

The following is from Laflamme 1993.

For a natural number m and a BA A, an *m-partition* of A is a set $P \in [A]^m$ such that $\sum P = 1$ and $a \cdot b = 0$ for all $\{a, b\} \in [P]^2$. A subset X of A is *(m, n)-reaped* by an m-partition P iff

$$\forall a \in X[|\{b \in P : a \cdot b \neq 0\}| \geq n].$$

Now we define

$$\mathfrak{r}_{mn}(A) = \min\{|X| : 0 \notin X \text{ and } X \text{ cannot be } (m, n)\text{-reaped}\}.$$

Proposition 3.45. $\mathfrak{r}_{mn}(A) = \min\{|X| : 0 \notin X$ *and for every* m*-partition* P *there is an* $a \in X$ *such that* $|\{b \in P : a \cdot b \neq 0\}| < n\}$. $\square$

Proposition 3.46. $\mathfrak{r}_{m2}(A) = \min\{|X| : 0 \notin X$ *and for every* m*-partition* P *there is an* $a \in X$ *such that* $a \leq b\}$.

Proof. $\leq$: let $|X|$ be minimum such that $0 \notin X$ and X cannot be $(m, 2)$-reaped. Then for any m-partition P there is an $a \in X$ such that $|\{b \in P : a \cdot b \neq 0\}| < 2$. Since $\sum P = 1$, there is a $b \in P$ such that $a \cdot b \neq 0$. This b is unique, so $a \leq b$.

$\geq$: let $|X|$ be minimum such that $0 \notin X$ and for every m-partition P there is an $a \in X$ such that $a \leq b\}$. Then for every m-partition P there is an $a \in X$ such that $|\{b \in P : a \cdot b \neq 0\}| < 2$. $\square$

Now we consider trees which consist of finite collections of finite sequences of natural numbers, closed under initial segments. A tree is *k-branching* iff each of its nonmaximal nodes has at least k immediate successors. For any tree T, $\mu(T)$ is the collection of maximal nodes of T.

For integers $k, l, m, n \geq 2$, $P(k, l, m, n)$ abbreviates the statement that for every k-branching tree T and every $c : \mu(T) \to l$, there is a subtree S of T such that S is m-branching, $\mu(S) \subseteq \mu(T)$, and $|c[\mu(S)]| \leq n$.

Proposition 3.47. $P(k, \lfloor \frac{k-1}{m-1} \rfloor, m, 1)$ *holds.*

Proof. Suppose that T is a k-branching tree and $c : \mu(T) \to \lfloor \frac{k-1}{m-1} \rfloor$. We may assume that T has more than one element. Let s be the maximum height of any element of T, and let a be an element of height s. Let b be the immediate predecessor of a. Then b has at least k immediate successors. There is an $i < \lfloor \frac{k-1}{m-1} \rfloor$ such that there is a set M of immediate successors of b with $\forall d \in M[c(d) = i]$ and $|M| = m$. Otherwise, $k \leq \lfloor \frac{k-1}{m-1} \rfloor (m-1) \leq k-1$, contradiction. Let T' be b together with m of its immediate successors d for which $c(d) = i$. This is as desired. $\square$

Theorem 3.48. $P(k, l, m, n)$ *holds iff* $\lceil \frac{l}{n} \rceil < \lceil \frac{k}{m-1} \rceil$.

Proof. $\Leftarrow$: Let $a = \lceil \frac{l}{n} \rceil$ and $b = \lceil \frac{k}{m-1} \rceil$ and assume that $a < b$; we show that $P(k, l, m, n)$ holds. Since $l \leq an$, there is a partition $\langle s_i : i < a' \rangle$ of l into sets s_i each of size $\leq n$, with $a' \leq a$. Now let T be a k-branching tree, and suppose that $c : \mu(T) \to l$. Define $c' : \mu(T) \to a'$ by $c'(\sigma) = i$ iff $c(\sigma) = i$. Now $b - 1 < \frac{k}{m-1}$, so $b(m-1) - (m-1) < k$, hence $(b-1)(m-1) \leq k - 1$ and so $a \leq b - 1 \leq \frac{k-1}{m-1}$. By Proposition 3.47, T has an m-branching subtree T' with $\delta(T') \subseteq \delta(T)$ and $c' \restriction \delta(T')$ has a constant value. Then $c \restriction \delta(T')$ has range of size at most n, as desired.

$\Rightarrow$: We show that $\forall n, k, m, l \left[\lceil \frac{l}{n} \rceil \geq \lceil \frac{k}{m-1} \rceil \text{ implies } \neg P(k, l, m, n) \right]$ by induction on n.

$n = 1$: Assume that $l \geq \lceil \frac{k}{m-1} \rceil$. Thus $l(m-1) \geq k$. Let $f : k \to l \times (m-1)$ be one-one. Let T be the tree with root r and k immediate successors $t_0, \ldots, t_{k-1}$. Define $c(t_i) = j$ where $f(i) = (j, s)$ for some s. Then for all $j < l$, $|\{i < k : c(t_i) = j\}| \leq m - 1$. This shows $\neg P(k, l, m, 1)$.

Now assume the result for n and suppose that $\lceil \frac{l}{n+1} \rceil \geq \lceil \frac{k}{m-1} \rceil$. Let $a = \lceil \frac{l}{n+1} \rceil$ and $b = \lceil \frac{k}{m-1} \rceil$. Let $l' = an - (n-1)$.

(1) $a = \lceil \frac{l'}{n} \rceil$.

In fact, $l' = an - (n-1) \leq an$, so $\frac{l'}{n} \leq a$. Also, $an - n < l'$, so $a - 1 < \frac{l'}{n}$; so (1) holds.

(2) If $s < l'$, then $\lceil \frac{s}{n} \rceil < a$.

In fact, $s < an - (n-1)$, so $s \leq an - n$, hence $\frac{s}{n} \leq a - 1$.

By (1), $\lceil \frac{l'}{n} \rceil \geq \lceil \frac{k}{m-1} \rceil$. Hence by the inductive hypothesis we have $\neg P(k, l', m, n)$. Note that $l' \leq l$. For each $s \in [l]^{l'}$ there exist a κ-branching tree T_s and a $c_s : \mu(T_s) \to s$ such that for every m-branching subtree S of T_s such that $\mu(S) \subseteq \mu(T_s)$, $|\text{rng}(c_s)| \geq n+1$. Note that if $k \leq m-1$, then obviously $\neg P(k, l', m, n)$. Hence we assume that $k > m-1$ and hence $a \geq b \geq 2$. Hence by (1), $l' > n$.

Now for each $\sigma \in {}^{an-2n+1}k$ we associate a one-one function $f_\sigma : \{1, \ldots, l'\} \to \{1, \ldots, l\}$ as follows. Let $f_\sigma \restriction \{1, \ldots, n\}$ be the identity. Now suppose that $f_\sigma \restriction \{1, \ldots, n+i\}$ has been defined and $n+i+1 \leq l'$. Let $t = \{1, \ldots, l\} \backslash f_\sigma[\{1, \ldots, n+i\}]$ and let $\pi : t \to \{1, \ldots, l-n-i\}$ be the order preserving bijection.

(3) $l \geq a(n+1) - n$.

For, $a - 1 < \frac{l}{n+1}$, so $a(n+1) - n - 1 < l$ and (3) follows.

(4) $k \leq a(m-1)$.

For, $\lceil \frac{k}{m-1} \rceil = b \leq a$, so $k \leq a(m-1)$.

(5) If $j < k$ and $i \leq an - 2n$, then $\lfloor \frac{j}{m-1} \rfloor < l - n - i$.

For, let $s = \lfloor \frac{k-1}{m-1} \rfloor$. Then $s \leq \frac{k-1}{m-1} < \frac{k}{m-1} \leq b \leq a$, so $\lfloor \frac{j}{m-1} \rfloor \leq s < a \leq l + n - an$ (by (3)) $= l - n - (an - 2n) \leq l - n - i$.

Now $\sigma(i) < k$ and $i \leq l' - n - 1 = an - n + 1 - n - 1 = an - 2n$. Hence by (5), $\lfloor \frac{\sigma(i)}{m-1} \rfloor < l - n - i$. We define $f_\sigma(n+i+1) = \pi^{-1}(\lfloor \frac{\sigma(i)}{m-1} + 1 \rfloor)$. Now if $\tau \in {}^{i}k$, let σ be a maximal node extending τ, and set $r_\tau = f_\sigma[\{1, \ldots, n+i\}]$.

(6) $\lfloor \frac{u}{m-1} \rfloor < \lfloor \frac{u+m-1}{m-1} \rfloor$.

In fact, let $\frac{u}{m-1} = \lfloor \frac{u}{m-1} \rfloor + f$ where $0 \leq f < 1$. Then $\frac{u+m-1}{m-1} = \frac{u}{m-1} + 1 = \lfloor \frac{u}{m-1} \rfloor + f + 1$ and so $\lfloor \frac{u+m-1}{m-1} = \lfloor \frac{u}{m-1} \rfloor + 1$, giving (6).

(7) Define $u \equiv v$ iff $\lfloor \frac{u}{m-1} \rfloor = \lfloor \frac{v}{m-1} \rfloor$. Then each $\equiv$ class has at most $m-1$ members.

For, the equivalence classes are clearly convex, so (7) follows from (6).

(8) If $\tau, \tau' \in {}^{i+1}k$, $\tau \restriction i = \tau' \restriction i$, and $r_\tau \neq r_{\tau'}$, then $r_\tau \cap r_{\tau'} = r_{\tau \restriction i}$.

For, say $\tau \subseteq \sigma \in {}^{an-2n+1}k$ and $r_\tau = f_\sigma[\{1, \ldots, n+i+1\}]$ and $\tau' \subseteq \sigma' \in {}^{an-2n+1}k$ and $r_{\tau'} = f_{\sigma'}[\{1, \ldots, n+i+1\}]$. Now

$$f_\sigma[\{1, \ldots, n+i+1\}] = \{1, \ldots, n\} \cup \left\{ \pi^{-1} \left(\left\lfloor \frac{\sigma(j)}{m-1} \right\rfloor + 1 \right) : 0 \leq j \leq i \right\}$$

$$= \{1,\ldots,n\} \cup \left\{\pi^{-1}\left(\left\lfloor \frac{\sigma(j)}{m-1}\right\rfloor + 1\right) : 0 \le j < i\right\}$$
$$\cup \left\{\left\lfloor \frac{\sigma(i)}{m-1}\right\rfloor + 1\right\}$$

Similarly,

$$f_{\sigma'}[\{1,\ldots,n+i+1\}] = \{1,\ldots,n\} \cup \left\{\pi^{-1}\left(\left\lfloor \frac{\sigma'(j)}{m-1}\right\rfloor + 1\right) : 0 \le j < i\right\}$$
$$\cup \left\{\left\lfloor \frac{\sigma'(i)}{m-1}\right\rfloor + 1\right\}$$

Hence

$$r_\tau \cap r_{\tau'} = \{1,\ldots,n\} \cup \left\{\pi^{-1}\left(\left\lfloor \frac{\sigma'(j)}{m-1}\right\rfloor + 1\right) : 0 \le j < i\right\} = r_{\tau\restriction i}.$$

The following is from Lagrange 1967.

Suppose that A is a BA, $a \in {}^I A$, and $\sum_{i\in I} a_i$ exists. We say that *a can be disjointed* iff there is a $b \in {}^I A$ such that b is disjointed, $\forall i \in I[b_i \le a_i]$, and $\sum_{i\in I} b_i = \sum_{i\in I} a_i$.

Let X be the set of all functions f such that $\mathrm{dmn}(f)$ is a successor ordinal less than ω_1 and $\mathrm{rng}(f) \subseteq \omega_2$. For each $g \in X$ let $a_g = \{f \in X : g \subseteq f\}$, and let A be the subalgebra of $\mathscr{P}(X)$ generated by $\{a_g : g \in X\}$.

Proposition 3.49. *For each $\alpha < \omega_1$ let h_α be the function with domain $\alpha + 1$ and range $\{0\}$. Then $\sum_{\alpha<\omega_1} -a_{h_\alpha} = 1$.*

Proposition 3.50. *A does not have a maximal disjoint subset of size ω_1.*

Corollary 3.51. *$\langle -a_{h_\alpha} : \alpha < \omega_1\rangle$ cannot be disjointed.*

The following is from Losada, Todorcevic 2000.

Proposition 3.52. *Every BA without an uncountable set of pairwise incomparable elements is isomorphic to a subalgebra of $\mathscr{P}(\omega)$.*

Proposition 3.53. (MA(ω_1)) *If A is a BA with an uncountable chain, then A has an uncountable antichain.*

The following is from Shelah 1984.

Theorem 3.54. (4.4) *It is relatively consistent that $2^\omega = 2^{\omega_1} = \omega_2 + \mathfrak{b} = \mathfrak{d} > \mathfrak{s}$.*

Theorem 3.55. (5.2) *It is relatively consistent that $2^\omega = 2^{\omega_1} = \omega_2 + \mathfrak{h} = \omega_1 + \mathfrak{b} = \mathfrak{s} = \omega_1$.*

The following is from Shelah, Spasojevic 2002.

Theorem 3.56. (3.7) *Let M be a ctm of GCH and $\kappa < \lambda \le \mu \le \theta$ such that κ, λ, μ are regular and $\lambda \le \mathrm{cf}(\theta)$. Then there is a cardinal preserving extension $M[G]$ such that*

$$M[G] \models \mathfrak{t}_\kappa = \lambda \text{ and } \mathfrak{b}_\kappa = \mu \text{ and } 2^\kappa = \theta.$$

The following is from Shelah, Spinas 1998.

Let $\mathfrak{h}(\lambda)$ be the least cardinal κ such that ${}^\lambda((\mathscr{P}(\omega)/fin)\backslash\{0\})$ is not κ-distributive.

Theorem 3.57. *For each $n \in \omega\backslash 1$ it is relatively consistent that $\mathfrak{h}(n) = \omega_2$ and $\mathfrak{h}(n+1) = \omega_1$.*

The following is from Spasojevic 1996.

For a poset P, $\Gamma(P)$ is the statement that for every increasing $a \in {}^{\omega_1}P$ there is a decreasing $b \in {}^{\omega_1}P$ such that $\forall\alpha, \beta < \omega_1[a_\alpha, b_\beta]$ and there is no $c \in P$ such that $\forall\alpha < \omega_1[a_\alpha < c < b_\alpha]$.

Theorem 3.58. *The following are equivalent:*

(i) $\mathfrak{t} > \omega_1$.

(ii) $\Gamma(\mathscr{P}(\omega), \subset^*)$.

(iii) $\Gamma({}^\omega\omega, \le^*)$.

The following is from Steprans 2001.

Define $\mathfrak{a}'(A)$ to be the least size of an uncountable partition of A.

Theorem 3.59. *In the iterated Lavel model, $\mathfrak{a}'(\mathscr{P}(\omega)/fin) = \omega_2$ and $\mathfrak{a}'(nwd) = \omega_1$, where nwd is the ideal of nowhere dense subsets of $\mathbb{Q}$.*

Theorem 3.60. *If I is any ideal on ω, then $\mathfrak{b} \le \mathfrak{a}'(Fin \times I)$.*

The following is from Zapletal 1997.

Proposition 3.61. *Let κ be uncountable and regular. Then κ is strongly inaccessible iff $\mathfrak{s}(\kappa) \ge \kappa$.*

Proposition 3.62. *Let κ be uncountable and regular. Then κ is weakly compact iff $\mathfrak{s}(\kappa) > \kappa$.*

Proposition 3.63. *It is relatively consistent that there is a regular cardinal $\kappa > \omega$ such that $\mathfrak{s}(\kappa) > \kappa^+$.*

Proposition 3.64. $\mathfrak{s}(\aleph_\omega) \le maxpcf\{\aleph_n : n \in \omega\}$.

The following is from Zapletal 1997a.

Functions $f, g : \omega_1 \to \omega$ are *strongly almost disjoint* iff $\{\alpha < \omega_1 : f(\alpha) \neq g(\alpha)\}$ is finite.

Theorem 3.65. *In M assum GCH and κ is a cardinal. Then there is a generic extension $M[G]$ preserving cardinals in which there is a strongly disjoint family of size κ.*

4. Cohen algebras

The following is from Balcar, Jech, Zapletal 1997.

A *Cohen algebra* is the completion of $\mathrm{Fr}(\kappa)$ for some κ. A BA A *has uniform density* iff $\pi(A \restriction a) = \pi(A)$ for every nonzero $a \in A$. A BA A is *semi-Cohen* iff A has uniform density and $[A]^\omega$ has a closed unbounded set of countable regular subalgebras of A.

Theorem 4.1. (Theorem 3.2) *Let A be a BA with uncountable uniform density. Then A is semi-Cohen iff $\{B : B \leq_{\mathrm{reg}} A\}$ contains a closed unbounded set C such that $\forall M, N \in C[\langle M \cup N\rangle \in C]$.*

Example 4.2. (Theorem 5.2) *There is a semi-Cohen algebra of uniform density ω_2 which cannot be embedded as a regular subalgebra of a Cohen algebra.*

Example 4.3. (Theorem 5.11) *There is an increasing chain $\langle A_n : n \in \omega\rangle$ of Cohen algebras, with $\forall n[A_n \leq_{\mathrm{reg}} A_{n+1}]$, such that $\bigcup_{n\in\omega} A_n$ is not a Cohen algebra.*

Example 4.4. (Theorem 5.1; Koppelberg, Shelah) it For every $\kappa \geq \omega_2$ the algebra $\overline{\mathrm{Fr}(\kappa)}$ has a complete subalgebra of uniform density κ which is not Cohen.

The following is from Koppelberg 1993.

If $A \leq B$, then $\pi(B/A)$ is the least $|X|$ such that $X \subseteq B$ and $A \cup X$ generates a dense subalgebra of B.

$\mathbb{S}$ is a *Cohen skeleton* for A iff the following conditions hold:

(1) The elements of $\mathbb{S}$ are regular subalgebras of A.
(2) There is an $S \in \mathbb{S}$ such that $\pi(S) \leq \omega$.
(3) $\forall S \in \mathbb{S}\forall X \in [A]^{\leq\omega}\exists S' \in \mathbb{S}[S \cup X \subseteq S'$ and $\pi(S'/S) \leq \omega]$.
(4) For every nonempty chain $\mathbb{C}$ in $\mathbb{S}$ there is some $S \in \mathbb{S}$ such that $\bigcup\mathbb{C}$ is dense in S.

Theorem 4.5. *For any BA A the following are equivalent:*

(i) A is a Cohen algebra.

(ii) A has a Cohen skeleton.

(iii) A is the union of a continuous chain $\langle A_\alpha : \alpha < \rho\rangle$ such that $\pi(A_0) \leq \omega$, A_α is regular in $A_{\alpha+1}$, and $\pi(A_{\alpha+1}/A_\alpha) \leq \omega$.

(iv) like (iii), but in addition $A_0 = 2$ and $A_{\alpha+1}$ is a simple extension of A_α.

(v) like (iv), but in addition A_α is dense in $A_{\alpha+1}$.

(vi) like (iv), but in addition A_α is relatively complete in $A_{\alpha+1}$.

(vii) A has a dense projective subalgebra.

The following is from Koppelberg, Shelah 1996.

Proposition 4.6. *For any $\kappa \geq \omega_2$ the BA $\overline{\mathrm{Fr}(\kappa)}$ has a complete regular subalgebra of π-weight κ which is not Cohen.*

5. Complete BAs

The following is proved in Argyros 1980.

Theorem 5.1. *For any complete BA A there exist finitely many cBAs $B_1, \ldots, B_m$ such that $A \cong B_1 \times \cdots \times B_m$ and for each i, $|B_i|^{<c'(B_i)} = |B_i|$.*

Arhangelski 1967 proves the following

Theorem 5.2. *If A is a cBA of size 2^ω, then* $\mathrm{Ult}(A)$ *is not homogeneous.*

The following summarizes results in Baker 2002.

Let $b_0, b_1, \ldots$ be pairwise disjoint nonzero elements of a complete BA A. A filter F on A is *nice over* $\{b_n : n \in \omega\}$ iff the following conditions hold:

(1) $\sum_{n\in\omega} b_n \in F$.

(2) $\forall n \in \omega[-b_n \in F]$.

(3) $\forall b \in F[\{n \in \omega : b \cdot b_n = 0\}$ is finite].

If F is any filter on a BA A, then K_F is the set of all ultrafilters G such that $F \subseteq G$.

A closed subset Y of a space X is 2^ω-*ok over* X iff whenever $U_1, U_2, \ldots$ are open supersets of Y ther are open supersets V_ζ for $\zeta < 2^\omega$ such that $\forall m \in \omega \forall \zeta_1 < \zeta_2 < \cdots < \zeta_m < 2^\omega[V_{\zeta_1} \cap V_{\zeta_2} \cap \ldots \cap V_{\zeta_m} \subseteq U_m]$.

Theorem 5.3. (Theorem 1.6) *Assume that A and B are complete BAs of size 2^ω. Let $b_0, b_1, \ldots$ be pairwise disjoint nonzero elements of B; let $\mathscr{Z} = \mathcal{S}(\sum_{n\in\omega} b_n) \setminus \bigcup_{n\in\omega} \mathcal{S}(b_n)$. Let F be a nice filter over $\{b_n : n \in \omega\}$. Then* $\mathrm{Ult}(A)$ *is homeomorphic to some $K \subseteq K_F$ such that K is 2^ω-ok in $\mathscr{Z}$.*

The following is from Balcar, Franek 87.

$\mathrm{Col}(\lambda, \kappa)$ is the completion of the poset of all functions f such that $\mathrm{dmn}(f) \in [\lambda]^{<\lambda}$ and $\mathrm{rng}(f) \subseteq \kappa$, ordered by $\supseteq$.

For I an ideal over κ we let $I^+ = \{X \subseteq \kappa : X \notin I\}$. If $S \in I^+$, then a set $P \backslash \mathscr{P}(S) \cap I^+$ is I-*disjoint* iff $\forall x, y \in P[x \neq y \to x \cap y \in I]$. P is an I-*partition of* S iff it is I-disjoint and maximal. $\langle P_\alpha : \alpha < \lambda\rangle$ is a *descending sequence of I-partitions of S* iff each P_α is an I-partition of S, and for all $\alpha, \beta < \kappa$, if $\alpha < \beta$ then P_β is a refinement of P_α. I is a *precipitous ideal* iff it is an ideal, and for every $S \in I^+$ and every descending sequence $\langle P_n : n \in \omega\rangle$ of I partitionns of S, there is an $x \in \prod_{n\in\omega} P_n$ such that $\bigcap_{n\in\omega} x_n \neq \emptyset$. I is *nowhere precipitous* iff for every $A \in I^+$, $\{x \cap A : x \in I\}$ is not precipitous.

Theorem 5.4. (Theorem 1) *Suppose that κ is uncountable and regular, and $2^\kappa = \kappa^+$. Let I be a κ-complete nowhere precipitous ideal over κ. Then $\overline{\mathscr{P}(\kappa)/I}$ is isomorphic to* $\mathrm{Col}(\omega, \kappa^+)$.

Theorem 5.5. (Theorem 3) *If κ is singular with* $\mathrm{cf}(\kappa) = \omega$, *$2^\kappa = \kappa^+$, and $2^\omega = \omega_1$, then for any precipitous ideal I on κ, $\overline{\mathscr{P}(\kappa)/I}$ is isomorphic to* $\mathrm{Col}(\omega_1, \kappa^+)$.

Theorem 5.6. (Theorem 3) *If κ is singular with* $\mathrm{cf}(\kappa) > \omega$, $2^\kappa = \kappa^+$, *and* $2^{\mathrm{cf}(\kappa)} = (\mathrm{cf}(\kappa))^+$, *then* $\overline{\mathscr{P}(\kappa)/[\kappa]^{<\kappa}}$ *is isomorphic to* $\mathrm{Col}(\omega, \kappa^+)$.

The following is from Balcar, Štěpánek 1977.

B is a *maximal subalgebra* of a complete BA A iff $\forall c \in A \exists a \leq c[B \restriction a = A \restriction a]$.

Theorem 5.7. (Theorem 5) *A complete BA A is rigid iff it does not have a proper maximal subalgebra.*

The following is from Balcar, Vopěnka 1972.

Theorem 5.8. *If κ is uncountable and regular and* $2^\kappa = \kappa^+$, *then* $\overline{\mathscr{P}(\kappa)/[\kappa]^{<\kappa}}$ *and* $\mathrm{Col}(\kappa^+, \omega)$ *are isomorphic.*

The following is from Baldwin 2002.

If A is a BA and I is an ideal in A, then (A, I) has the *hull property* iff $\forall X \subseteq A \exists Y \in A[X \subseteq Y$ and $\forall Z \in A[X \subseteq Z \to Y \backslash Z \in I]]$.

Theorem 5.9. (Theorem 1) $(\mathscr{P}(\mathbb{R}), [\mathbb{R}]^{\leq\omega})$ *has the hull property, but* $\mathscr{P}(\mathbb{R})/I$ *is not complete.*

Theorem 5.10. (Theorem 2) *If I is an ideal in A, A/I is atomless, and $\kappa \geq |A|$, then there exist a BA B and an ideal J in B such that (B, J) does not have the hull property, $|B| = \kappa$, and $A/I \cong B/J$.*

Corollary 5.11. (Corollary 3) *There exist a BA A and an ideal I of A such that A/I is complete, but (A, I) does not have the hull property.*

Theorem 5.12. (Theorem 5) *There exist a σ-algebra A and a σ-ideal I of A such that A/I is complete but (A, I) does not satisfy the hull property.*

Let I be an ideal in a BA A. A subset S of $A \backslash I$ is *predense* in $A \backslash I$ iff $\forall a \in A \backslash I \exists b \in S[a \cdot b \in A \backslash I]$. (A, I) *has the density property* iff for every predense $S \subseteq A \backslash I$ we have $A \backslash \bigcup S \subseteq I$.

Theorem 5.13. (Theorem 7) *If A/I is a complete BA and (A, I) has the density property, then (A, I) has the hull property.*

The following is from Baumgartner, Erdös, Higgs 1984.

A *cross-cut* in a poset P is a maximal antichain C in P such that for all $x, y \in P$, if $x \leq y$, $\exists z \in C[x \leq z]$, and $\exists z \in C[z \leq y]$, then $\exists z \in C[x \leq z \leq y]$.

Proposition 5.14. *If E is infinite, and k is a positive integer, then $[E]^k$ is a cross-cut in $\mathscr{P}(E)$.*

Theorem 5.15. (MA, Theorem 3) *There is a cross-cut of $\mathscr{P}(\omega_1)$ consisting of uncountable sets whose complements are also incountable.*

Theorem 5.16. (MA, Theorem 4) *There is a cross-cut of $\mathscr{P}(\omega_2)$ consisting of countable sets.*

The following is from Beazer 1972.

An *inverse system* of BAs is a triple (I, A, f) such that I is an upwards directed poset, $A = \langle A_i : i \in I\rangle$ is a system of BAs, and $f = \langle f_{ij} : i, j \in I,\ i \le j\rangle$ is a system of homomorphisms $f_{ij} : A_j \to A_i$ such that f_{ii} is the identity on A_i and if $i \le j \le k$ then $f_{ik} = f_{ij} \circ f_{jk}$.

The *limit* of an inverse system (I, A, f) is the subset of $\prod_{i\in I} A_i$ consisting of all $x \in \prod_{i\in I} A_i$ such that $\forall i, j \in I[i \le j \to f_{ij}(x(j)) = x(i)$.

Theorem 5.17. *Any complete BA is isomorphic to an inverse limit of BAs such that no f_{ij}, $i \ne j$, is an isomorphism.*

The following is from Blaszczyk, Shelah 2001.

A filter D on ω is *nowhere dense* iff for every function $f : \omega \to {}^\omega 2$ there is an $A \in D$ such that $f[A]$ is nowhere dense in ${}^\omega 2$.

Theorem 5.18. (Theorem 1) *There exists an atomless, complete, σ-centered Boolean algebra without a countable atomless regular subalgebra iff there is a nowhere dense ultrafilter.*

The following is from Ciesielski, Galvin 1987.

If $F \subseteq \mathscr{P}(X)$ is closed under intersections, we define $[F]_\alpha$ for $\alpha < \omega_1$ by

$$[F]_\alpha = \begin{cases} F & \text{if } \alpha = 0;\\ \text{all countable unions of members of } \bigcup_{\beta<\alpha}[F]_\beta & \text{if } \alpha \ne 0 \text{ and } \alpha \text{ is odd;}\\ \text{all countable intersections of members of } \bigcup_{\beta<\alpha}[F]_\beta & \text{if } \alpha \ne 0 \text{ and } \alpha \text{ is even.}\end{cases}$$

If $n \le m < \omega$ and $i_0 < \cdots < i_{n-1} < m$ we let

$$C^m_{\langle i_0,\ldots,i_{n-1}\rangle}(X) = \{\{x \in {}^m X : \langle x_{i_0}, \ldots, x_{i_{n-1}}\rangle \in S\} : S \subseteq {}^n X\}.$$

Let

$$C^m_n = \bigcup\{C^m_{\langle i_0,\ldots,i_{n-1}\rangle}(X) : i_0 < \cdots < i_{n-1} < m\}.$$

Then the n-dimensional cylinder statement is the assertion

$$P_n(X), \text{ saying that } \mathscr{P}({}^{n+1}X) = [C^{n+1}_n(X)].$$

Theorem 5.19. (Corollary 1) *If $1 \le n < \omega$ and $P_n(\kappa)$, then $P_{n+1}(\kappa^+)$.*

Theorem 5.20. (Theorem 2) *If $1 \le n < \omega$ and $P_n(\kappa)$ holds, then $\kappa \le \beth_n$.*

Theorem 5.21. (Corollary 3) (GCH) *For $1 \le n < \omega$, $P_n(\kappa)$ holds iff $\kappa \le \omega_n$.*

The following is from Day 1965.

Given a BA A, a *free complete extension* of A is a complete BA B which extends A such that A completely generates B and for any homomorphism f of A into a complete BA C there is an extension $g : B \to C$ of f to a complete homomorphism of B into C.

Theorem 5.22. *If A is an infinite free BA, then A does not have a free complete extension.*

Proof. Suppose to the contrary that B is a free complete extension of A. Say A is freely generated by X. We claim that $(\mathrm{id} \restriction X, B)$ is a free complete BA, contradicting the Gaifman, Hales theorem. For, let $f : X \to C$ be any mapping, with C a complete BA. Since A is freely generated by X, there is a homomorphism $g : A \to C$ which extends f. By assumption, there is a complete homomorphism $h : B \to C$ which extends g. So h extends f. Since A competely generates B, h is unique. $\square$

Theorem 5.23. *If B is a free complete extension of A and A' is a subalgebra of A, then A' has a free complete extension.*

Proof. Let B' be the complete subalgebra of B completely generated by A'. We claim that B' is a free complete extension of A'. Suppose that $f : A' \to C$ is a homomorphism of A' into a complete BA C. By Sikorski's extension theorem, let $f' : A \to C$ be a homomorphism extending f. Let $g : B \to C$ be an extension of f' to a complete homomorphism of B into C. Then clearly $g \restriction B'$ is a homomorphism from B' into C. It is complete, since if $X \subseteq B'$, then $(g \restriction B')(\sum^{B'} X) = g(\sum^{B} X) = \sum^{C} g[X] = \sum^{C}(g \restriction B')[X]$. $\square$

Corollary 5.24. *If A has a free complete extension, then A is superatomic.*

Proof. Suppose that A is not superatomic. Then A has an atomless subalgebra, and hence it has a denumerable atomless subalgebra A'. A' is free, and so A' does not have a free complete extension, by Theorem 5.22. Hence by Theorem 5.23, A does not have a free complete extension. $\square$

Theorem 5.25. *If A is superatomic, then A has a free complete extension.*

Proof. It suffices to show that $\mathscr{P}(\mathrm{Ult}(A))$ is a free complete extension of $\mathcal{S}[A]$. Suppose that $f : \mathcal{S}[A] \to C$ is a homomorphism with C complete. For $F \in \mathrm{Ult}(A)$ and F an isolated point of $(\mathrm{Ult}(A))^{(\alpha)}$, let $b_F \in A$ be such that $\mathcal{S}(b_F) \cap (\mathrm{Ult}(A))^{(\alpha)} = \{F\}$. For $F \in \mathrm{Ult}(A)$ we define $g(F) \in C$ by induction on the rank of F; recall the definition of rank given just before Proposition 12.29. We define

$$g(F) = f(\mathcal{S}(b_F)) \cdot \prod\{-g(G) : G \in \mathcal{S}(b_F), \mathrm{rank}(G) < \mathrm{rank}(F)\}.$$

Then for any $X \subseteq \mathrm{Ult}(A)$ let $g'(X) = \sum_{F \in X} g(F)$. Now we claim that for any β,

(1) For all $b \in A$, if $\mathcal{S}(b) \cap (\mathrm{Ult}(A))^{(\beta)} = \emptyset$, then $g'(\mathcal{S}(b)) = f(\mathcal{S}(b))$.

(2) For all $b \in A$, if $\mathcal{S}(b) \cap (\mathrm{Ult}(A))^{(\beta)}$ is finite and nonempty, then

$$f(\mathcal{S}(b)) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \mathrm{rank}(G) < \beta\} = g'(\mathcal{S}(b) \cap (\mathrm{Ult}(A))^{(\beta)}).$$

We prove (1) and (2) by induction on β. Assume that $b=0$. For (1), if $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(0)}=\emptyset$, then $\mathcal{S}(b)\cap\mathrm{Ult}(A)=\emptyset$, and so $b=0$ and $g'(\mathcal{S}(b))=g'(\emptyset)=0=f(\mathcal{S}(b)$. For (2), if $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(0)}=\{F\}$, then $\mathcal{S}(b)\cap\mathrm{Ult}(A)=\{F\}$, which is impossible. So (2) holds.

Now suppose inductively that $\beta>0$. For (1), suppose that $b\in A$ and $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta)}=\emptyset$.

Case 1. β is a limit ordinal. Thus $\mathcal{S}(b)\cap\bigcap_{\alpha<\beta}(\mathrm{Ult}(A))^{(\alpha)}=\emptyset$. Now each $(\mathrm{Ult}(A))^{(\alpha)}$ is closed and $\mathcal{S}(b)$ is clopen, so by compactness there is an $\alpha<\beta$ such that $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\alpha)}=\emptyset$. Hence by the inductive hypothesis, $g'(\mathcal{S}(b))=f(\mathcal{S}(b))$.

Case 2. β is not a limit ordinal.

Subcase 2.1. $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}=\emptyset$. Then by the inductive hypothesis, $g'(\mathcal{S}(b))=f(\mathcal{S}(b))$.

Subcase 2.2. $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}\neq\emptyset$. Now

$$\emptyset=\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta)}=\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}\backslash\mathrm{Is}((\mathrm{Ult}(A))^{(\beta-1)}),$$

so $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}\subseteq\mathrm{Is}((\mathrm{Ult}(A))^{(\beta-1)})$. Thus for each $F\in\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}$ there is a $c_F\in A$ such that $(\mathrm{Ult}(A))^{(\beta-1)}\cap\mathcal{S}(c_F)=\{F\}$. So

$$\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}\subseteq\bigcup\{\mathcal{S}(c_F):F\in\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}\}.$$

Since $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}$ is compact, it follows that $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)}$ is finite. By the case assumption it is nonempty. Hence by (2) in the inductive hypothesis, $f(\mathcal{S}(b))\cdot\prod\{-g(G):G\in\mathcal{S}(b),\mathrm{rank}(G)<\beta-1\}=g'(\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)})$. Hence

$$f(\mathcal{S}(b))\leq g'(\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta-1)})+\sum\{g(G):G\in\mathcal{S}(b),\mathrm{rank}(G)<\beta-1\}=g'(\mathcal{S}(b)).$$

Now suppose that $F\in\mathcal{S}(b)$. Since $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta)}=\emptyset$, it follows that $\gamma\stackrel{\text{def}}{=}\mathrm{rank}(F)<\beta$. Now F is an isolated point of $(\mathrm{Ult}(A))^{(\gamma)}$. Hence $\mathcal{S}(b_F)\cap(\mathrm{Ult}(A))^{(\gamma)}=\{F\}$. Hence also $\mathcal{S}(b\cdot b_F)\cap(\mathrm{Ult}(A))^{(\gamma)}=\{F\}$. So by the inductive hypothesis with (2), $g(F)\leq f(\mathcal{S}(b\cdot b_F))\leq f(\mathcal{S}(b))$. Since F is arbitrary, $g'(\mathcal{S}(b))\leq f(\mathcal{S}(b)$. Hence by the above, $g'(\mathcal{S}(b))=f(\mathcal{S}(b))$. This proves (1) for β.

Now for (2), suppose that $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta)}$ is finite and nonempty; say $\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta)}=\{F_0,\ldots,F_n\}$. Then

$$b=b\cdot b_{F_0}+b\cdot-b_{F_0}\cdot b_{F_1}+\cdots+b\cdot-b_{F_0}\cdot-b_{F_1}\cdot b_{F_2}+\cdots+b\cdot-b_{F_0}\cdot\ldots\cdot-b_{F_n}.$$

If $i<n$, then

$$\begin{aligned}&f(\mathcal{S}(b\cdot-b_{F_0}\cdot\ldots\cdot-b_{F_{i-1}}\cdot b_{F_i}))\cdot\prod\{-g(G):G\in\mathcal{S}(b),\ \mathrm{rank}(G)<\beta\}\\&\quad\leq f(\mathcal{S}(b_{F_i}))\cdot\prod\{-g(G):G\in\mathcal{S}(b),\ \mathrm{rank}(G)<\beta\}\\&\quad=g(F_i)\leq g'(\mathcal{S}(b)\cap(\mathrm{Ult}(A))^{(\beta)}).\end{aligned}$$

Now $\mathcal{S}(b\cdot-b_{F_0}\cdot\ldots\cdot-b_{F_n})\cap(\mathrm{Ult}(A))^{(\beta)}=\emptyset$, so by (1)

$$g'(\mathcal{S}(b\cdot-b_{F_0}\cdot\ldots\cdot-b_{F_n}))=f(\mathcal{S}(b\cdot-b_{F_0}\cdot\ldots\cdot-b_{F_n})).$$

Now if $G \in \mathcal{S}(b)$ and $\gamma \stackrel{\text{def}}{=} \text{rank}(G) \geq \beta$, then $G \in (\text{Ult}(A))^{(\gamma)} \subseteq (\text{Ult}(A))^{(\beta)}$. Hence each member of $\mathcal{S}(b \cdot -b_{F_0} \cdot \ldots - b_{F_n})$ has rank less than β. It follows that

$$g'(\mathcal{S}(b \cdot -b_{F_0} \cdot \ldots \cdot -b_{F_n})) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\} = 0.$$

Hence

$$f(\mathcal{S}(b \cdot -b_{F_0} \cdot \ldots \cdot -b_{F_n})) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\} = 0.$$

Now putting all this together, we get

$$f(\mathcal{S}(b)) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\} \leq g'(\mathcal{S}(b) \cap (\text{Ult}(A))^{(\beta)}).$$

For the other inequality, we first claim

(3) $\forall i \leq n \forall G \in \mathcal{S}(b_{F_i} \cdot -b)[\text{rank}(G) < \beta]$.

In fact, suppose that $i \leq n$, $G \in \mathcal{S}(b_{F_i} \cdot -b)$, and $\text{rank}(G) \geq \beta$. Then $G \in \mathcal{S}(b_{F_i}) \cap (\text{Ult}(A))^{(\beta)} = \{F_i\}$, so $G = F_i$ and $b \in G$, contradiction. So (3) holds.

Now by (3) we have $\mathcal{S}(b_{F_i} \cdot -b)) \cap (\text{Ult}(A))^{(\beta)} = \emptyset$, so by (1), $g'(\mathcal{S}(b_{F_i} \cdot -b))) = f(\mathcal{S}(b_{F_i} \cdot -b))$. Moreover,

$$g'(\mathcal{S}(b_{F_i} \cdot -b) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\} = 0$$

Hence for any $i \leq n$,

$$\begin{aligned} g(F_i) &= f(\mathcal{S}(b_{F_i})) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\} \\ &\leq f(\mathcal{S}(b)) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\}, \end{aligned}$$

It follows that

$$g'(\mathcal{S}(b) \cap (\text{Ult}(A))^{(\beta)}) \leq f(\mathcal{S}(b)) \cdot \prod\{-g(G) : G \in \mathcal{S}(b),\ \text{rank}(G) < \beta\}.$$

This finishes the proof of (1) and (2).

Next we claim

(4) For all distinct $F, G \in \text{Ult}(A)$ we have $g(F) \cap g(G) = \emptyset$.

In fact, choose e so that $e \in F$ and $-e \in G$. Choose b, c, β, γ so that $b \leq e$, $c \leq -a$, $\mathcal{S}(b) \cap (\text{Ult}(A))^{(\beta)} = \{F\}$, and $\mathcal{S}(c) \cap (\text{Ult}(A))^{(\gamma)} = \{G\}$. Then $f(\mathcal{S}(b)) \cdot f(\mathcal{S}(c)) = 0$, and

$$\begin{aligned} g'(\{F\}) &= g'(\mathcal{S}(b) \cap (\text{Ult}(A))^{(\beta)}) \leq f(\mathcal{S}(b)); \\ g'(\{G\}) &= g'(\mathcal{S}(c) \cap (\text{Ult}(A))^{(\gamma)}) \leq f(\mathcal{S}(c)). \end{aligned}$$

Now (4) follows.

Clearly g' preserves finite and infinite sums. If $X \subseteq \text{Ult}(A)$, then $g'(X) + g'(\text{Ult}(A)\backslash X) = 1$, and by (4), $g'(X) \cap g'(\text{Ult}(A)\backslash X) = 0$. So g' is a complete homomorphism of $\mathscr{P}(\text{Ult}(A))$ into C. Applying (1) with β large, we see that g' extends f. $\square$

The following is from Dobrinen 2002.

B satisfies the (η, κ)-*distributive law* iff for all I, J with $|I| \leq \eta$ and $|J| \leq \kappa$, and for all $b \in {}^{I\times J}B$,

$$\prod_{i\in I}\sum_{j\in J} b_{ij} = \sum_{f\in {}^I J}\prod_{i\in I} b_{if(i)}.$$

Let κ, η be cardinals, with η infinite, and let A be an η^+-complete BA. The game $\mathscr{G}_1^\eta(\kappa)$ goes as follows. There are η rounds. At the beginning, P1 chooses $b \in A^+$. At round $\alpha < \eta$, P1 chooses a partition W_α of α with $|W_\alpha| \leq \kappa$, and P2 chooses $b_\alpha \in W_\alpha$. The game gives a play

$$\langle b, W_0, b_0, W_1, b_1, \ldots, W_\alpha, b_\alpha, \ldots\rangle_{\alpha<\eta}.$$

P1 wins the play iff $\prod_{\alpha<\eta} b_\alpha = 0$.

Theorem 5.26. *If A is η^+-complete and P1 has a winning strategy for $\mathscr{G}_1^\eta(\kappa)$, then:*
(i) The $(\kappa^{<\eta}, \kappa)$-distributive law fails.
(ii) The $(\eta, \kappa^{<\eta})$-distributive law fails.

The following is from Dobrinen 2004.

The κ-*Cohen algebra* is the completion of $\text{Fr}(\kappa)$.

Theorem 5.27. (Galvin, Hajnal) *There is a family $\langle S_\alpha : \alpha < 2^\omega\rangle$ with the following properties*
(i) $\forall\alpha < 2^\omega[S_\alpha \subseteq \alpha]$.
(ii) $\forall\alpha < 2^\omega[[S_\alpha]^2 \subseteq \bigcup_{\gamma<2^\omega}\{\{\beta,\gamma\} : \beta \in S_\gamma\}]$.
(iii) $\forall\alpha < 2^\omega[ot(S_\alpha) \leq \omega$.
(iv) $\forall S \subseteq 2^\omega[[S]^2 \subseteq \bigcup_{\gamma<2^\omega}\{\{\beta,\gamma\} : \beta \in S_\gamma\}$ *and* $ot(S) \leq \omega \to \exists\alpha < 2^\omega[S = S_\alpha]]$.

Now we define the Galvin, Hajnal poset $\mathbb{P}_{GH}$:

$$\forall\alpha < 2^\omega[W_\alpha = \{f \in {}^{2^\omega}2 : f(\alpha = 1) \text{ and } \forall\beta \in S_\alpha[f(\beta) = 0]\};$$
$$\mathbb{P}_{GH} = \left\{\bigcap_{\alpha\in F} W_\alpha : F \in [2^\omega]^{<\omega} \text{ and } \bigcap_{\alpha\in F} W_\alpha \neq \emptyset\right\}.$$

Theorem 5.28. (Theorem 2.5) $\overline{\text{Fr}(\text{cf}(2^\omega))}$ *can be completely embedded in* $\text{RO}(\mathbb{P}_{GH})$.

Argyros constructed a rather complicated poset $\mathbb{P}_A$.

Theorem 5.29. (Theorem 3.5) $\overline{\text{Fr}(\omega)}$ *can be completely embedded in* $\mathbb{P}_A$.

Gaifman constructed a rather complicated poset $\mathbb{P}_G$.

Theorem 5.30. (Theorem 5.2) $\overline{\mathrm{Fr}(\omega)}$ *can be completely embedded in* $\mathbb{P}_G$.

The following is from van Douwen, van Mill 1980.

Lemma 5.31. *Suppose that B is countably complete, $g : B \to C$ and $f : C \to \mathscr{P}(\omega)$ are surjections. Then there is an isomorphic embedding $e : \mathscr{P}(\omega) \to C$ such that $f \circ e$ is the identity.*

Proof. For each $n \in \omega$ choose $a_n \in C$ such that $f(a_n) = \{n\}$. Then choose $b_n \in B$ so that $g(b_n)) = a_n$. Thus $f(g(b_n)) = \{n\}$ for all $n \in \omega$. Let $b'_n = b_n \cdot \prod_{m<n} -b_m$. Thus $f(g(b'_n)) = \{n\}$ for all $n \in \omega$, and the b'_n are pairwise disjoint. Let $c = \sum_{n\in\omega} b_n$. Define $e : \mathscr{P}(\omega) \to C$ by setting, for each $Y \subseteq \omega$,

$$e(Y) = \begin{cases} g(\sum_{n\in Y} b'_n) & \text{if } 0 \notin Y; \\ g(-c + \sum_{n\in Y} b'_n) & \text{if } 0 \in Y. \end{cases}$$

If $n \in Y$, then $\{n\} = f(g(b'_n)) \leq f(g(Y))$. If $n \notin Y$, then $\{n\} \cap \{m\} = \emptyset$ for all $m \in Y$, hence $f(g(b'_n)) \cdot f(g(b'_m)) = 0$ and so $f(g(b'_n)) \cdot f(g(Y)) = 0$. Hence $f \circ e$ is the identity.

For $Y, Z \subseteq \omega$ we have

$$\begin{aligned} e(Y \cup Z) &= \begin{cases} g(\sum_{n\in Y\cup Z} b'_n) & \text{if } 0 \notin Y \cup Z, \\ = g(-c + \sum_{n\in Y\cup Z} b'_n) & \text{if } 0 \notin Y \text{ and } 0 \in Z, \\ = g(-c + \sum_{n\in Y\cup Z} b'_n) & \text{if } 0 \in Y \text{ and } 0 \notin Z, \\ = g(-c + \sum_{n\in Y\cup Z} b'_n) & \text{if } 0 \in Y \text{ and } 0 \in Z, \end{cases} \\ &= \begin{cases} g(\sum_{n\in Y} b'_n) + g(\sum_{n\in Z} b'_n) & \text{if } 0 \notin Y \cup Z, \\ g(\sum_{n\in Y} b'_n) + g(-c + \sum_{n\in Z} b'_n) & \text{if } 0 \notin Y \text{ and } 0 \in Z, \\ g(-c + \sum_{n\in Y} b'_n) + g(\sum_{n\in Z} b'_n) & \text{if } 0 \in Y \text{ and } 0 \notin Z, \\ g(-c + \sum_{n\in Y} b'_n) + g(-c + \sum_{n\in Z} b'_n) & \text{if } 0 \in Y \text{ and } 0 \in Z, \end{cases} \\ &= e(Y) + e(Z). \end{aligned}$$

Clearly $e(\omega) = 1$, and $e(X) \cdot e(\omega \backslash X) = 0$. So e is a homomorphism. Clearly e is injective. □

The following is from Dow, Gubbi, Szymanski 1988.

For each $s \in {}^{<\omega}\omega$ let F_s be a nonprincipal ultrafilter on ω. Let

$$\mathscr{O} = \{V \subseteq {}^{<\omega}\omega : \forall s \in V[\{n : s^\frown\langle n\rangle \in V\} \in F_s]\}.$$

Proposition 5.32. *$\mathscr{O}$ is a topology on ${}^{<\omega}\omega$.*

Proof. Clearly $\emptyset, {}^{<\omega}\omega \in \mathscr{O}$. Suppose that $V_1, V_2 \in \mathscr{O}$. Take any $s \in V_1 \cap V_2$. Then $\{n : s^\frown\langle n\rangle \in V_1\} \in F_s$ and $\{n : s^\frown\langle n\rangle \in V_2\} \in F_s$. Hence

$$\{n : s^\frown\langle n\rangle \in V_1 \cap V_2 = \{n : s^\frown\langle n\rangle \in V_1\} \cap \{n : s^\frown\langle n\rangle \in V_2\} \in F_s.$$

Thus $V_1 \cap V_2 \in \mathscr{O}$.

Suppose that $\mathscr{A} \subseteq \mathscr{O}$. Take any $s \in \bigcup \mathscr{A}$. Say $s \in V \in \mathscr{A}$. Since $V \in \mathscr{O}$, we have $\{n : s^\frown\langle n\rangle \in V\} \in F_s$. This shows that $\bigcup \mathscr{A} \in \mathscr{O}$. $\square$

Theorem 5.33. *${}^{<\omega}\omega$ is extremally disconnected Hausdorff.*

Proof. For brevity let $seq = {}^{<\omega}\omega$.

Hausdorff: Suppose that $s, t \in {}^{\omega}\omega$ and $s \neq t$.

Case 1. $s \subset t$. Say $\mathrm{dmn}(s) = m$. Let

$$V_1 = \{s\} \cup \{u \in seq : s \subset u \text{ and } u(m) \neq t(m)\};$$
$$V_2 = \{u \in seq : t \subseteq u\}.$$

Clearly V_1 and V_2 are open, $s \in V_1$, $t \in V_2$, and $V_1 \cap V_2 = \emptyset$.

Case 2. $t \subset s$. Similarly.

Case 3. $\exists m \in \mathrm{dmn}(s) \cap \mathrm{dmn}(t)[s(m) \neq t(m)]$. Let $V_1 = \{u \in seq : s \restriction (m+1) \subseteq u\}$ and $V_2 = \{u \in seq : t \restriction (m+1) \subseteq u\}$.

Extremally disconnected: Let V be open and $s \in \overline{V}$; we want to show that $\{n : s^\frown\langle n\rangle \in V\} \in F_s$. ???

Proposition 5.34. *There are 2^{2^ω} pairwise non homeomorphic extremally disconnected compact Hausdorff spaces.*

The following is from Foreman 1983.

G_ω is a game of length ω, played on a complete BA A. I moves first, and I and II choose in turn $a_0 \geq a_1 \geq \cdots$. If at some stage a player cannot move, then I wins. If the game goes all the way, then II wins if there is a nonzero b such that $b \leq a_i$ for all i.

Theorem 5.35. *For any complete BA A, and any successor cardinal κ, assume that*

(i) II has a winning strategy in G_ω.

(ii) A has a dense subset of size κ.

(iii) A is (κ, ∞)-distributive.

Then A has a dense ω-closed subset.

The following is from Jech 1974.

A complete BA A is *simple* iff it is atomless but has no proper atomless complete subalgebra.

Proposition 5.36. *Every simple complete BA is rigid.*

Proof. Suppose that A is complete but is not rigid. Say f is a nontrivial automorphism of A. Then there is an $a \in A^+$ such that $a \cdot f(a) = 0$. Then $\{b + f(b) + c : b \leq a,\ c \cdot (a + f(a)) = 0\}$ is a proper atomless complete subalgebra of A. $\square$

Theorem 5.37. *If κ is weakly compact, then there is no simple complete BA of size κ.*

Theorem 5.38. (GCH) *If κ is singular, then there is no simple complete BA of size κ.*

Theorem 5.39. $(V = L)$ *If κ is uncountable, regular, and not weakly compact, then there is a simple complete BA of size κ.*

The following is from Jech, Shelah 2001.

Theorem 5.40. *Let κ be an uncountable regular cardinal. Then there is a simple complete BA with κ generators.*

The following is from Jipsen, Pinus, Rose 2001.

The Rudin, Keisler order on ultrafilters is extended to arbitrary complete BAs.

Theorem 5.41. *If there is a κ^+-complete ultrafilter on A, then $\forall\lambda \leq \kappa[RK(\mathscr{P}(\lambda))$ can be isomorphically embedded in $RK({}^\kappa A)$.*

The following is from Koppelberg 1980.

If A is a complete BA and $X \subseteq A$, then $[X]$ is the complete subalgebra of A generated by X. If A is a complete BA, then $\mathrm{cf}_c(A)$ is the least κ such that there is an increasing sequence of complete subalgebras of A with union A, if such a sequence exists; ∞ if no such sequence exists. Also, we define

$$\tau(A) = \min\{|X| : [X] = A\}.$$

Theorem 5.42. *If A is a complete BA and $\mathrm{cf}(A) < \infty$, then*
(i) $\mathrm{cf}(A)$ *is regular.*
(ii) $\omega_1 \leq \mathrm{cf}(A)$.
(iii) $\mathrm{cf}(A) \leq \tau(A)$.

For A a complete BA, let

$$t(A) = \sup\{|D|^+ : D \text{ is a disjointed subset of } A\}.$$

Proposition 5.43. *If A is a complete BA and $t(A) \leq \mathrm{cf}(\tau(A))$, then $\mathrm{cf}(A) \leq \mathrm{cf}(\tau(A))$.*

The following is from Jech 1972.

Theorem 5.44. *If A is complete and atomless, and if A has a proper complete atomless subalgebra, then A has a complete atomless subalgebra which is not rigid.*

Proof. Let A be complete and atomless, and let B be a proper complete atomless subalgebra of A. Fix $u \in A \backslash B$. Let $B[u]$ be the set of all elements of A of the form $a \cdot u + b \cdot -u$ with $a, b \in B$. Clearly $B[u]$ is a subalgebra of A. To see that it is a complete subalgebra, suppose that $\langle x_i : i \in I\rangle$ is a system of elements of $B[u]$. Say $x_i = a_i \cdot u + b_i \cdot -u$ with $a_i, b_i \in B$. Then $(\sum^B_{i\in I} a_i) \cdot u + (\sum^B_{i\in I} b_i) \cdot -u$ is an element of $B[u]$, and

$$\left(\sum_{i\in I}^{B} a_i\right) \cdot u + \left(\sum_{i\in I}^{B} b_i\right) \cdot -u = \left(\sum_{i\in I}^{A} a_i\right) \cdot u + \left(\sum_{i\in I}^{A} b_i\right) \cdot -u = \sum_{i\in I}^{A} (a_i \cdot u + b_i \cdot -u) = \sum_{i\in I}^{A} x_i.$$

Thus since $\sum_{i\in I}^A x_i$ is in $B[u]$, it follows that $\sum_{i\in I}^{B[u]} x_i$ exists and equals $\sum_{i\in I}^A x_i$. So $B[u]$ is a complete subalgebra of A.

To show that $B[u]$ is atomless it suffices to take a nonzero element of the form $a\cdot u$ with $a\in B$ and find $c\in B[u]$ with $0\neq c<a\cdot u$. Now $0\neq a\cdot u\leq \prod^A\{b\in B : a\cdot u\leq b\}$, so it follows from B being a complete subalgebra of A that $\prod^B\{b\in B : a\cdot u\leq b\} = \prod^A\{b\in B : a\cdot u\leq b\}\neq 0$. Choose $c\in B$ such that $0\neq c<\prod^B\{b\in B : a\cdot u\leq b\}$. Now $a\in\{b\in B : a\cdot u\leq b\}$, so $c\leq a$. Hence $c\cdot u\leq a\cdot u$. If $c\cdot u=a\cdot u$, then $\prod^B\{b\in B : a\cdot u\leq b\}\leq c$, contradiction. Thus $c\cdot u<a\cdot u$. If $c\cdot u=0$, then $a\cdot u\leq u\leq -c$, and it follows that $c\leq \prod^B\{b\in B : a\cdot u\leq b\}\leq -c$ and hence $c=0$, contradiction. So $c\cdot u$ is the required nonzero element of $B[u]<a\cdot u$.

It remains only to show that $B[u]$ is not rigid. Let

$$c=-\left(\sum^B\{a\in B : a\leq u\}+\sum^B\{a\in B : a\leq -u\}\right).$$

(1) $c\neq 0$.

In fact, suppose that $u=0$. Then $\sum^B\{a\in B : a\leq u\}\leq u$, $\sum^B\{a\in B : a\leq -u\}\leq -u$, $\left(\sum^B\{a\in B : a\leq u\}\right)\cdot\left(\sum^B\{a\in B : a\leq -u\}\right)=0$ and $\sum^B\{a\in B : a\leq u\}+\sum^B\{a\in B : a\leq -u\}=1$, so $u=\sum^B\{a\in B : a\leq u\}\in B$, contradiction. So $(*)$ holds.

Let $c^+=c\cdot u$ and $c^-=c\cdot -u$.

(2) $c^+\neq 0\neq c^-$.

In fact, suppose that $c^+=0$. Then $c\leq -u$, so $c\leq -c$, hence $c=0$, contradicting (1). Similarly $c^-\neq 0$.

(3) If $x\in B[u]$ and $0\neq x\leq c^+$, then there is an $a\in B$ such that $x=a\cdot u$.

In fact, suppose that $x\in B[u]$ and $0\neq x\leq c^+$. Write $x=a\cdot u+b\cdot -u$ with $a,b\in B$. Since $x\leq c^+$, it follows that $b\cdot -u=0$, and (3) follows.

(4) If $a,a'\in B$ and $a\cdot u=a'\cdot u$, then $a\cdot c^-=a'\cdot c^-$.

In fact, suppose that $a,a'\in B$ and $a\cdot u=a'\cdot u$. Then $(a\triangle a')\cdot u=0$, so $(a\triangle a')\cdot c=0$, hence $a\cdot c=a'\cdot c$, and (4) follows.

Similarly to (3) and (4) we have

(5) If $x\in B[u]$ and $0\neq x\leq c^-$, then there is an $a\in B$ such that $x=a\cdot -u$.

(6) If $a,a'\in B$ and $a\cdot -u=a'\cdot -u$, then $a\cdot c^+=a'\cdot c^+$.

Now if $x\in B[u]$ and $0\neq x\leq c^+$, by (3) we choose $a\in B$ such that $x=a\cdot u$, and we define $\tilde{x}=a\cdot c^-$. This does not depend on a, by (4). Similarly, if $x\in B[u]$ and $0\neq x\leq c^-$, by (5) we choose $a\in B$ such that $x=a\cdot -u$, and we define $\tilde{x}=a\cdot c^+$. This does not depend on a, by (6).

(7) If $x\in B[u]$ and $0\neq x\leq c^+$, then $\tilde{\tilde{x}}=x$.

For, assume that $x \in B[u]$ and $0 \neq x \leq c^+$. By (3) choose $a \in B$ such that $x = a \cdot u$. Then $\tilde{x} = a \cdot c^- = a \cdot c \cdot -u$. Hence $\tilde{\tilde{x}} = a \cdot c \cdot u = a \cdot u = x$.

Similarly,

(8) If $x \in B[u]$ and $0 \neq x \leq c^-$, then $\tilde{\tilde{x}} = x$.

(9) If $x, y \in B[u]$ and $0 \neq x, y \leq c^-$, then $(x + y)\tilde{} = \tilde{x} + \tilde{y}$.

For, suppose that $x, y \in B[u]$ and $0 \neq x, y \leq c^-$. Choose $a, b \in B$ such that $x = a \cdot u$ and $y = b \cdot u$. Then $x + y = (a + b) \cdot u$, and so $(x + y)\tilde{} = (a + b) \cdot c^+ = a \cdot c^+ b \cdot c^+ = \tilde{x} + \tilde{y}$.

Similarly,

(10) If $x, y \in B[u]$ and $0 \neq x, y \leq c^+$, then $(x + y)\tilde{} = \tilde{x} + \tilde{y}$.

(11) If $x \in B[u]$ and $0 \neq x \leq c^+$, then $(c^+ \cdot -x)\tilde{} = c^- \cdot -\tilde{x}$.

For, suppose that $x \in B[u]$ and $0 \neq x \leq c^+$. Choose $a \in B$ such that $x = a \cdot u$. Then $c^+ \cdot -x = c \cdot u \cdot (-a + -u) = c \cdot -a \cdot u$, and so $(c^+ \cdot -x)\tilde{} = c \cdot -a \cdot -u = c \cdot -u \cdot -(a \cdot -u) = c^- \cdot -\tilde{x}$.

Similarly,

(12) If $x \in B[u]$ and $0 \neq x \leq c^-$, then $(c^- \cdot -x)\tilde{} = c^= \cdot -\tilde{x}$.

Now by the above, $\tilde{}$ is an isomorphism from $B[u] \restriction c^+$ onto $B[u] \restriction c^-$. So this isomorphism induces a nontrivial automorphism of $B[u]$ which is the identity on $B[u] \restriction -c$. $\square$

The following is from Kurilic, Sobot 2008.

The game $\mathscr{G}_{ls}(\kappa)$ runs as follows. It is played on a complete BA A. White starts by choosing a nonzero $p \in A$. At the α-th move ($\alpha < \kappa$), white chooses $p_\alpha < p$ and black chooses $i(\alpha) \in 2$. White wins the play $\langle p, p_0, i(0), \ldots\rangle$ iff $\prod_{\beta\in\kappa} \sum_{\alpha\geq\beta} p_\alpha^{i(\alpha)} = 0$. The game $\mathscr{G}_{c\&c}(\kappa)$ has the same rules, but white wins if $\prod_{\alpha\in\kappa} p_\alpha^{i(\alpha)} = 0$.

Proposition 5.45. *Let $\langle p, p_0, i(0), \ldots\rangle$ be a play in the game $\mathscr{G}_{ls}(\kappa)$.*

(i) If white has a winning strategy for $\mathscr{G}_{ls}(\kappa)$, then white has a winning strategy for $\mathscr{G}_{c\&c}(\kappa)$.

(ii) If black has a winning strategy for $\mathscr{G}_{c\&c}(\kappa)$, then black has a winning strategy for $\mathscr{G}_{ls}(\kappa)$.

Proposition 5.46. *If A is a complete BA with a λ-closed dense subset, then for each $\kappa < \lambda$ black has a winning strategy in the games $\mathscr{G}_{c\&c}(\kappa)$ and $\mathscr{G}_{ls}(\kappa)$.*

Proposition 5.47. *If A is a complete BA and $\kappa \geq \pi(A)$, then black has a winning strategy in the game $\mathscr{G}_{ls}(\kappa)$.*

The following is from Mansfield 1971.

If A is a complete BA, an *A-valued $\mathscr{L}$-structure* is a pair (M,f) such that M is a nonempty set and for each m-ary relation symbol $\mathbf{R}$, $f_{\mathbf{R}} : {}^mM \to A$. Satisfaction is defined as follows. For each formula φ and each $a \in {}^\omega M$ we define $||\varphi, a||$ by recursion.

$$||v_i = v_j, a|| = \begin{cases} 1 & \text{if } a_i = a_j, \\ 0 & \text{otherwise;} \end{cases}$$

$$||\mathbf{R}v_{i(0)} \dots v_{i(m-1)}, a|| = f_{\mathbf{R}}(a_{i(0)}, \dots, a_{i(m-1)});$$

$$||\neg\varphi, a|| = -||\varphi, a||;$$

$$||\varphi \vee \psi, a|| = ||\varphi, a|| + ||\psi, a||;$$

$$||\exists v_i \varphi, a|| = \sum\{||\varphi, a^i_x|| : x \in M\}.$$

Now if M is an ordinary $\mathscr{L}$-structure, we define an A-structure $M^{(A)}$ as follows. The universe of $M^{(A)}$ is

$$\left\{ f \in {}^MA : \forall a, b \in M[a \neq b \to f(a) \cdot f(b) = 0] \text{ and } \sum_{a \in M} f(a) = 1 \right\}.$$

For any m-ary relation symbol $\mathbf{R}$,

$$\mathbf{R}^{M^{(A)}} = \left\langle \sum_{(a_0,\dots,a_{m-1}) \in \mathbf{R}^M} \bigwedge_{i<m} f_i(m_i) : (f_0, \dots, f_{m-1}) \in {}^m(M^{(A)}) \right\rangle.$$

Theorem 5.48. *For any complete BA A, any $\mathscr{L}$-structure M, any $f \in {}^m(M^{(A)})$, and any formula $\varphi(v_0, \dots, v_{m-1})$,*

$$||\varphi, f|| = \bigvee_{M \models \varphi[a]} \bigwedge_{i<m} f_i(a_i).$$

The following is from Koppelberg 1981.

For any complete BA A, let $\tau(A)$ be the isomorphism type of A, and $T(A) = \{\tau(A \restriction a) : a \in A\}$. Let $\tau(A) \le \tau(B)$ iff A is isomorphic to $B \restriction b$ for some $b \in B$.

Theorem 5.49. *For any complete BA A, $T(A)$ is a distributive lattice with 0 and 1; it is a Stone algebra and a Heyting algebra.*

The following is from Monro 1974.

Theorem 5.50. *If A, B, C are complete BAs, A is a complete subalgebra of both B and C, and $A = B \cap C$, then there is a complete BA D such that B and C are complete subalgebras of D.*

The proof uses the correspondence between forcing and complete BAs.

The following is from Pierce 1961.

Let κ, λ, μ be cardinals, with κ, λ infinite and $\mu \ge 2$. $\Phi(\kappa, \lambda, \mu)$ is the set of finite functions $\subseteq \lambda \times \mu$ of size less than κ. $\Phi(\kappa, \lambda, \mu)$ is a dense subset of a complete BA $B_{\kappa\lambda\mu}$.

Proposition 5.51. *For $\kappa \le \lambda$, $\mathrm{c}'(B_{\kappa\lambda\mu}) = \sup\{\gamma^\delta : \delta < \alpha\}$.*

Under GCH the values of several cardinal functions on the algebras $B_{\kappa\lambda\mu}$ are determined.

6. Free algebras

The following results are from Argyros 1982.

Theorem 6.1. *Let λ be an uncountable regular cardinal, and let A be generated by a set X of size λ. Assume that for every filter F on A we have $|F \cap X| < \lambda$. Then A does not have an independent subset of size λ.*

Theorem 6.2. (GCH) *For every singular cardinal κ there is a BA A of size κ^+ such that A has the $(\mathrm{cf}(\kappa))^+$-chain condition and A does not have an independent subset of size κ^+.*

The following is from Arhangelski, Buzyakova 2009,

For any set X, $\mathrm{ord}(X)$ is the set of all linear orders on X.

Proposition 6.3. *Let X be any nonempty set. For each $L \in \mathrm{ord}(X)$ let*

$$B(L) = \bigcup_{F \in [X]^{<\omega}} \{M \in \mathrm{ord}(X) : M \cap (F \times F) = L \times (F \times F)\}$$

Then $\langle B(L) : L \in \mathrm{ord}(X)\rangle$ is a neighborhood system for a topology on $\mathrm{ord}(X)$

Proposition 6.4. (Theorem 4) $\mathrm{ord}(\omega)$ *is homeomorphic to* ${}^{\omega}2$.

Proposition 6.5. (Theorem 6) $\mathrm{ord}(\omega_1)$ *is homeomorphic to* ${}^{\omega_1}2$.

The following is from Banaschewski 2010.

Proposition 6.6. (Comparison principle) *In $A \oplus B$, if $a, a' \in A$, $b, b' \in B$, $a, b \neq 0$, and $a \cdot b \leq a' \cdot b'$, then $a \leq a'$ and $b \leq b'$.*

Proof. Assume the hypotheses. Then $a \cdot b \cdot (-a' + -b') = 0$, and the conclusion follows from Definition 11.3 in Koppelberg. □

Proposition 6.7. *Suppose that $f : C \to A$, $g : C \to B$, and $I = \langle\{f(a) \triangle g(a)\}\rangle^{\mathrm{id}}$, an ideal in $A \oplus B$. Let $h : A \oplus B \to (A \oplus B)/I$ be the natural map. Then $h \circ f = h \circ g$.*

Proof. $(h \circ f)(a) = h(f(a)) = [f(a)]_I = [g(a)]_I = h(g(a)) = (h \circ g)(a)$. □

Proposition 6.8. (LaGrange) *Let A, B, C be BAs with C a subalgebra of A and B. Let $J = \langle\{a \otimes (-a) : a \in A\}\rangle$. Suppose that $c \in A$, $b \in B$, and $c \otimes b \in J$. Then there is an $a \in A$ such that $c \otimes b \leq a \otimes (-a)$.*

Proof. Choose $n \in \omega$ and $a \in {}^nA$ such that $c \otimes b \leq \sum_{i<n}(a_i \otimes (-a_i))$. Let P be the set of atoms of the subalgebra of A generated by $\mathrm{rng}(a)$. Then for each $i < n$,

$$\begin{aligned} a_i \otimes (-a_i) &= \sum\{u \otimes v : u, v \in P,\ u \leq a_i,\ v \leq (-a_i)\} \\ &\leq \sum\{u \otimes (-u) : u \in P,\ u \leq a_i\}; \end{aligned}$$

the last $\leq$ is because $u \leq a_i$ implies that $-a_i \leq -u$. Then

$$c \otimes b \leq \sum\{u \otimes (-u) : u \in P,\ u \leq a_i,\ i < n\}.$$

Hence

$$\begin{aligned}
c \otimes b &\leq c \cdot \sum\{u \otimes (-u) : u \in P,\ u \leq a_i,\ i < n\} \\
&= \sum\{(c \cdot u) \otimes (-u) : u \in P,\ u \leq a_i,\ i < n\} \\
&\leq \sum\{u \otimes (-u) : u \in P,\ u \leq a_i,\ i < n,\ c \cdot u \neq 0\} \\
&= b \cdot \sum\{u \otimes (-u) : u \in P,\ u \leq a_i,\ i < n,\ c \cdot u \neq 0\} \\
&= \sum\{u \otimes b \cdot (-u) : u \in P,\ u \leq a_i,\ i < n,\ c \cdot u \neq 0\} \\
&\leq \sum\{u \otimes (-u) : u \in P,\ u \leq a_i,\ i < n,\ c \cdot u \neq 0,\ b \cdot (-u) \neq 0\}.
\end{aligned}$$

Let $P' = \{u \in P : \exists i < n[u \leq q_i,\ c \cdot u \neq 0, b \cdot (-u) \neq 0]\}$. Now if $u, v \in P'$ then, since the members of P' are pairwise disjoint, $(c \cdot u) \otimes (b \cdot -v) \leq u \otimes (-u \cdot -v)$. By the comparison principle, $b \cdot -v \leq -u \cdot -v$. This is true for all $v \in P'$, so $b \cdot -v \leq -\sum P'$. Hence

$$\begin{aligned}
c \otimes b &\leq \sum_{u \in P'} ((c \cdot u) \otimes (b \cdot -u)) \\
&\leq \sum_{u \in P'} \left((c \cdot u) \otimes (-\sum P')\right) \\
&\leq \left(\sum P'\right) \otimes \left(-\sum P'\right).
\end{aligned}$$

□

Several somewhat complicated normal forms for elements of free poset algebras are given in Alami, Bekkali, Faouzi, Zhani 2007.

The following is from Bekkali, Zhani 2004.

Proposition 6.9. *For any BA A the following are equivalent:*

(i) A is a free poset algebra.

(ii) A has a set X of generators such that $1 \in X$, and for all $m, n \in \omega$ and all $a \in {}^m X$ and $b \in {}^n X$ we have

(a) $\prod_{i<m} a_i \neq 0$.

(b) If $\prod_{i<m} a_i \cdot \prod_{j<n} -b_j = 0$, then there exist $i < m$ and $j < n$ such that $a_i \leq b_j$.

The following is from Blaszczyk, Kucharski, Turek 2014

A subgroup H of $\mathrm{Aut}(A)$ *acts minimally on* A iff for every $a \in A^+$ there exist $h_0, \ldots, h_{n-1} \in H$ such that $h_0(a) + \cdots + h_{n-1}(a) = 1$.

Lemma 6.10. *If H acts minimally on A, then* $\mathrm{c}(A) \leq |H|$.

Theorem 6.11. *If a countable group of automorphisms acts minimally on A, then A has a dense projective subalgebra of size $\pi(A)$.*

The following is from Bonnet, Rubin 2004.

A poset P has *finite width* iff there is an $n \in \omega$ such that P is the union of n chains. P is *scattered* iff it does not have a subset isomorphic to $\mathbb{Q}$. P is *semi*-well ordered iff for every $a \in {}^{\omega}P$ there exist $m < n$ such that $a_m \leq a_n$.

Theorem 6.12. (Theorem 1.1) *If P is a scattered poset with finite width, then there is a semi-well ordered poset Q such that the free BA over Q can be embedded in the free BA over P.*

The following is from Cramer 1974.

$\mathscr{C}(\lambda)$ is the class of all BAs which have no subalgebra isomorphic to $\mathrm{Fr}(\lambda)$. $\mathscr{D}(\mu)$ is the class of all BAs which have no homomorphic image isomorphic to $\mathscr{P}(\mu)$.

Theorem 6.13. (Proposition 2.2) $\forall\mu[\mathscr{D}(\mu) = \mathscr{C}(2^{\mu})$.

For any BA A we let $\mathscr{D}(\lambda, A)$ be the set of all elements $a \in A$ such that $A \upharpoonright a \in \mathscr{D}(\lambda)$.

Theorem 6.14. (Proposition 3.6) *$A \in \mathscr{C}(\lambda)$ iff $D(\lambda, A) = A$.*

The following is from Fuchino, Koppelberg, Shelah 1996.

A has the *weak Freese-Nation property* (WFN) iff there is an $f : A \to [A]^{\leq\omega}$ such that

$$\forall a, b \in A[a \leq b \to \exists c \in f(a) \cap f(b)[a \leq c \leq b]].$$

A has the *κ-Freese-Nation property* (κ-FN) iff there is an $f : A \to [A]^{<\kappa}$ such that

$$\forall a, b \in A[a \leq b \to \exists c \in f(a) \cap f(b)[a \leq c \leq b]].$$

Proposition 6.15. (Proposition 4.1) *If A has the κ-FN, then* $\mathrm{Depth}(A) \leq \kappa$.

Proposition 6.16. (Theorem 4.2) *If κ is regular, A has the κ-FN, $\lambda = \lambda^{<\kappa}$, $X \subseteq A$, and $|X| > \lambda$, then X has an independent subset of size $> \lambda$.*

Proposition 6.17. (Proposition 5.3) *If κ is uncountable and regular, then $\mathscr{P}(\kappa)$ and $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ do not have the κ-FN.*

The following is from Grygiel 1989

Theorem 6.18. (Theorem) *Every countably generated proper filter in an atomless BA has an independent set of generators.*

The following is from Grygiel 1995.

For a filter F on a BA A, $\mu(F)$ is the least size of a generating set for F.

Theorem 6.19. (Theorem 2) *If F is a filter in a free BA A and $\mathrm{cf}(\mu(F)) > \omega$, then F is generated by an independent set.*

The following is from Grygiel 1990.

Theorem 6.20. (Theorem 3) *If H is a proper countably generated filter with finitely many coatoms, then H is generated by an independent set.*

The following is from Koszmider, Shelah 2013.

A has the *weak subsequential separation property* iff for every disjoint $a \in {}^{\omega}A$ there is a $b \in A$ such that both of the sets

$$\{n \in \omega : a_n \leq b\} \quad \text{and} \quad \{n \in \omega : a_n \cdot b = 0\}$$

are infinite.

Theorem 6.21. *If A has the weak subsequential separation property, then A has an independent subset of size 2^{ω}.*

A has the *subsequential separation property* iff for every disjoint $a \in {}^{\omega}A$ there is a $b \in A$ such that

$$\{n \in \omega : a_{2n} \leq b \text{ and } a_{2n+1} \cdot b = 0\}$$

is infinite.

Corollary 6.22. *If A has the weak subsequential separation property, then $\beta\omega$ is a subspace of* $\mathrm{Ult}(A)$.

The following is from Kunen 1983.

A subset $\mathscr{A}$ of $\mathscr{P}(\kappa)$ is *θ-independent* iff for every $\mathscr{B} \in [\mathscr{A}]^{<\theta}$ and every $\varepsilon \in {}^{\mathscr{B}}2$ we have

$$\left|\bigcap_{A\in\mathscr{B}} A^{\varepsilon(A)}\right| = \kappa.$$

Theorem 6.23. *If $\kappa = \kappa^{<\theta}$ then there is a θ-independent subset of $\mathscr{P}(\kappa)$ of size 2^{κ}.*

Proof. Let $\mathscr{F} = [\kappa]^{<\theta}$ and $\Phi = [\mathscr{F}]^{<\theta}$. Thus $|\mathscr{F} \times \Phi| = \kappa$. For each $\Gamma \subseteq \kappa$ let

$$b_{\Gamma} = \{(\Delta, \varphi) \in \mathscr{F} \times \Phi : \Delta \cap \Gamma \in \varphi\}.$$

Suppose that $H, K \subseteq \mathscr{P}(\kappa)$ are disjoint and of size less than θ; we claim

$$(*) \qquad \left|\left(\bigcap_{A\in H} b_A\right) \cap \left(\bigcap_{B\in K} ((\mathscr{F} \times \Phi)\backslash b_B\right)\right| = \kappa.$$

This will complete the proof.

For distinct $A, B \in H \cup K$ choose $\alpha_{AB} \in A \triangle B$. Let $\Delta = \{\alpha_{AB} : A, B \in H \cup K, A \neq B\}$, $\beta \in \kappa \backslash \Delta$, and $\varphi = \{\Delta \cap A : A \in H\} \cup \{\{\beta\}\}$. Note that by varying β we get κ many such φ. We claim that $(\Delta, \varphi) \in (*)$. If $A \in H$, then $\Delta \cap A \in \varphi$, and so $(\Delta, \varphi) \in b_A$. Suppose that $B \in K$ and $(\delta, \varphi) \in b_B$. Then $\Delta \cap B \in \varphi$. Since $\beta \notin \Delta$, it follows that there is an $A \in H$ such that $\delta \cap B = \Delta \cap A$. Now $\alpha_{AB} \in A \triangle B$ and $\alpha_{AB} \in \Delta$, contradiction. $\square$

Theorem 6.24. *If θ is uncountable and regular and there is a maximal θ-independent family $\mathscr{A} \subseteq \mathscr{P}(\kappa)$ of size $\geq \theta$, then:*

(i) $2^{<\theta} = \theta$.

(ii) There is a λ with $\sup\{(2^\alpha)^+ : \alpha < \theta\} \leq \lambda \leq \min(\kappa, 2^\theta)$ such that there is a nontrivial θ^+-saturated λ-complete ideal over λ.

Theorem 6.25. *If ZFC plus the existence of a measurable cardinal is consistent, then so is ZFC plus the existence of a maximal σ-independent subset of $\mathscr{P}(2^{\omega_1})$.*

The following is from Koppelberg 1997. Proofs of the following theorems are given using σ-filtered BAs.

An ideal I is *σ-directed* iff every countable subset of I has an upper bound in I.

Theorem 6.26. (Mokobodzki) (CH) *If $p : B \to A$ is an epimorphism, B is CSP,* $\ker(p)$ *is σ-directed, $|A| \leq \omega_2$, and A satisfies ccc, then there is a homomorphism $f : A \to B$ such that $p \circ f$ is the identity.*

Theorem 6.27. (Dow, Vermeer) (CH) *If $|A| \leq \omega_2$ and A satisfies CSP, then A is a homomorphic image of a complete BA.*

$\mathrm{Bor}(\mathbb{R})$ is the algebra of Borel subsets of $\mathbb{R}$.

Theorem 6.28. (Carlson, Frankiewicz, Zbierski) *In the Cohen model there is a homomorphism $f : \mathrm{Bor}(\mathbb{R})/meag \to \mathrm{Bor}(\mathbb{R})$ such that $\pi \circ f$ is the identity, where $\pi : \mathrm{Bor}(\mathbb{R}) \to \mathrm{Bor}(\mathbb{R})/meag$ is the natural mapping.*

Theorem 6.29. (Carlson, Frankiewicz, Zbierski) *In the Cohen model there is a homomorphism $f : \mathrm{Bor}(\mathbb{R})/null \to \mathrm{Bor}(\mathbb{R})$ such that $\pi \circ f$ is the identity, where $\pi : \mathrm{Bor}(\mathbb{R}) \to \mathrm{Bor}(\mathbb{R})/null$ is the natural mapping.*

Theorem 6.30. (Frankiewicz, Zbierski) *In the Cohen model, $\mathscr{P}(\omega_1$ does not embed in $\mathscr{P}(\omega)/fin$.*

Theorem 6.31. (Frankiewicz, Zbierski) *In the Cohen model, A is a homomorphic image of $\mathscr{P}(\omega)/fin$ iff $|A| \leq 2^\omega$ and A has CSP.*

The following is from Trnkova 1980.

Theorem 6.32. *If A is a countable BA and ${}^n \oplus A \cong {}^m \oplus B$ with $n < m$, then ${}^n \oplus A \cong {}^{n+1} \oplus A$.*

7. Homogeneous BAs

In Arhangelski 1970 the following is proved.

Theorem 7.1. (GCH) *If* $\mathrm{Ult}(A)$ *is homogeneous, then* $|\mathrm{Ult}(A)|$ *is not a limit cardinal.*

The following is from van Douwen 1978.

$U(\kappa)$ is the set of all uniform ultrafilters on κ. Let X be a space, κ an infinite cardinal, and $\mathscr{I}$ a collection of subsets of X. We define

$$\forall x \in X \forall \varphi \in {}^{\kappa}\mathscr{I}\left[w(x,\varphi) = \left\{a \subseteq \kappa : x \in \overline{\bigcup\{\varphi(\alpha) : \alpha \in a\}}\right\}\right];$$
$$\forall x \in X[W(x,\kappa,\mathscr{I}) = \{w(x,\varphi) : \varphi \in {}^{\kappa}\mathscr{I}\}].$$

A family $\mathscr{I}$ of subsets of X is *invariant* iff for every $I \in \mathscr{I}$ and every autohomeomorphism h of X, $h[I] \in \mathscr{I}$.

Proposition 7.2. (Criterion 2.2) *If there exist a familyy* $\mathscr{I}$ *of subsets of* X*, an infinite cardinal* κ*, a* $\varphi \in {}^{\kappa}\mathscr{I}$*, and a* $p \in X$ *such that*

$$|W(p,\kappa,\mathscr{I})| < |\{w(x,\varphi) : x \in S\}|,$$

then X *is not homogeneous.*

Theorem 7.3. *For any infinite cardinal* κ*, no power of* $\beta(\kappa)$*,* $\beta(\kappa)\backslash\kappa$*, or* $U(\kappa)$ *is homogeneous.*

The following is from Geschke, Shelah 2003.

Theorem 7.4. (Theorem 1.1) *If* A *is a BA such that every ultrafilter on* A *is countably generated, and* A *has a dense subset* D *such that* $\forall a \in D[A \restriction a \cong A]$*, then* A *is homogeneous.*

Theorem 7.5. (Corollary 2.3) *If* A *is an atomic BA, then* ${}^{\omega}\oplus A$ *is homogeneous.*

Theorem 7.6. (Corollary 2.6) *If* X *is a first countable Boolean space and every point in* X *has a dense* $\mathrm{aut}(X)$*-orbit, then* X *and* $\mathrm{clop}(X)$ *are homogeneous.*

The following is from Koppelberg 1985.

Theorem 7.7. (CH) *There is a homogeneous BA* A *such that the following conditions hold:*

(i) $|A| = \omega_1$.
(ii) A *has a countable dense subalgebra.*
(iii) $\mathrm{Aut}(A)$ *is not simple.*

Theorem 7.8. (MA) *If A is homogeneous, $|A| < 2^\omega$, and A has a countable dense subalgebra, then* $\mathrm{Aut}(A)$ *is simple.*

Theorem 7.9. *For every infinite free BA A,* $\mathrm{Aut}(A)$ *is simple.*

The following is from Morozov 1982.

See Chapter 14 for the Tarski invariants; we write $\mathrm{inv}(A) = (\mathrm{inv}_1(A), \mathrm{inv}_2(A), \mathrm{inv}_3(A))$. A countable BA A is *model-homogeneous* iff

$$\forall a, b \in A[\mathrm{inv}(A \restriction a) = \mathrm{inv}(A \restriction b) \text{ and } \mathrm{inv}(A \restriction -a) = \mathrm{inv}(A \restriction -b) \rightarrow$$
$$\forall a_0 \leq a \exists b_0 \leq b[\mathrm{inv}(a_0) = \mathrm{inv}(b_0) \text{ and } \mathrm{inv}(a \cdot -a_0) = \mathrm{inv}(b \cdot -b_0)]].$$

For any BA A let $A^i = A/T_i(A)$. Let $A_\omega = \{a \in A : \mathrm{inv}(A \restriction a) = (\omega, 0, 0)\}$.

Proposition 7.10. *The model-homogeneous countable BAs A with* $\mathrm{inv}_1(A) = 0$ *are up to isomorphism the following:*
(i) $\mathrm{intalg}(\omega + \eta)$*;*
(ii) $\mathrm{intalg}(\omega + \eta + 1 + \eta)$*;*
(iii) $\mathrm{intalg}(n)$ *for $n \in \omega$;*
(iv) $\mathrm{intalg}(n + \eta)$ *for $n \in \omega$;*
(v) $\mathrm{intalg}(\omega)$*;*
(vi) $\mathrm{intalg}(\eta + \omega)$*;*
(vii) $\mathrm{intalg}(\omega + \omega)$*;*
(viii) $\mathrm{intalg}(\omega + \omega + 1 + \eta)$.

Proposition 7.11. *If* $\mathrm{inv}_1(A) > 0$ *then one of the following holds:*
(i) $A_\omega = \emptyset$.
(ii) $A_\omega = \{a \in A : A \restriction a \cong$ $\mathrm{intalg}(\omega + \eta)\} \neq \emptyset$.
(iii) $A_\omega = \{a \in A : A \restriction a \cong$ $\mathrm{intalg}(\omega)\} \neq \emptyset$.

The type $\rho(A)$ of a countable model-homogeneous BA A is defined as follows. If $\mathrm{inv}_1(A) > 0$ then $\rho(A)$ is 1,2, or 3 as in Proposition 7.11. If $\mathrm{inv}_1(A) = 0$ then $\rho(A)$ is one of the orders given in Proposition 1.10.

Proposition 7.12. *If A is a countable model-homogeneous BA with* $\mathrm{inv}_1(A) = \infty$*, then one of the following holds:*
(i) $\forall a \in A[\mathrm{inv}_1(A \restriction a) < \infty$ *or* $\mathrm{inv}_1(-a) < \infty]$.
(ii) $\exists b \in A[\mathrm{inv}_1(A \restriction a) = \infty$ *and* $\mathrm{inv}_1(-a) = \infty]$ *and (ii) of Proposition 7.11 holds.*
(iii) $\forall a \in A[\mathrm{inv}_1(A \restriction a) = \infty \rightarrow \exists c \leq a[\mathrm{inv}_1(c) = \mathrm{inv}_1(a \cdot -c) = \infty]]$.

The case (i)–(iii) is denoted by $t(A)$; if $\mathrm{inv}_1(A) < \infty$ then $t(A) = 0$.

Theorem 7.13. *Countable model-homogeneous BAs A and B are isomorphic iff $t(A) = t(B)$ and $\rho(A^i) = \rho(B^i)$ for all i.*

elementary characteristic	number of countable model homogeneous BAs
$(\infty, 0, 0)$	2^{ω}
(n, ∞, ε)	$3 \cdot 2^{n}$
$(n.l.\varepsilon)$, $1 \neq l < \infty$	2^{n}
$(n, 1, 1)$	2^{n}
$(n, 1, 0)$	$3 \cdot 2^{n-1}$
$(0, 1, \varepsilon)$	1

8. Homomorphisms

The following is from Bacsich 1972.

Proposition 8.1. *If I is an ideal of A, F is a filter of A, $I \cap F = \emptyset$, and $F^* = \{a \in A : -a \in F\}$, then $I \cup F^*$ generates a proper ideal.*

Theorem 8.2. *If A is a subalgebra of B, D is a maximal ideal of A, I is an ideal of B, and $I \cap A \subseteq D$, then there is a maximal ideal E of B such that $E \cap A = D$.*

A *semimorphism* from A to B is a function $f : A \to B$ such that f preserves 0, 1, and $+$. $\mathrm{Sem}(A, B)$ is the set of all semimorphisms from A to B. For $f, g \in \mathrm{Sem}(A, B)$ we define $f \leq g$ iff $\forall a \in A[f(a) \leq g(a)]$.

Proposition 8.3. $(\mathrm{Sem}(A, 2), \leq)$ *is isomorphic to* $(\mathscr{I}^*, \supseteq)$, *where* $\mathscr{I}^* = \{I : I$ *is a proper ideal of* $A\}$.

Theorem 8.4. (Monteiro) *If C is a complete BA, A is a subalgebra of B, $f : A \to C$ is a homomorphism, $d \in \mathrm{Sem}(B, C)$, and $f \leq (d \restriction A)$, then there is a homomorphism $g; B \to C$ such that $g \leq d$ and $(g \restriction A) = f$.*

The following is from Dwinger 1963.

If D is a poset directed upwards, then a *D-inverse system* is a pair (B, f) such that $B = \langle B_\alpha : \alpha \in D\rangle$ is a system of BAs and for $\alpha, \beta \in D$ with $\alpha < \beta$, $f_{\alpha\beta} : B_\beta \to B_\alpha$ is a homomorphism, $f_{\alpha\alpha}$ is the identity, and for $\alpha < \beta < \gamma$, $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$. The *limit* of this system is

$$B_\infty \stackrel{\text{def}}{=} \left\{x \in \prod_{\alpha \in D} B_\alpha : \forall \alpha, \beta \in D[\alpha \leq \beta \to x_\alpha = f_{\alpha\beta}(x_\beta)]\right\}.$$

The article is concerned with the topological dual of this construction.

The following is from Geschke 2006.

Let D be the subalgebra of $\mathscr{P}(\omega) \times \mathscr{P}(\omega)$ consisting of all pairs (a, b) such that $a \triangle b$ is finite.

Theorem 8.5. (Corollary 3.5) *For every BA A, $\mathrm{cf}(A) = \omega$ iff there is a homomorphism $f : A \to D$ such that* $\mathrm{rng}(f)$ *contains all atoms of D.*

The following is from Hyttinen, Shelah 2002.

Theorem 8.6. (Conclusion 5) *Con(ZFC) implies Con(ZFC+$\exists A[A$ is an atomic BA, $|A| = \aleph_1$ and $|\mathrm{Aut}(A)| = \aleph_\omega$ and $\aleph_\omega < 2^\omega$).*

The following is from McKenzie 1977.

Proposition 8.7. *If A and B are denumerable BAs such that* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$, *then* $\sum \mathrm{at}(A)$ *exists iff* $\sum \mathrm{at}(B)$ *exists.*

Proposition 8.8. *If A and B are BAs not isomorphic to* $\mathrm{Fr}(\omega)$, *if* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$, *and if* $\sum \mathrm{at}(A)$ *exists, then* $A \cong B$.

Proposition 8.9. *There exist denumerable BAs A, B each having denumerably many atoms, such that* $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ *but* $A \not\cong B$.

The following is from Perovic 1999.

Members f, g of $\mathrm{Aut}(A)$ are *strongly distinct* iff for every nonzero $b \in A$ there is an $s \in A$ such that $f(s) \cdot b \neq g(s) \cdot b$. If $C \leq B$, then $\mathrm{Aut}_C(B)$ is the set of all automorphisms of B that fix C pointwise. If G is a subgroup of $\mathrm{Aut}(A)$, then $\mathrm{Fix}(G) = \{a \in A : \forall f \in G[f(a) = a]\}$. B is *Galois over* C iff B is a finite extension of C and there exists a finite subgroup of strongly distinct members of $\mathrm{Aut}_C(B)$ such that $\mathrm{Fix}(G) = C$. For the notion of a sheaf, see the Handbook, pp. 116ff. For Boolean powers see Burris 1975.

Theorem 8.10. *If B is a finite extension of C then the following are equivalent:*

(i) B is Galois over C.

(ii) The sheaf of B over C is Hausdorff with all stalks of equal cardinality.

(iii) B is a Boolean power of a finite BA by C.

(iv) There is a natural number k such that $B = C \oplus {}^k 2$.

The following is from Palchunov, Trofimov 2012.

For any BA A and any $f \in \mathrm{Aut}(A)$, let $\mathrm{fix}(f, A) = \{a \in A : f(a) = a\}$. Clearly this is a subalgebra of A.

Theorem 8.11. *For any BA A and any subalgebra B of A the following are equivalent:*

(i) There is an automorphism f of A such that $B = \mathrm{fix}(f, A)$.

(ii) For any $a \in A$ there exist $b, c, d \in A$ such that the following conditions hold:

(a) $a = b + c$ and $b \cdot c = 0$ and $a \cdot d = 0$ and $c, b + c \in B$.

(b) $\forall e \leq b[e \neq 0 \to e \notin B]$.

(c) $\forall p \leq d[p \neq 0 \to p \notin B]$.

(d) $\forall e \leq b \exists p \leq d[e + p \in B]$.

(e) $\forall p \leq d \exists e \leq b[e + p \in B]$.

Theorem 8.12. *Let f be an automorphism of A. Then the following are equivalent:*

(i) For every automorphism g of A, if $\mathrm{fix}(f, A) = \mathrm{fix}(g, A)$, *then $f = g$.*

(ii) $f = f^{-1}$.

The following is from Pinus 2014.

For G a subgroup of $\mathrm{Aut}(A)$, $\mathrm{fix}(G) = \bigcap_{f \in G} \mathrm{fix}(f)$. For B a subalgebra of A, $\mathrm{Stab}(B) = \{f \in \mathrm{Aut}(A) : B \subseteq \mathrm{fix}(f)\}$. Also, $\overline{B} = \mathrm{fix}(\mathrm{Stab}(B))$. B is *Galois-closed* iff $\overline{B} = B$. A is *separable* iff all of its subalgebras are Galois-closed.

Theorem 8.13. *A is separable iff A is finite.*

The following is from Roitman 1981. A rather complicated theorem is proved of which a special case is the following.

Theorem 8.14. *It is consistent to have an atomic BA A of size $\aleph_\omega$ such that A has $\aleph_\omega$ atoms and $\aleph_{\omega+1}$ automorphisms.*

The following is from Rubin 1996.

Theorem 8.15. *Let A be a complete atomless BA. For each $f \in \mathrm{Aut}(A)$ let $\mathrm{var}(f) = \sum\{a \in A : a \cdot f(a) = 0\}$. A subgroup G of $\mathrm{Aut}(A)$ is* **locally moving** *iff $\{\mathrm{var}(g) : g \in G\}$ is dense in A. Then every locally moving subgroup of $\mathrm{Aut}(A)$ determines A up to isomorphism. That is, if A and B are complete atomless BAs, G and H are locally moving subgroups of $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ respectively, and φ is an isomorphism from G to H, then there is an isomorphism τ of A onto B such that $\forall f \in G[\varphi(f) = \tau \circ f \circ \tau^{-1}]$.*

The following is from Senf, Vladimirov 1987.

Proposition 8.16. *A complete BA is not the free product of other algebras.*

Proposition 8.17. *A system $\langle f_i : i \in I\rangle$ of BAs $\langle A_i : i \in I\rangle$ can be extended to an automorphism of the free product $\oplus_{i\in I} A_i$.*

Proposition 8.18. *A system $\langle f_i : i \in I\rangle$ of BAs $\langle A_i : i \in I\rangle$ can be extended to an automorphism of the completion $\overline{\oplus_{i\in I} A_i}$.*

The following is from Steprans 2003. He shows that it is consistent for $\mathrm{Aut}(\mathscr{P}(\omega)/fin)$ to be any regular cardinal between 2^ω and 2^{2^ω}.

9. Ideals

The following is from Baumgartner, Taylor, Wagon 1982.

An *infinitary ideal* on a regular uncountable cardinal κ is a proper κ-complete ideal on κ containing all singletons. If $\langle X_\alpha : \alpha < \kappa\rangle$ is a sequence of subsets of κ, then the *diagonal union* of $\langle X_\alpha : \alpha < \kappa\rangle$ is

$$\nabla_{\alpha<\kappa}X_\alpha \stackrel{\text{def}}{=} \{\beta < \kappa : \exists\alpha < \beta[\beta \in X_\alpha]\}.$$

For I an infinitary ideal on κ we let $\nabla(I) = \{Y : \exists X \in {}^\kappa I[Y = \nabla_{\alpha<\kappa}X_\alpha]\}$. An ideal I is *normal* iff $\nabla(I) = I$. NS_κ is the ideal of nonstationary subsets of κ.

Theorem 9.1. (Fodor) NS_κ *is a normal ideal on* κ*.*

If $A \notin I$ and $f : A \to \kappa$, then f is *I-small* iff $\forall\alpha \in \kappa[f^{-1}[\{\alpha\}] \in I$. An ideal I on κ is

a *P-point* iff for every I-small $f : \kappa \to \kappa$ there is an X with $\kappa\backslash X \in I$ such that $f \restriction X$ is $[\kappa]^{<\kappa}$-small.

a *Q-point* iff for every $[\kappa]^{<\kappa}$-small $f : \kappa \to \kappa$ there is an X with $\kappa\backslash X \in I$ such that $f \restriction X$ is one-one.

selective iff for every I-small $f : \kappa \to \kappa$ there is an X with $\kappa\backslash X \in I$ such that $f \restriction X$ is one-one.

Theorem 9.2. (Theorem 3.9) *(i) Every normal ideal is selective.*
(ii) Every extension of NS_κ *is a Q-point.*

The following is from Galvin, Jech, Magidor 1978.

Let I be a σ-complete ideal on a set S containing all singletons. Players empty and nonempty successively choose $S \supseteq T_0 \supseteq T_1 \cdots$ with each $T_i \notin I$. If $\bigcap_{i\in\omega} T_i$ is empty, then empty wins; otherwise nonempty wins.

Theorem 9.3. (Theorem 1) *If* $|S| \le 2^\omega$*, then nonempty does not have a winning strategy.*

Theorem 9.4. (Theorem 2) *If* $S = \kappa$*, an infinite cardinal, then empty has a winning strategy iff* I *is not precipitous.*

The following is from Foreman 1983a.

An ideal I on κ is *normal* iff for every $X \subseteq \kappa$ with $X \notin I$ and every regressive function f defined on X there is a $\beta \in \kappa$ such that $\{\alpha \in X : f(\alpha) = \beta\} \notin I$. I is *λ-saturated* iff it is normal and $\mathscr{P}(\kappa)/I$ has the λ-cc.

Theorem 9.5. *Con(ZFC+there is a huge cardinal) implies Con(ZFC+*$\forall n \in \omega$*[there is a normal* $\aleph_n$*-complete,* $\aleph_{n+1}$*-saturated ideal on* $\aleph_n$*]+there is a normal* $\aleph_{\omega+1}$*-complet,* $\aleph_{\omega+2}$*-saturated ideal on* $\aleph_{\omega+1}$*.*

Theorem 9.6. *Con(ZFC+there is a huge cardinal) implies Con(ZFC+every regular cardinal carries a κ^+-saturated ideal).*

The following is from Huberich 1996.

If A is κ^+-complete, then $\mathrm{Part}_\kappa(A)$ is the set of all partitions of A of size κ. If F is a filter on A and $a, b \in {}^\tau A$ are partitions of A, then $a \equiv_F b$ iff $\sum_{\alpha<\tau}(a(\alpha) \cdot b(\alpha)) \in F$.

Theorem 9.7. *If κ is uncountable and regular and A is a κ-complete BA with a dense subset of size $\leq \kappa$, then there is a filter F on A such that $\forall \tau < \kappa[|\{a/\equiv_F : a \in \mathrm{Part}_\tau(A)\}| \leq 2^{<\kappa}$.*

The following is from Hrusak 2011. This is a survey of results on filters and ideals over ω.

An ideal $\mathscr{I}$ is *tall* iff $\forall Y \in [\omega]^\omega \exists I \in \mathscr{I}[I \cap Y$ is infinite].

Orders on ideals: Katetov, Katetov-Blass, Rudin-Keisler, Tukey.

Ultrafilters: selective, P-points, Q-points, rapid, nowhere dense.

Frechet ideal, eventually different ideal.

Fubini product.

Random graph ideal.

The following is from Jech 1977.

Let I be an ideal on a cardinal κ. If $S \notin I$, then an *I-partition* of S is a maximal collection W of subsets of S such that each member of W is not in I, and $\forall X, Y \in W[X \neq y \to X \cap Y \in I]$. An I-partition W' is a *refinement* of an I-partition W, in symbols $W' \leq W$, iff $\forall X \in W' \exists Y \in W[X \subseteq Y]$. A κ-complete ideal I on a cardinal κ is *precipitous* iff for every $S \subseteq \kappa$ which is not in I and for every sequence

$$W_0 \geq W_1 \geq \cdots$$

of I partitions of S there is a sequence

$$X_0 \supseteq X_1 \supseteq \cdots$$

with each $X_i \in W_i$, such that $\bigcap_{n\in\omega} X_n \neq \emptyset$.

Proposition 9.8. $[\kappa]^{<\kappa}$ *is not precipitous.*

Proposition 9.9. *If I is κ^+-saturated, then I is precipitous.*

Theorem 9.10. *If there is a precipitous ideal, then there is a transitive model with a measurable cardinal.*

Theorem 9.11. (Mitchell) *If κ is a measurable cardinal in M, then there is a generic extension $M[G]$ in which $\kappa = \omega_1$ and κ has a precipitous ideal.*

The following is from Kunen 1978.

Let $S^*(\kappa, \lambda)$ abbreviate: there is a nontrivial κ-complete ideal on κ which is λ-saturated, but not λ'-saturated for any $\lambda' < \lambda$.

$S^*(\kappa,\lambda)$ for κ not measurable			
	$\omega < \lambda < \kappa$	$\lambda = \kappa$	$\lambda = \kappa^+$
κ successor	FALSE (Ulam [5])	FALSE (Ulam [5])	con: sec. 4
κ w.i., not s.i.	con (Prikry [3])	con ([2])	con
κ s.i., not w.c.	FALSE (Tarski [4])	con; sec. 3	con (Boos [1])
κ s.i., w.c.	FALSE (Tarski [4])	FALSE (Levy, Silver)	con ([2])

Here w.i. means weakly inaccessible; s.i. means strongly inaccessible; w.c. means weakly compact; con means consistent

[1] Boos *Boolean extensions which efface the Mahlo property.* JSL 39 (1974), 254–268.
[2] Kunen, Paris *Boolean extensions and measurable cardinals.* Annals Math. Logic 2 (1971), 359–378.
[3] Prikry *Changing measurable into accessible cardinals.* Rosp. Math. 68 (1970).
[4] Tarski *Ideale in vollstandigen Mengenkörper.* FM 33 (1945), 51-65.
[5] Ulam *Zur Masstheorie in der algemeinen Mengenlehre.* FM 16 (1930), 140–150.

The following is from Matet 1997.

If J is a nontrivial ideal over X, $A \subseteq X$, and $A \notin J$, then we set $J \restriction A = \{B \subseteq X : A \cap B \in J\}$. J is *nowhere prime* iff $\forall A \in \mathscr{P}(X)\backslash J[J \restriction A$ is not prime].

Let I be a nontrivial ideal on κ such that $\kappa \subseteq I$. I is *tall* iff $\forall A \in \mathscr{P}(\kappa)\backslash I[I \cap [A]^\kappa \neq \emptyset]$. I is *nowhere tall* iff $\forall A \in \mathscr{P}(\kappa)\backslash I \exists B \in \mathscr{P}(A) \cap \mathscr{P}(\kappa)\backslash I[[B]^\kappa \subseteq I]$. An ideal I is *feeble* iff there is an increasing $f \in {}^\kappa\kappa$ such that $\{f^{-1}[E] : E \in [\kappa]^\kappa\} \subseteq \mathscr{P}(\kappa)\backslash I$. Several other notions are considered.

There are several papers by Y. Abe concerning ideals on $[\lambda]^{<\kappa}$.

Aragon gives some model-theoretic results concerning BAs with a distinguished ideal.

10. Interval algebras

The following theorem is from Alami, Zhani 2004.

Theorem 10.1. *If A is an infinite interval algebra, consider the following statements:*

(i) Every subalgebra of A is isomorphic to an interval algebra.

(ii) $\pi(A)=\omega$.

(iii) A is isomorphic to an interval algebra on a subset of $\mathbb{R}$

Then (ii) and (iii) are equivalent, and (i) implies (ii).

The following is from Bekkali 1994.

Theorem 10.2. (Theorem 1.4) *If A is a subalgebra of an interval algebra and $|B|$ is singular, and $\pi(B)=\omega$, then B has a chain or antichain of size $|B|$.*

Theorem 10.3. (Theorem 2.3) *It is consistent that there is a BA A with the following properties:*

(i) A is a subalgebra of an interval algebra.

(ii) $|A|=\aleph_{\omega_1}$.

(iii) $\pi(A)=\omega_1$.

(iv) $\mathrm{Inc}(A)=\omega_1$.

(v) $\mathrm{Length}(A)=\aleph_{\omega_1}$ *not attained.*

The following is from Bekkali, Todorcevic 20110.

Proposition 10.4. (Theorem 2.5) *If A is a subalgebra of an interval algebra and A is not σ-centered, then A has a subalgebra which is not isomorphic to an iterval algebra.*

Theorem 10.5. *Any σ-centered pseudotree algebra of size less than $\mathfrak{b}$ is isomorphic to an interval algebra.*

Example 10.6. *There is a σ-centered pseudotree algebra of cardinality 2^ω which is not isomorphic to an interval algebra.*

The following is from Bell, M. 1978.

A space X is *supercompact* iff there is a subbase $\mathscr{O}$ for the topology on X such that every cover of X using elements of $\mathscr{O}$ has a subcover using just 1 or 2 elements. A BA A is *supercompact* iff $\mathrm{Ult}(A)$ is supercompact.

Proposition 10.7. *Every interval algebra is supercompact.*

Proof. Let $A=\mathrm{intalg}(L)$, where L is an infinite linear order with first element 0. Let $X=\{[0,a):0<a\le\infty\}\cup\{[a,\infty):0\le a<\infty\}$. Let $\mathscr{O}=\{\mathcal{S}(x):x\in X\}$. Clearly $\mathscr{O}$ is a subbase for the topology on $\mathrm{Ult}(A)$. Now suppose that $Y\subseteq X$ and $\{\mathcal{S}(x):x\in Y\}$ covers $\mathrm{Ult}(A)$. Let $Y'\in[Y]^{<\omega}$ be such that $\{\mathcal{S}(x):x\in Y'\}$ covers $\mathrm{Ult}(A)$. If $[0,\infty)\in Y'$, then $\{\mathcal{S}([0,\infty))\}$ covers $\mathrm{Ult}(A)$. Suppose that $[0,\infty)\notin Y'$. There is some $a\in L$ such that $[0,a)\in Y'$; let a be maximum with this property. There is some $b\in L$ such that

$[b,\infty) \in Y'$; let b be minimum with this property. Clearly $b \le a$. Then $\{[0,a),[b,\infty)\}$ covers $\mathrm{Ult}(A)$. $\square$

In the paper it is shown that no infinite complete BA is supercompact.

11. Scott rank

We describe notions and results from Alaev 1998, 1999.

Translation of Tarski invariants in chapter 2 to the invariants given here, with $\infty = \omega$:

$\text{ch}_1(A)$ is the least k such that $I_{k+1}(A) = A$, or ∞ if there is no such k. Thus $\text{ch}_1(A)$ is k if $\text{inv}(A) = (k, l, m)$ for some l, m.

$\text{ch}_2(A)$ is 0 if $\text{ch}_1(A) = \infty$, while if $\text{ch}_1(A) = m \in \omega$, then $\text{ch}_2(A)$ is k if $A/I_m(A)$ has exactly $k < \omega$ atoms, or ∞ if $A/I_m(A)$ has infinitely many atoms. Thus $\text{ch}_2(A)$ is m if $\text{inv}(A) = (k, l, m)$.

$\text{ch}_3(A)$ is 0 if $\text{ch}_1(A) = \infty$, while if $\text{ch}_1(A) = m \in \omega$, then $\text{ch}_3(A) = 1$ if $A/I_m(A)$ has a nonzero atomless element, and 0 otherwise. Thus $\text{ch}_1(A)$ is l if $\text{inv}(A) = (k, l, m)$ with $k \in \omega$.

$\text{ch}(A) = (\text{ch}_1(A), \text{ch}_2(A), \text{ch}_3(A))$. Thus $\text{ch}(A) = (k, m, l)$ iff $\text{inv}(A) = (k, l, m)$.

Recall also the ideals $I_\alpha(A)$ defined after Proposition 1.13. A BA A is α-atomic iff for each $\beta < \alpha$ $A/I_\beta(A)$ is atomic. Thus every BA is 0-atomic. A is 1-atomic iff it is atomic.

A system $\langle a_i : i \in I\rangle$ is a *partition of unity* iff $\sum_{i\in I} a_i = 1$ and $a_i \cdot a_j = 0$ for $i \neq j$. Note that some $a_i = 0$ is allowed. If $b \in {}^nA$ then $B(b)$ is the system $\langle \prod_{i<n}^{\varepsilon(i)} : \varepsilon \in {}^n2\rangle$. Thus $B(b)$ is a partition of unity.

For φ a formula in $L_{\infty\omega}$ we define the *quantifier rank* $\text{qr}(\varphi)$ as follows:

If φ is atomic, then $\text{qr}(\varphi) = 0$.

If φ is $\neg\psi$, then $\text{qr}(\varphi) = \text{qr}(\psi)$.

If φ is $\bigwedge \Phi$ or $\bigvee \Phi$, then $\text{qr}(\varphi) = \sup_{\psi\in\Phi} \text{qr}(\psi)$.

If φ is $\forall x\psi$ or $\exists x\psi$, then $\text{qr}(\varphi) = \text{qr}(\psi) + 1$.

If A and B are $\mathscr{L}$-structures, then $A \equiv^\alpha B$ iff the following holds:

$$\forall \text{ sentences } \theta[\text{qr}(\theta) \leq \alpha \to [A \models \theta \leftrightarrow B \models \theta]].$$

If A and B are $\mathscr{L}$-structures, then a *partial isomorphism* from A to B is an isomorphism from a substructure of A to a substructure of B.

For an ordinal α, an *α-regular chain of partial isomorphisms from A to B* is a chain $I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\alpha$ such that each I_β is a set of partial isomorphisms of A to B, $I_\alpha \neq \emptyset$, and the following conditions hold:

$$\forall \beta_1 < \beta_2 \leq \alpha \forall a \in A \forall f \in I_{\beta_2} \exists g \in I_{\beta_1}[f \subseteq g \text{ and } a \in \text{dmn}(g)];$$
$$\forall \beta_1 < \beta_2 \leq \alpha \forall b \in B \forall f \in I_{\beta_2} \exists g \in I_{\beta_1}[f \subseteq g \text{ and } b \in \text{rng}(g)].$$

The following results are attributed to Goncharov, Countable Boolean algebras and decidability.

Theorem 11.1. *$A \equiv^\alpha B$ iff there is an σ-regular chain of partial isomorphisms from A to B.*

Theorem 11.2. *If λ is a limit ordinal, then $A \equiv^\lambda B$ iff $\forall \beta < \lambda[A \equiv^\beta A]$.*

Theorem 11.3.

$$A \equiv^{\alpha+1} B \quad \text{iff} \quad \forall x \in A \exists y \in B[(A,x) \equiv^{\alpha} (B,y)]$$
$$\text{and } \forall y \in B \exists x \in A[(A,x) \equiv^{\alpha} (B,y)]$$

We say that A is *partially isomorphic* to B, and write $A \cong_p B$ iff there is an $F \neq \emptyset$ such that F is a set of partial isomorphisms from A to B and the following conditions hold:

$$\forall f \in F \forall a \in A \exists g \in F[f \subseteq g \text{ and } a \in \mathrm{dmn}(g)]$$
$$\forall f \in F \forall b \in B \exists g \in F[f \subseteq g \text{ and } b \in \mathrm{rng}(g)]$$

Two more results from Goncharov:

Theorem 11.4. $A \cong_p B$ iff $\forall \alpha[A \equiv^{\alpha} B]$.

Theorem 11.5. *If A and B are countable, then $A \cong_p B$ iff $A \cong B$.*

The following fact is basic for the definition of Scott rank.

Theorem 11.6. *For any structure A there is an ordinal α such that for all $n \in \omega$ and all $a, b \in {}^nA[(A,a) \equiv^{\alpha} (A,b) \rightarrow (A,a) \equiv^{\alpha+1} (A,b)]$.*

Now for any structure A, the *Scott rank* of A is

$$\mathrm{sr}(A) = \min\{\alpha : \forall n \in \omega \forall a, b \in {}^nA[(A,a) \equiv^{\alpha} (A,b) \rightarrow (A,a) \equiv^{\alpha+1} (A,b)]\}.$$

Another result from Goncharov is:

Theorem 11.7. *If $\mathrm{sr}(A) = \alpha$ then $\forall n \in \omega \forall a, b \in {}^nA[(A,a) \equiv^{\alpha} (A,b) \rightarrow (A,a) \cong_p (A,b)]$.*

If I is an ideal in a BA A and $a \in A$, let $I \restriction a = \{x \in I : x \leq a\}$.

Proposition 11.8. *For any $a \in A$ we have $(E(A)) \restriction a = E(A \restriction a)$.*

For any BA A, let $o(A)$ be the least ordinal such that $A/I_{\alpha}(A)$ is atomless, where we count the one-element BA as atomless.

Proposition 11.9. *For all $n \in \omega$, all $a, b \in {}^nA$, and all ordinals α the following are equivalent:*

(i) $(A,a) \equiv^{\alpha} (A,b)$.

(ii) $(A,B(a)) \equiv^{\alpha} (A,B(b))$.

Lemma 11.10. (Lemma 1) *Suppose that A and B are BAs, $n \in \omega$, $a \in {}^nA$ and $b \in {}^nB$ are partitions of unity, and α is an ordinal. Then*

$$(A,a) \equiv^{\alpha} (B,b) \quad \text{iff} \quad \forall i < n[(A,a_i) \equiv^{\alpha} (B,b_i)].$$

Lemma 11.11. (Lemma 1) *Suppose that A and B are BAs, $a \in A$, $b \in B$, and α is an ordinal. Then*

$$(A, a) \equiv (B, b) \quad (A \upharpoonright a) \equiv^\alpha (B \upharpoonright b) \text{ and } (A \upharpoonright (-a)) \equiv^\alpha (B \upharpoonright (-b)).$$

Lemma 11.12. (Lemma 2) *Suppose that A and B are BAs and α is an ordinal. Then $A \equiv^\alpha B$ iff the following two conditions hold:*

(i) $\forall a \in A \exists b \in B[(A \upharpoonright a) \equiv^\alpha (B \upharpoonright b)$ and $(A \upharpoonright (-a)) \equiv^\alpha (B \upharpoonright (-b))]$.

(ii) $\forall b \in B \exists a \in A[(A \upharpoonright a) \equiv^\alpha (B \upharpoonright b)$ and $(A \upharpoonright (-a)) \equiv^\alpha (B \upharpoonright (-b))]$.

Lemma 11.13. (Lemma 3) *Suppose that A is a BA, I is an ideal of A, and $a \in I$. Then $A/I \cong (A \upharpoonright (-a))/(I \upharpoonright (-a))$.*

(Definition 4) Let Φ be a set of $L_{\infty\omega}$-sentences in the language of BAs. We say that Φ is a *characterizing set for rank* α iff

$$\forall A, B[A \equiv^\alpha B \quad \text{iff} \quad \forall \varphi \in \Phi[A \models \varphi \text{ iff } B \models \varphi]].$$

Lemma 11.14. (Lemma 5) *Let Φ_α be a characterizing set for rank α. Define $\Phi'_\alpha = \{\bigwedge_{\varphi \in \Phi_\alpha} \varphi^{\varepsilon(\varphi)} : \varepsilon \in {}^{\Phi_\alpha}2\}$. For all $\psi_1, \psi_2 \in \Phi'_\alpha$ let*

$$\begin{aligned} \Theta(\psi_1, \psi_2) = \{\theta : \theta \textit{ is a sentence and } \forall BA\ A[A \models \theta \\ \leftrightarrow \exists a \in A[A \upharpoonright a \models \psi_1 \textit{ and } (A \upharpoonright (-a)) \models \psi_2]]\}. \end{aligned}$$

Then $\Theta(\psi_1, \psi_2) \neq \emptyset$.

Moreover, let $\Phi_{\alpha+1}$ be a collection of sentences such that the following two conditions hold:

(i) $\forall \psi_1, \psi_2 \in \Phi'_\alpha \exists \theta \in \Phi_{\alpha+1} \cap \Theta(\psi_1, \psi_1)$.

(ii) $\forall \theta \in \Phi_{\alpha+1} \exists \psi_1, \psi_2 \in \Phi'_\alpha[\theta \in \Theta(\psi_1, \psi_2)]$.

Then $\Phi_{\alpha+1}$ is a characterizing set for $\alpha + 1$.

Lemma 11.15. (Lemma 6) *Suppose that $\varphi(x)$ is a formula with quantifier rank β. Also suppose that $R \stackrel{\text{def}}{=} \{x \in A : A \models \varphi(x)\}$ is an ideal of A. Then for each formula $\psi(x_1, \ldots, x_n)$ of quantifier rank α there is a formula $\psi'(x_1, \ldots, x_n)$ of quantifier rank $\beta+\alpha$ such that for all $a_1, \ldots, a_n \in A$,*

$$A/R \models \psi(a_1/R, \ldots, a_n/R) \quad \text{iff} \quad A \models \psi'(a_1, \ldots, a_n).$$

Lemma 11.16. (Lemma 7) *If I is an ideal in A and $a \in A$, then $(A/I) \upharpoonright (a/I) \cong (A \upharpoonright a)/(I \upharpoonright a)$.*

If I is an ideal on A and J is an ideal on A/I, then $I \circ J = \{a \in A : (a/I) \in J\}$. This is an ideal on A. For any BA A, $I_{\text{at}}(A)$ is the ideal generated by the set of atoms of A.

Lemma 11.17. (Lemma 8) *If I is an ideal on A and $a \in A$, then $(I \circ I_{\mathrm{at}}(A/I)) \cap (A \upharpoonright a) = (I \upharpoonright a) \circ I_{\mathrm{at}}((A \upharpoonright a)/(I \upharpoonright a))$.*

Lemma 11.18. (Lemma 9) *If A is a BA, $a \in A$, and α is an ordinal, then $((I_\alpha(A)) \upharpoonright a) = I_\alpha(A \upharpoonright a)$.*

Lemma 11.19. (Lemma 10) *If A is a BA, $a \in A$, and α is an ordinal, then $((A/I_\alpha(A)) \upharpoonright (a/I_\alpha(A))) \cong ((A \upharpoonright a)/I_\alpha(A \upharpoonright a))$.*

For any $k \in \omega$, M_k is a finite BA with exactly k atoms; $|M_0| = 1$.

Lemma 11.20. (Lemma 11) *For any BAs A, B the following are equivalent:*
(i) $A \equiv^0 B$.
(ii) $A \cong M_0$ iff $B \cong M_0$.

Lemma 11.21. (Lemma 11) *For any BAs A, B the following are equivalent:*
(i) $A \equiv^1 B$.
(ii) The following conditions hold:
(a) $A \cong M_0$ iff $B \cong M_0$.
(b) $A \cong M_1$ iff $B \cong M_1$.

Lemma 11.22. (Lemma 11) *For any BAs A, B the following are equivalent:*
(i) $A \equiv^2 B$.
(ii) The following conditions hold:
(a) $\forall k < 4[A \cong M_k$ iff $B \cong M_k]$.
(b) A is atomless iff B is atomless.

Lemma 11.23. (Lemma 11) *For any BAs A, B the following are equivalent:*
(i) $A \equiv^3 B$.
(ii) The following conditions hold:
(a) $\forall k < 8[A \cong M_k$ iff $B \cong M_k]$.
(b) $\forall l = 1, 2, 3, 4[A$ has exactly l atoms iff B has exactly l atoms$]$.
(c) A is atomic iff B is atomic.

Lemma 11.24. (Lemma 11) *For any BAs A, B and any $k \in \omega \backslash 4$ the following are equivalent:*
(i) $A \equiv^k B$.
(ii) The following conditions hold:
(a) $\forall j < 2^k[A \cong M_j$ iff $B \cong M_j]$.
(b) $\forall j = 1, \ldots, 2^k - 4[A$ has exactly j atoms iff B has exactly j atoms$]$.
(c) A is atomic iff B is atomic.
(d) $(A/E(A)) \equiv^{k-4} (B/E(B))$.

Lemma 11.25. (Lemma 12) *For any BA A,*

$$\mathrm{sr}(A) = \min\{\alpha : \forall a, b \in A[(A, a) \equiv^\alpha (A, b) \rightarrow (A, a) \equiv^{\alpha+1} (A, b)]\}.$$

Lemma 11.26. (Lemma 13) *Suppose that* $\mathrm{sr}(A) = \alpha$. *Then* $\forall a \in A[\mathrm{sr}(A \restriction a) \leq \alpha]$.

For each $k \in \omega$ let

$$M_k = \{A : \mathrm{ch}(A) = (0, k, 0)\};$$
$$M_k^* = \{A : \mathrm{ch}(A) = (0, k, 1)\}.$$

Theorem 11.27. (Theorem 14) *Each BA of finite Scott rank belongs to one of the classes* M_k, M_k^*.

Theorem 11.28. (Theorem 14) *(i) If* $A \in M_0$ *then* $\mathrm{sr}(A) = 0$.
(ii) If $A \in M_1$ *then* $\mathrm{sr}(A) = 0$.
(iii) If $k \geq 2$ *and* $A \in M_k$ *then* $\mathrm{sr}(A) = \lfloor \log_2(k-1) \rfloor$.
(iv) If $A \in M_0^*$ *then* $\mathrm{sr}(A) = 0$.
(v) If $A \in M_1^*$ *then* $\mathrm{sr}(A) = 2$.
(vi) If $A \in M_2^*$ *then* $\mathrm{sr}(A) = 2$.
(vii) If $k \geq 3$ *and* $A \in M_k^*$ *then* $\mathrm{sr}(A) = \lfloor \log_2(k+7) \rfloor$.

Lemma 11.29. (Lemma 15) $A \equiv^\omega B$ *iff* $\mathrm{ch}(A) = \mathrm{ch}(B)$.

Lemma 11.30. (Lemma 16) *Let* A *and* B *be BAs and* α *and* γ *ordinals. Assume that* A *and* B *are* α*-atomic,* $|A/I_\alpha(A)| > 1$ *and* $|B/I_\alpha(B)| > 1$. *Then*

$$A \equiv^{\omega\cdot\alpha+\gamma} B \quad \text{iff} \quad (A/I_\alpha(A)) \equiv^\gamma (B/I_\alpha(B)).$$

Lemma 11.31. (Lemma 17) *If* A *and* B *are* α*-atomic and* $A \equiv^{\omega\cdot\alpha+\gamma} B$, *the* $(A/I_\alpha(A)) \equiv^\gamma (B/I_\alpha(B))$.

Lemma 11.32. (Lemma 17) *If* A *and* B *are* α*-atomic,* $A \equiv^{\omega\cdot\alpha+\gamma} B$, *and* $(A/I_\alpha(A)) \equiv^\beta (B/I_\alpha(B))$, *then* $A \equiv^{\omega\cdot\alpha+\beta} B$.

Lemma 11.33. (Proposition 18) *If* A *and* B *are BAs,* $\mathrm{sr}(A) \leq \alpha$, *and* $A \equiv^{\alpha+1} B$, *then* $A \cong_p B$.

Theorem 11.34. (Theorem 19) *If* A *is* α*-atomic and* $|A/I_\alpha(A)| > 1$, *then* $\mathrm{sr}(A) = \omega \cdot \alpha + \mathrm{sr}(A/I_\alpha(A))$.

Lemma 11.35. (Lemma 20) *If* $|A/I_\alpha(A)| = 1$, *then* A *is* α*-atomic.*

Lemma 11.36. (Theorem 21) *If* $o(A)$ *is a limit ordinal* α, *then* $\mathrm{sr}(A) \geq \omega \cdot \alpha$.

Lemma 11.37. (Theorem 21) *If* $o(A)$ *is a successor ordinal* $\alpha + 1$ *and* $A/I_\alpha(A)$ *has infinitely many atoms, then* $\mathrm{sr}(A) \geq \omega \cdot (\alpha + 1)$.

Lemma 11.38. (Theorem 21) *If $o(A)$ is a successor ordinal $\alpha+1$ and $A/I_\alpha(A)$ has exactly $k \in \omega$ atoms, then* $\mathrm{sr}(A) \geq \omega + \alpha + \mathrm{sr}(B)$*, where $B \in M_k$.*

Lemma 11.39. (Theorem 21) *If A is a superatomic BA, then $o(A)$ is a successor ordinal $\alpha+1$ and there is a $k \in \omega$ such that $A/I_\alpha(A)$ has exactly k atoms, and* $\mathrm{sr}(A) = \omega\cdot\alpha+\mathrm{sr}(B)$ *where $B \in M_k$.*

A BA A is *decomposable* iff $1 \in E(A)$.

Lemma 11.40. (Lemma 22) *Suppose that A and B are decomposable; say $1_A = t_A + p_A$ and $1_B = t_B + p_B$, where t_A and T_B are sums of atoms and $p_A = 0$ iff $p_B = 0$. Then*

$$A \equiv^\alpha B \quad \text{iff} \quad (A \restriction t_A) \equiv^\alpha (B \restriction t_B).$$

Theorem 11.41. (Theorem 23) *If A is a decomposable BA, with say $1 = t_A + p_A$ with t_A atomic and p_A atomless, then*

$$\mathrm{sr}(A) = \alpha \quad \text{iff} \quad \mathrm{sr}(A \restriction t_A) = \alpha.$$

Theorem 11.42. (Proposition 24) *If* $\max(\mathrm{sr}(A), \mathrm{sr}(B)) = \alpha$*, then* $\alpha \leq \mathrm{sr}(A \times B) \leq \alpha+4$.

(End of summary of Alaev 1998; begin summary of Alaev 1999.)

For $n \geq 1$ we define $B \in {}^nA$ to mean that there are pairwise disjoint $a_i \in A$ for $i < n$ such that $B \cong (A \restriction a_i)$ for all $i < n$. Let $B \in {}^\omega A$ abbreviate that for all $k \in \omega\backslash 1$ and all $a \in {}^kA$ with $a_i \cdot a_j = 0$ for $i \neq j$ we have:

(1) $\forall i < k[B \cong (A \restriction a_i)]$.

(2) $B \in {}^1(A \restriction -\sum_{i<k} a_i)$.

$B \in {}^\infty A$ abbreviates that $B \in {}^1A$ and for all $a \in A\backslash\{0,1\}$, if $(A \restriction a) \cong B$ then $A \cong (A \restriction (-a))$.

If (D, f) is a semi-tree in a BA A and $\alpha \in D$, then we define $D^{(\alpha)} = \{\beta \in {}^{<\omega}2 : \alpha\beta \in D\}$ and $f^{(\alpha)} : D^{(\alpha)} \to A$ is defined by $f^{(\alpha)}(\beta) = f(\alpha\beta)$.

Proposition 11.43. (Lemma 1) *Let A be an atomic BA. Then:*

(i) If $B_0, \ldots, B_{n-1} \in {}^\infty A$, then $B_0 \times \cdots \times B_{n-1} \in {}^\infty A$.

(ii) If $\forall i < m[B_i \in {}^1A$ and $B_i \ncong A]$, then $A \times B_0 \times \cdots \times B_{n-1} \cong A$.

(iii) If $B \in {}^\infty A$ and $b \in B$, then $(B \restriction b) \in {}^\infty A$.

(iv) If $B \in {}^n(A_0 \times \cdots \times A_{n_{k-1}})$, then there exist $m_0, \ldots, m_{k-1}$ such that $m_0 + \cdots + m_{k-1} = n$ and $\forall i < k[B \in {}^{m_i}A]$.

(v) If $\forall n \in \omega[B \in {}^nA]$, then $B \in {}^\omega A$.

(vi) If $B \in {}^\omega(A \times C)$, then $B \in {}^\omega A$ or $B \in {}^\omega C$.

(vii) If A is a countable BA and $B \in {}^\omega A$, then $B \in {}^\infty A$.

We define $\alpha \preceq \beta$ iff $\alpha, \beta \in {}^{<\omega}2$ and there is a $\gamma \in {}^{<\omega}2$ such that $\alpha = \beta\gamma$. A set $D \subseteq {}^{<\omega}2$ is a *semi-tree* iff $\emptyset \in D$ and $\forall \alpha, \beta[\alpha \in D$ and $\alpha \preceq \beta \to \beta \in D]$. A *semi-tree in a Boolean algebra* A is a pair (D, f) such that D is a semi-tree and the following conditions hold:

(1) $f(\emptyset) = 1$, and $f(\alpha) \neq 1$ for all $\alpha \in D\backslash\{\emptyset\}$.

(2) If $\alpha, \alpha 0, \alpha 1 \in D$, then $f(\alpha 0) \cdot f(\alpha 1) = 0$ and $f(\alpha)) + f(\alpha 1) = f(\alpha)$.

(3) If $\alpha \in D$, $\varepsilon \in \{0, 1\}$. $\alpha\varepsilon \in D$, and $\alpha(1 - \alpha 0) \notin D$, then $f(\alpha\varepsilon) = f(\alpha)$.

$(D.f)$ is a *generating semi-tree* in A iff it is a semi-tree in A and $\text{rng}(f)$ generates A.

A *tree* is a semi-tree such that for any $a \in {}^{<\omega}2$ and any $\varepsilon \in 2$, $\alpha\varepsilon \in D$ iff $\alpha(1-\varepsilon) \in D$.

Lemma 11.44. (Lemma 2) *(i) If $(D.f)$ is a tree in A and $\alpha \preceq \beta$, then $f(\alpha) \leq f(\beta)$.*

(ii) If $(D.f)$ is a tree in A and α and β are incomparable, then $f(\alpha) \cdot f(\beta) = 0$.

(iii) If (D, f) is a generating semi-tree, then any $a \neq 0$ in A can be written in the form $a = f(b_0) + \cdots + f(b_{n-1})$ with each $b_i \in D$ and b_i and b_j incomparable for $i \neq j$.

(iv) If D is a semi-tree, then there exist a BA A and a function $f : D \to A$ such that (D, f) is a generating semi-tree in A.

(v) If (D, f_1) and (D, f_2) are generating semi-trees in A, B respectively, then there is an isomorphism $g : A \to B$ such that $f_2 = g \circ f_1$.

Lemma 11.45. (Lemma 3) *Suppose that (D, f) is a semi-tree in a BA A, I is an ideal of A, and $f[D] \cup I$ generates A. Then:*

(i) Any element $a \in A$ can be written in the form $a = (f(\alpha_0) \cdot -b_0) + \cdots + (f(\alpha_{k-1}) \cdot -b_{k-1}) + c$, where $k \geq 0$, each $\alpha_i \in D$, each $b_i \in I$, and $c \in I$.

(ii) If $I \neq A$ and $E = \{\alpha \in D : f(\alpha) \in I\}$, then $(D\backslash I, g)$ is a semi-tree in A/I, where $g(\alpha) = f(\alpha)/I$ for all $\alpha \in D\backslash I$.

Lemma 11.46. (Lemma 4) *Suppose that (D, f) is a semitree in a BA A, I is an ideal of A, and $f[D] \cap I = \emptyset$. Then:*

(i) If $\sum_{i<m} f(\alpha_i)/I \leq \sum_{i<n} f(\beta_i)/I$ with each $\alpha_i, \beta_j \in D$, then $\sum_{i<m} f(\alpha_i) \leq \sum_{i<n} f(\beta_i)$.

(ii) If $\sum_{i<m} f(\alpha_i)/I = \sum_{i<n} f(\beta_i)/I$ with each $\alpha_i, \beta_j \in D$, then $\sum_{i<m} f(\alpha_i) = \sum_{i<n} f(\beta_i)$.

(iii)

12. Superatomic BAs

Theorem 12.1. (Abraham, Bonnet 1992) *Every superatomic BA which is embeddable in an interval algebra is embeddable in the interval algebra of some ordinal.*

The following definitions and results are essentially from Abraham, Bonnet, Kubiś, Rubin 2003.

Theorem 12.2. *Let P be a poset, and let A be a BA freely generated by P. Let I be the ideal in A generated by $\{p \cdot -q : p \leq_P q\}$.*

Suppose that B is a BA and $f : P \to B$ is such that $\forall p, q \in P[p \leq_P q$ implies that $f(p) \leq_B f(q)]$.

Then there is a homomorphism $g : A/I \to B$ such that $g(p/I) = f(p)$ for all $p \in P$. Moreover, such a homomorphism g is unique.

Proof. The function f extends to a homomorphism $f^+ : A \to B$. If $p, q \in P$ and $p \leq_P q$, then $f^+(p \cdot -q) = f(p) \cdot -f(q) = 0$. Thus $I \subseteq \ker(f^+)$, so g exists as indicated. Clearly g is unique. $\square$

Theorem 12.3. *Let P be a poset, and let $\langle a_p : p \in P\rangle$ be a system of elements of a BA C such that*

(i) $\{a_p : p \in P\}$ generates C, and

(ii) $\forall p, q \in P[p \leq_P q \to a_p \leq_C q_q]$.

and for any BA B and function $f : P \to B$ such that $\forall p, q \in P[p \leq_P q$ implies that $f(p) \leq_B f(q)]$ there is a unique homomorphism $g : C \to B$ such that $g(a_p) = f(p)$ for all $p \in P$.

Then C is isomorphic to the algebra A/I described in Theorem 12.2.

Proof. By Theorem 12.2, there is a unique homomorphism $g : A/I \to C$ such that $\forall p \in P[g(p/I) = a_p]$. Now $\forall p, q \in P[p \leq_P q \to p/I \leq q/I]$, so by the condition on C, there is a homomorphism $h : C \to A/I$ such that $\forall p \in P[h(a_p) = p/I]$. Now $\forall p \in P[h(g(p/I)) = p/I$, so $(h \circ g)(x) = x$ for all $x \in A/I$. Similarly $(g \circ h)(y) = y$ for all $y \in C$. $\square$

We call the algebra A/I of Theorem 12.2 the *free* P-algebra and denote it by $\text{free}(P)$.

A BA A is *well-generated* iff A has a subset L which is closed under $+$ and $\cdot$ such that L generates A and $(L, \leq_A)$ is well-founded.

Theorem 12.4. *For any poset P the following are equivalent:*

(i) P does not contain an infinite subset consisting of pairwise incomparable elements, and does not contain a subset isomorphic to $\mathbb{Q}$.

(ii) $\text{free}(P)$ *is superatomic.*

(iii) $\text{free}(P)$ *is well-generated.*

If P is a poset, then $X \subseteq P$ is a *final segment* of P iff $\forall p \in X \forall q \in P[p \leq q \to q \in X]$. $\text{fs}(P)$ is the set of all final segments of P.

Theorem 12.5. *For any poset P, the set* $\mathrm{fs}(P)$ *is a closed subspace of* $\mathscr{P}(P)$.

Proof. Suppose that $X \in \mathscr{P}(P)\backslash\mathrm{fs}(P)$. Then there exist $p \in X$ and $q \in P$ such that $p \le q$ but $q \notin X$. Then $X \in U_{\{p\}\{q\}} \subseteq \mathscr{P}(P)\backslash\mathrm{fs}(P)$. $\square$

Let P be a poset. For each $p \in P$ let $V_p = \{X \in \mathrm{fs}(P) : p \in X\}$. Let $\mathrm{free}'(P)$ be the subalgebra of $\mathscr{P}(\mathrm{fs}(P))$ generated by $\{V_p : p \in P\}$.

Theorem 12.6. $\mathrm{free}'(P)$ *is the set of all clopen subsets of* $\mathrm{Fs}(P)$.

Proof. If $p \in P$ and $X \in V_p$, then $X \in U_{\{p\}\emptyset} \cap \mathrm{fs}(P) \subseteq V_p$. So V_p is open in $\mathrm{fs}(P)$. If $X \in \mathrm{fs}(P)\backslash V_p$, then $X \in U_{\emptyset\{p\}} \cap \mathrm{fs}(P) \subseteq \mathrm{fs}(P)\backslash V_p$. Thus V_p is clopen in $\mathrm{fs}(P)$. It follows that $\mathrm{free}'(P)$ is a subset of the collection of all clopen subsets of $\mathrm{fs}(P)$.

Now suppose that W is any clopen subset of $\mathrm{fs}(P)$. Note that if F, G are finite disjoint subsets of $\mathrm{fs}(P)$, then

$$(*) \qquad \bigcap_{p\in F} V_p \cap \bigcap_{q\in G}(\mathrm{fs}(P)\backslash V_q) = \{W \in \mathrm{fs}(P) : F \subseteq W \text{ and } G \cap W = \emptyset\} = U_{FG} \cap \mathrm{fs}(P).$$

Now since W is open, there is a set $\mathscr{H}$ of pairs (F, G) with F, G finite and disjoint subsets of $\mathrm{fs}(P)$ such that $W = \bigcup_{(F,G)\in\mathscr{H}} U_{FG}$; hence by $(*)$,

$$W = \bigcup_{(F,G)\in\mathscr{H}} \left(\bigcap_{p\in F} V_p \cap \bigcap_{q\in G}(\mathrm{fs}(P)\backslash V_q)\right).$$

Now compactness of W shows that $W \in \mathrm{free}'(P)$. $\square$

Theorem 12.7. *Suppose that F, G are finite disjoint subsets of P. Then*

$$\bigcap_{p\in F} V_p \cap \bigcap_{q\in G}(\mathrm{fs}(P)\backslash V_q) = \emptyset \quad \textit{iff} \quad \exists p \in F \exists q \in G[p \le_P q].$$

Proof. $\leftarrow$: clear. $\Rightarrow$: Suppose that $\bigcap_{p\in F} V_p \cap \bigcap_{q\in G}(\mathrm{fs}(P)\backslash V_q) = \emptyset$. Let $X = \{r \in P : \exists p \in F[p \le_P r]\}$. Then $X \in \bigcap_{p\in F} V_p$, so there is a $q \in G$ such that $X \in V_q$. So $q \in X$, hence there is a $p \in F$ such that $p \le_P q$. $\square$

Theorem 12.8. *For any poset P,* $\mathrm{free}(P) \cong \mathrm{free}'(P)$.

Proof. If $p \le q$, then $V_p \subseteq V_q$. Hence by Theorem 12.2 there is an epimorphism $g : \mathrm{free}(P) \to \mathrm{free}'(P)$ such that $g(p/I) = V_p$ for all $p \in P$. Suppose that F, G are disjoint finite subsets of P and $g(\prod_{p\in F}(p/I) \cdot \prod_{q\in G} -(q/I)) = 0$. Thus $\bigcap_{p\in F} V_p \cap \prod_{q\in G}(\mathrm{fs}(P)\backslash V_q) = \emptyset$. By Theorem 12.7, there exist $p \in F$ and $q \in G$ such that $p \le_P q$. So $(p/I) \le (q/I)$, and hence $\prod_{p\in F}(p/I) \cdot \prod_{q\in G} -(q/I) = 0$. $\square$

The following comes from an unpublished paper of Bonnet, Rubin, Si-Kadoor.

If $f : A \to B$ is a homomorphism, we denote by f^d the dual of f, mapping $\mathrm{Ult}(B)$ into $\mathrm{Ult}(A)$, defined by $f^d(F) = f^{-1}[F]$. This is a continuous map. If I is an ideal of A, then $I^d = \bigcup_{a\in I} \mathcal{S}(a)$; this is an open set in $\mathrm{Ult}(A)$.

Proposition 12.9. *If $G \in \mathrm{Ult}(A)$, then $G \notin I^d$ iff $\pi[G]$ is an ultrafilter on A/I, where $\pi : A \to A/I$ is the natural map.*

Proof. Assume that $G \in \mathrm{Ult}(A)$

First suppose that $G \notin I^d$. If $a, b \in G$ then $\pi(a)\cdot\pi(b) = \pi(a\cdot b) \in \pi[G]$. If $a \in G$ and $\pi(a) \le \pi(b)$, then $\pi(a\cdot -b) = 0$, so $a\cdot -b \in I$, hence $G \notin \mathcal{S}(a\cdot -b)$, so $a\cdot -b \notin G$, hence $b \in G$. For any $a \in A$, either $a \in G$ and hence $\pi(a) \in \pi[G]$, or $-a \in G$, hence $-\pi(a) \in G$. $0 \notin \pi[G]$, as otherwise there is an $a \in G$ with $\pi(a) = 0$, hence $a \in I$, hence $\mathcal{S}(a) \subseteq I^d$, and $G \in \mathcal{S}(a)$, contradiction. All this shows that $\pi[G]$ is an ultrafilter on A/I.

Second suppose that $\pi[G]$ is an ultrafilter on A/I. If $G \in I^d$, say $G \in \mathcal{S}(a)$ with $a \in I$. Then $a \in G$. But $\pi(-a) = 1 \in \pi[G]$, so there is a $b \in G$ such that $\pi(-a) = \pi(b)$. Then $-\pi(a) + -\pi(b) = \pi(-a + -b) = 1 \in G$ and $\pi(a) \in \pi[G]$, so $-\pi(b) \in \pi[G]$. But also $\pi(b) \in \pi[G]$, contradiction. □

Proposition 12.10. *Let I be an ideal of A, and let $\pi : A \to A/I$ be the natural homomorphism. Then π^d is a homeomorphism from $\mathrm{Ult}(A/I)$ onto the closed subset $\mathrm{Ult}(A)\backslash I^d$ of $\mathrm{Ult}(A)$.*

Proof. Since π is a surjection, it follows that π^d is an injection; so we just need to show that the range of π^d is $\mathrm{Ult}(A)\backslash I^d$.

$\mathrm{rng}(\pi^d) \subseteq (\mathrm{Ult}(A)\backslash I^d)$: Suppose that $F \in \mathrm{Ult}(A/I)$. For each $a \in I$ we have $\pi(-a) = -a/I = 1 \in F$, so $-a \in \pi^{-1}[F] = \pi^d(F)$. Hence $\pi^d(F) \in \mathcal{S}(-a)$. Hence $\pi^d(F) \in \bigcap_{a\in I} \mathcal{S}(-a) = \mathrm{Ult}(A)\backslash I^d$, as desired.

$\mathrm{Ult}(A)\backslash I^d \subseteq \mathrm{rng}(\pi^d)$: Suppose that $G \in (\mathrm{Ult}(A)\backslash I^d)$. By Proposition 1, $\pi[G]$ is an ultrafilter on A/I. We have $\pi^d(\pi[G]) = \pi^{-1}[\pi[G]]$. Now $\pi^{-1}[\pi[G]] = G$ (as desired). In fact, if $a \in G$ then $\pi(a) \in \pi[G]$, hence $a \in \pi^{-1}[\pi[G]]$. Thus $G \subseteq \pi^{-1}[\pi[G]]$, so $G = \pi^{-1}[\pi[G]]$. □

For all $a \in A$ let $\mathscr{F}_a$ be the filter generated by a.

Proposition 12.11. *a is an atom iff $\mathscr{F}_a$ is an ultrafilter.* □

Proposition 12.12. *For any ultrafilter F, F is principal iff F is isolated in $\mathrm{Ult}(A)$.* □

Proposition 12.13. *$\mathcal{S}(a)$ is a singleton iff a is an atom.* □

We define the standard sequence of ideals $\langle I_\alpha(A) : \alpha \in \mathbf{On}\rangle$ of a BA A:

$$\begin{aligned} I_0(A) &= \{0\};\\ I_{\alpha+1}(A) &= \{a \in A : a/I_\alpha(A) \text{ is a finite sum of atoms of } A/I_\alpha(A)\};\\ I_\lambda(A) &= \bigcup_{\alpha<\lambda} I_\alpha(A) \quad \text{for } \lambda \text{ limit.}\end{aligned}$$

Proposition 12.14. $A \cong A/I_0(A)$. $\square$

Let $I_{\rm at}(A)$ be the ideal of A generated by the atoms of A.

Proposition 12.15. $A/I_{\alpha+1}(A) \cong (A/I_\alpha(A))/I_{\rm at}(A/I_\alpha(A))$.

Proof. For each $a \in A$ let $f(a) = a/I_\alpha(A)$, and for each $b \in A/I_\alpha(A)$ let $g(b) = b/I_{\rm at}(A/I_\alpha(A))$. Thus $g \circ f : A \to (A/I_\alpha(A))/I_{\rm at}(A/I_\alpha(A))$ is a surjection. Its kernel is $I_{\alpha+1}(A)$, and the proposition follows. $\square$

If X is a topological space, then $\mathrm{is}(X)$ is the set of all isolated points of X; it is an open subset of X. Then we set $X' = X\backslash \mathrm{is}(X)$; further,

$$\begin{aligned} X^{(0)} &= X; \\ X^{(\alpha+1)} &= (X^{(\alpha)})'; \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

Proposition 12.16. $(I_{\rm at}(A))^d$ *is the set of all isolated points of* $\mathrm{Ult}(A)$.

Proof. Suppose that $G \in (I_{\rm at}(A))^d$. Choose $a \in I_{\rm at}(A)$ such that $G \in \mathcal{S}(a)$. Then $a \in G$. There is a finite collection F of atoms of A such that $a \leq \sum F$. Thus $\sum F \in G$, so there is an $x \in F$ such that $x \in G$. So G is principal, and hence by Proposition 4 it is isolated in $\mathrm{Ult}(A)$.

Conversely, suppose that G is isolated in $\mathrm{Ult}(A)$. By Proposition 4, it is principal; say it is generated by the atom a. Then $G \in \mathcal{S}(a)$, so $G \in (I_{\rm at}(A))^d$. $\square$

Proposition 12.17. $\mathrm{Ult}(A/I_{\rm at}(A))$ *is homeomorphic to* $(\mathrm{Ult}(A))'$.

Proof. By Proposition 2, $\mathrm{Ult}(A/I_{\rm at}(A))$ is homeomorphic to $\mathrm{Ult}(A)\backslash(I_{\rm at}(A))^d$. So our proposition follows from Proposition 12.16. $\square$

Proposition 12.18. $(I_{\alpha+1}(A))^d = (I_\alpha(A))^d \cup \bigcup\{\mathcal{S}(a) : a/I_\alpha(A)$ *is an atom*$\}$.

Proof. First suppose that $F \in (I_{\alpha+1}(A))^d$. Choose $a \in I_{\alpha+1}(A)$ such that $F \in \mathcal{S}(a)$. Say $a/I_\alpha(A) = \sum_{b\in M} b/I_\alpha(A)$ with M a finite subset of A and each $b/I_\alpha(A)$ an atom of $A/I_\alpha(A)$. Then $a \cdot \prod_{b\in M} -b \in I_\alpha(A)$.

Case 1 $\forall b \in M[-b \in F]$. Then $a \cdot \prod_{b\in M} -b \in F$, and F is in the right side.

Case 2. There is a $b \in M$ such that $b \in F$. Then $F \in \mathcal{S}(b)$ and $b/I_\alpha(A$ is an atom, so again F is in the right side.

Second suppose that $F \in (I_\alpha(A))^d$. Say $F \in \mathcal{S}(a)$ with $a \in I_\alpha(A)$. Since clearly $I_\alpha(A) \subseteq I_{\alpha+1}(A)$, F is in the left side.

Finally, suppose that $F \in \mathcal{S}(a)$ with $a/I_\alpha(A)$ an atom. Then $a \in I_{\alpha+1}(A)$ and again F is in the left side. $\square$

Proposition 12.19. $(I_\alpha(A))^d = \mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^\alpha$.

Proof. By induction on α. The case $\alpha = 0$ is obvious. Now we assume the condition for α and prove it for $\alpha+1$. By Proposition 12.18 and the inductive hypothesis,

$$\begin{aligned}
(I_{\alpha+1}(A))^d &= (I_\alpha(A))^d \cup \bigcup\{\mathcal{S}(a) : a/I_\alpha(A) \text{ is an atom}\} \\
&= (\mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^\alpha) \cup \bigcup\{\mathcal{S}(a) : a/I_\alpha(A) \text{ is an atom}\} \\
\mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^{(\alpha+1)} &= \mathrm{Ult}(A)\backslash((\mathrm{Ult}(A))^{(\alpha)})'
\end{aligned}$$

Suppose that $F \in (I_{\alpha+1}(A))^d$

Case 12.18 $F \in (\mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^\alpha)$. Since $((\mathrm{Ult}(A))^{(\alpha)})' \subseteq (\mathrm{Ult}(A))^{(\alpha)}$, it follows that $F \in \mathrm{Ult}(A)\backslash((\mathrm{Ult}(A))^{(\alpha)})'$.

Case 2. $F \notin (I_\alpha(A))^d$, but $F \in \mathcal{S}(a)$ for some a such that $a/I_\alpha(A)$ is an atom. Thus $a \in F$. Let $\pi : A \to A/I_\alpha(A)$ be the natural map. Then by Proposition 2, π^d is a homeomorphism from $\mathrm{Ult}(A/I_\alpha(A))$ onto $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d$, Say $\pi^d(G) = F$. Thus $\pi^{-1}[G] = F$. Since $a \in F$, we have $\pi(a) \in G$, i.e. $a/I_\alpha(A) \in G$. Thus G is isolated in $\mathrm{Ult}(A/I_\alpha(A))$, so F is isolated in $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d$. Now $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d = (\mathrm{Ult}(A))^{(\alpha)}$ by the inductive hypothesis, so $F \in \mathrm{is}((\mathrm{Ult}(A))^{(\alpha)})$ and hence $F \notin (\mathrm{Ult}(A))^{(\alpha)})'$.

Now suppose conversely that $F \in \mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^{(\alpha+1)}$, and $F \in (\mathrm{Ult}(A))^{(\alpha)}$; we want to find an $a \in A$ such that $a/I_\alpha(A)$ is an atom and $F \in \mathcal{S}(a)$. Now $(\mathrm{Ult}(A))^{(\alpha+1)} = ((\mathrm{Ult}(A))^{(\alpha)})'$, and $((\mathrm{Ult}(A))^{(\alpha)})' = (\mathrm{Ult}(A))^{(\alpha)}\backslash\mathrm{is}(\mathrm{Ult}(A))^{(\alpha)})$, so $F \in \mathrm{is}(\mathrm{Ult}(A))^{(\alpha)})$. Let $\pi : A \to A/I_\alpha(A)$ be the natural map. Then by Proposition 12.10, π^d is a homeomorphism from $\mathrm{Ult}(A/I_\alpha(A))$ onto $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d$. Now $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d = (\mathrm{Ult}(A))^{(\alpha)}$ by the inductive hypothesis. Hence there is an isolated point G of $\mathrm{Ult}(A/I_\alpha(A))$ such that $\pi^d(G) = F$. Say G is determined by the atom $a/I_\alpha(A)$ of $A/I_\alpha(A)$. Then $a \in \pi^{-1}[G] = \pi^d(G) = F$, as desired.

Now assume that λ is limit, and $\forall\alpha < \lambda[(I_\alpha(A))^d = \mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^\alpha]$, Then

$$\begin{aligned}
(I_\lambda(A))^d &= \left(\bigcup_{\alpha<\lambda} I_\alpha(A)\right)^d = \bigcup\left\{\mathcal{S}(a) : a \in \bigcup_{\alpha<\lambda} I_\alpha(A)\right\} \\
&= \bigcup_{\alpha<\lambda}\bigcup\{\mathcal{S}(a) : a \in I_\alpha(A)\} = \bigcup_{\alpha<\lambda}(I_\alpha(A))^d \\
&= \bigcup_{\alpha<\lambda}[\mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^{(\alpha)}] = \mathrm{Ult}(A)\backslash\bigcap_{\alpha<\lambda}(\mathrm{Ult}(A))^{(\alpha)} \\
&= \mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^{(\lambda)}.
\end{aligned}$$

□

Corollary 12.20. $(\mathrm{Ult}(A))^{(\alpha)} = \mathrm{Ult}(A)\backslash(I_\alpha(A))^d$. □

Proposition 12.21. $\mathrm{Ult}(A/I_\alpha(A))$ *is homeomorphic to* $(\mathrm{Ult}(A))^{(\alpha)}$.

Proof. By Corollary 12.20 we have $(\mathrm{Ult}(A))^{(\alpha)} = \mathrm{Ult}(A)\backslash(I_\alpha(A))^d$. By Proposition 2, $\mathrm{Ult}(A)/I_\alpha(A)$ is homeomorphic to $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d$. □

A subspace Y of a space S is *dense in itself* iff it has no isolated points in the relative topology. A space X is *scattered* iff it has no nonempty dense in itself subspace.

Proposition 12.22. *A is superatomic iff* $\mathrm{Ult}(A)$ *is scattered.*

Proof. $\Rightarrow$: Suppose that $\emptyset \neq Y \subseteq \mathrm{Ult}(A)$ is dense in itself. Define $f(a) = \mathcal{S}(a) \cap Y$ for all $a \in A$. Thus f is a homomorphism, so $f[B]$ is atomic. Let a be such that $f(a)$ is an atom. Since Y is dense in itself, there are distinct $F, G \in (\mathcal{S}(a) \cap y)$. Choose $b \in F \backslash G$. Then $a \cdot b \in F$, and $a \cdot -b \in G$. So

$$f(a) = \mathcal{S}(a) \cap Y = (\mathcal{S}(a \cdot b) \cap Y) \cup (\mathcal{S}(a \cdot -b) \cap Y) = f(a \cdot b) \cup f(a \cdot -b),$$

contradicting $f(a)$ being an atom.

$\leftarrow$: Suppose that A is not superatomic. Let f be a homomorphism from A onto an atomless BA B. Then f^d is a homeomorphism from $\mathrm{Ult}(B)$ onto a subspace of $\mathrm{Ult}(A)$. $\mathrm{Ult}(B)$ has no isolated points by Proposition 8. So A is not scattered. $\square$

Proposition 12.23. *A is superatomic iff there is an α such that $I_\alpha(A) = A$. If A is superatomic, then the least α such that $I_\alpha(A) = A$ is a successor ordinal $\beta + 1$, and $A/I_\beta(A)$ is finite.*

Proof. $\Rightarrow$: Suppose that A is superatomic. Let α be minimum such that $I_\alpha(A) = I_{\alpha+1}(A)$. By the definition of $I_\alpha(A)$, it follows that $A/I_\alpha(A)$ does not have any atoms; so $|A/I_\alpha(A)| = 1$ or $A/I_\alpha(A)$ is atomless. Since A is superatomic, it follows that $|A/I_\alpha(A)| = 1$. Then α cannot be a limit ordinal, since $1 \notin I_\beta(A)$ for all $\beta < \alpha$. So $\alpha = 0$ and $|A| = 1$, or α is a successor ordinal $\beta + 1$ and $A/I_\beta(A)$ is finite.

$\Leftarrow$: Suppose that A is not superatomic. Let $f : A \to A/I$ be a homomorphism such that A/I is atomless. Then $f^d[A/I]$ is a dense in itself subspace of $\mathrm{Ult}(A)$. By induction, $f^d[A/I] \subseteq (\mathrm{Ult}(A))^{(\alpha)}$ for all α, and so there is no α such that $I_\alpha(A) = A$. $\square$

If A is superatomic, then the *rank* of A is $\mathrm{rk}(A)$ =least $\alpha[I_{\alpha+1}(A) = A]$. The *cardinal sequence* of A is $\mathrm{CS}(A) = \langle |\mathrm{At}(A/I_\alpha(A)| : \alpha < \mathrm{rk}(A) \rangle$. Thus if A is infinite, then this is a sequence of infinite cardinals.

Proposition 12.24. $|\mathrm{At}(A)| = |\mathrm{is}(\mathrm{Ult}(A))|$.

Proof. We claim that $\mathscr{F}$ is a bijection from $\mathrm{At}(A)$ onto $\mathrm{is}(\mathrm{Ult}(A))$; see the definition preceding Proposition 12.112. If a is an atom of A, then $\mathscr{F}_a$ is an isolated ultrafilter by Propositions 12.11 and 12.12. If F is isolated, then by Proposition 12.12 F is principal; hence there is an atom a such that $\mathscr{F}_a = F$. So $\mathscr{F}$ maps $\mathrm{At}(A)$ onto $(\mathrm{Ult}(A))$. Clearly it is one-one. $\square$

Proposition 12.25. *If A is superatomic and $\alpha < \mathrm{rk}(A)$, then*

$$|\mathrm{At}(A/I_\alpha(A))| = |\mathrm{is}((\mathrm{Ult}(A))^{(\alpha)})|.$$

Proof. For $a/I_\alpha(A)$ an atom of $A/I_\alpha(A)$ let $g(a/I_\alpha(A)) = f^d(\mathscr{F}_{a/I_\alpha(A)})$, where $f : A \to A/I_\alpha(A)$ is the natural map. Thus $\mathscr{F}_{a/I_\alpha(A)}$ is an isolated point of $\mathrm{Ult}(A/I_\alpha(A))$ by Propositions 12.11 and 12.12, and so $f^d(\mathscr{F}_{a/I_\alpha(A)})$ is an isolated point of $\mathrm{Ult}(A)\backslash(I_\alpha(A))^d = (\mathrm{Ult}(A))^{(\alpha)}$ by Propositions 12.10 and 12.20. Thus g maps $\mathrm{At}(A/I_\alpha(A))$ into $\mathrm{is}((\mathrm{Ult}(A))^{(\alpha)})$. g is one-one since f is onto.

Conversely, if G is an isolated point of $(\mathrm{Ult}(A))^{(\alpha)}$, then $f^{d-1}(G)$ is an isolated point of $\mathrm{Ult}(A/I_\alpha(A))$, and so there is an atom $a/I_\alpha(A)$ of $A/I_\alpha(A)$ such that $f^{d-1}(G) = \mathscr{F}_{a/I_\alpha(A)}$. Then $g(a/I_\alpha(A)) = f^d(\mathscr{F}_{a/I_\alpha(A)}) = G$. □

Note by Proposition 12.21 that $\mathrm{rk}(A)$ =least $\alpha[X^{(\alpha+1)} = \emptyset]$. By Proposition 12.25, $\mathrm{CS}(A) = \langle|\mathrm{is}((\mathrm{Ult}(A))^{(\alpha)})| : \alpha < \mathrm{rk}(A)\rangle$.

Proposition 12.26. *A is not superatomic iff there is an α such that $(\mathrm{Ult}(A))^{(\alpha)}$ is dense in itself.*

Proof. ⇒: Suppose that A is not superatomic. Let α be minimum such that $I_\alpha(A) = I_{\alpha+1}(A)$. By Corollary 12.20, $(\mathrm{Ult}(A))^{(\alpha)} = (\mathrm{Ult}(A))^{(\alpha+1)}$. Hence by definition, $\mathrm{is}((\mathrm{Ult}(A))^{(\alpha)}) = \emptyset$, so $(\mathrm{Ult}(A))^{(\alpha)}$ is dense in itself.

⇐: Assume that $(\mathrm{Ult}(A))^{(\alpha)}$ is dense in itself. Now $(\mathrm{Ult}(A))^{(\alpha)}$ is a subspace of $\mathrm{Ult}(A)$, so $\mathrm{Ult}(A)$ is not scattered. By Proposition 14, A is not superatomic. □

Proposition 12.27. *A is superatomic iff there is an ordinal α such that $(\mathrm{Ult}(A))^{(\alpha)} = \emptyset$.*

Proof. ⇒: suppose that A is superatomic. By Proposition 12.23 there is an α such that $I_\alpha(A) = A$. By Proposition 12.21 we then have $(\mathrm{Ult}(A))^{(\alpha)} = \emptyset$.

⇐: suppose that $(\mathrm{Ult}(A))^{(\alpha)} = \emptyset$. By Proposition 13, $I_\alpha(A) = A$. By Proposition 12.23, A is superatomic. □

Proposition 12.28. *Let F be an ultrafilter on a nontrivial superatomic BA A. Then there is an α such that $F \notin (\mathrm{Ult}(A))^{(\alpha)}$. The least such α is a successor ordinal.*

Proof. The first statement holds by Proposition 12.27. Now let α be minimum such that $F \notin (\mathrm{Ult}(A))^{(\alpha)}$. Since A is nontrivial, $\alpha \neq 0$. Clearly α is not a limit ordinal. □

If A is a nontrivial superatomic BA and $F \in \mathrm{Ult}(A)$, then the *rank* of F is $\mathrm{rk}(F) = \min\{\alpha : F \notin (\mathrm{Ult}(A))^{(\alpha+1)}\}$.

Proposition 12.29. *Suppose that A is a nontrivial superatomic BA and $F \in \mathrm{Ult}(A)$. Then $\mathrm{rk}(F) = \alpha$ iff the following condition holds:*

$$F \in (\mathrm{Ult}(A))^{(\alpha)} \text{ and } \exists a \in F[a/I_\alpha(A) \text{ is an atom}].$$

Proof. ⇒: Assume that $\mathrm{rk}(F) = \alpha$. By definition we have $F \in \mathrm{Ult}(A)\backslash(\mathrm{Ult}(A))^{(\alpha+1)}$. Hence by Proposition 12.19 we have $F \in (I_{\alpha+1}(A))^d$. So, choose $a \in I_{\alpha+1}(A)$ such that $F \in \mathcal{S}(a)$. By definition there is a finite subset G of A such that $a/I_\alpha(A) = \sum_{b\in G} b/I_\alpha(A)$ with each $b/I_\alpha(A)$ an atom. Thus $a \cdot \prod_{b\in G} -b \in I_\alpha(A)$. Now $F \in (\mathrm{Ult}(A))^{(\alpha)}$, so by Proposition 12.19, $F \notin (I_\alpha(A))^d$. It follows that $a \cdot \prod_{b\in G} -b \notin F$, so $-a + \sum_{b\in G} b \in F$. Hence there is a $b \in G$ such that $b \in F$, as desired.

⇐: Assume the indicated condition. Then $a \in I_{\alpha+1}(A)$, so $F \in (I_{\alpha+1}(A))^d$. Hence $F \notin (\mathrm{Ult}(A))^{(\alpha+1)}$ by Proposition 12.19. Since $F \in (\mathrm{Ult}(A))^{(\alpha)}$, it follows that $\mathrm{rk}(F) = \alpha$. □

If A is a nontrivial superatomic BA and $a \in A$, then there is a least ordinal α such that $a \in I_\alpha(A)$. If $a \neq 0$, then this ordinal is a successor ordinal $\beta + 1$, and we define the *rank* of A, $\text{rk}(a)$ to be β.

Proposition 12.30. *If A is a nontrivial superatomic BA, F is an ultrafilter on A, and $\text{rk}(F) = \alpha$, then $\text{rk}(a) \geq \alpha$ for all $a \in F$.*

Proof. We have $F \in (\text{Ult}(A))^{(\alpha)} \backslash (\text{Ult}(A))^{(\alpha+1)}$. Suppose that $a \in F$ and $\text{rk}(a) < \alpha$; say $\text{rk}(a) = \beta$. Thus $a \in I_{\beta+1}(A) \backslash I_\beta(A)$. Say $a/I_\beta(A) = b_1/I_\beta(A) + \cdots - b_n/I_\beta(A)$, with each $b_i/I_\beta(A)$ an atom. Then $a \cdot -b_1 \cdot \ldots \cdot -b_n \in I_\beta(A)$. Now $F \in (\text{Ult}(A))^{(\alpha)} \subseteq (\text{Ult}(A))^{(\beta)}$, so $F \notin (I_\beta(A))^d$ by Corollary 12.20. Thus $a \cdot -b_1 \cdot \ldots \cdot -b_n \notin F$, so $-a + b_1 + \cdots + b_n \in F$. Hence $b_i \in F$ for some i. Then $\text{rk}(F) = \beta$ by Proposition 19, contradiction. □

Proposition 12.31. *If X is an open subset of $\text{Ult}(A)$, then $X^{(\alpha)} = X \cap (\text{Ult}(A))^{(\alpha)}$.*

Proof. We use induction on α. It is obvious for $\alpha = 0$. For α limit,

$$X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)} = \bigcap_{\beta<\alpha} (X \cap (\text{Ult}(A))^{(\beta)}) = X \cap \bigcap_{\beta<\alpha} (\text{Ult}(A))^{(\beta)} = X \cap (\text{Ult}(A))^{(\alpha)}.$$

Now assume $X^{(\alpha)} = X \cap (\text{Ult}(A))^{(\alpha)}$. Suppose that $F \in X^{(\alpha+1)}$ while $F \notin (\text{Ult}(A))^{(\alpha+1)}$. Now $F \in X^{(\alpha)}$, so $F \in (\text{Ult}(A))^{(\alpha)}$ by the inductive hypothesis. Hence F is an isolated point of $(\text{Ult}(A))^{(\alpha)}$. Thus $\{F\}$ is a clopen subset of $(\text{Ult}(A))^{(\alpha)}$. Say U is open in $(\text{Ult}(A))^{(\alpha)}$ and $U \cap (\text{Ult}(A))^{(\alpha)} = \{F\}$. Thus $U \cap X^{(\alpha)} = U \cap X \cap (\text{Ult}(A))^{(\alpha)} = \{F\}$. Thus F is isolated in $X^{(\alpha)}$, contradiction,

Conversely, suppose $F \in X \cap (\text{Ult}(A))^{(\alpha+1)}$ but $F \notin X^{(\alpha+1)}$. Now $F \in (\text{Ult}(A))^{(\alpha)}$, so $F \in X \cap (\text{Ult}(A))^{(\alpha)}$, hence by the inductive hypothesis, $F \in X^{(\alpha)}$. So F is an isolated point of $X^{(\alpha)}$. Thus $\{F\}$ is a clopen subset of $X^{(\alpha)}$, hence by the inductive hypothesis also of $X \cap (\text{Ult}(A))^{\alpha)}$. Say U is open in $(\text{Ult}(A))^{(\alpha)}$ and $U \cap X \cap (\text{Ult}(A))^{\alpha)} = \{F\}$. Since $U \cap X$ is open in $(\text{Ult}(A))^{(\alpha)}$, this shows that F is isolated in $(\text{Ult}(A))^{(\alpha)}$, contradiction. □

The following is from Bagaria 2002.

If $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$ is a sequence of infinite cardinals, then a θ-poset is a poset $(T, \leq)$ such that the following conditions hold:

(1) $T = \bigcup_{\alpha<\lambda} S_\alpha$, where each S_α has the form $\{\alpha\} \times Y_\alpha$, with Y_α of size κ_α.

(2) For all distinct $s, t \in T$ there is a finite $i(\{s, t\}) \subseteq T$ such that
   (a) $\forall u \in i(\{s, t\})[u \leq s, t]$;
   (b) $\forall u \leq s, t \exists v \in i(\{s, t\})[u \leq v]$.

(3) $\forall s \in S_\alpha \forall t \in S_\beta [s < t \rightarrow \alpha < \beta]$.

(4) $\forall \alpha, \beta \forall t [\alpha < \beta < \lambda \wedge t \in S_\beta \rightarrow \{s \in S_\alpha : s < t\}$ is infinite].

A topological space X is *locally compact* iff $\forall x \in X \exists$open $U[x \in U$ and $\overline{U}$ is compact]. For X locally compact, its *one-point compactification* is $X \cup \{\Omega\}$, where a subset V of $X \cup \{\Omega\}$ is called open iff it is an open subset of X, or else has the form $\{\omega\} \cup (X \backslash F)$ with F closed and compact in X.

Proposition 12.32. *The definition does give a topology on $X \cup \{\Omega\}$.*

Proof. Clearly $\emptyset$ and $X \cup \{\Omega\}$ are open. Suppose that V_1 and V_2 are open. If one of them is a subset of X, clearly $U_1 \cap U_2 \subseteq X$ and $U_1 \cap U_2$ is open in X, hence in $X \cup \{\Omega\}$. Suppose that $V_1 = \{\Omega\} \cup (X \backslash F_1)$ and $V_2 = \{\Omega\} \cup (X \backslash F_2)$ with F_1 and F_2 closed and compact in X. Then $V_1 \cap V_2 = \{\Omega\} \cup (X \backslash (F_1 \cup F_2))$, and $F_1 \cup F_2$ is closed and compact in X. So $V_1 \cap V_2$ is open.

Now suppose that $\mathscr{A}$ is a family of open subsets of $X \cup \{\Omega\}$. If Ω is not in the union, then the union is open in X, and hence in $X \cup \{\Omega\}$. If Ω is in the union, say $\{\omega\} \cup (X \backslash F) \in \mathscr{A}$ with F closed and compact in X. Then $\bigcup \mathscr{A}$ has the form $\{\Omega\} \cup W$ with $W \subseteq X$ and W open, so $\bigcup \mathscr{A} = \{\Omega\} \cup (X \backslash (X \backslash W))$, and $X \backslash W \backslash F$, so $X \backslash W$ is closed and compact in X. $\square$

Proposition 12.33. *$X \cup \{\Omega\}$ is compact.*

Proof. Suppose that $\mathscr{O}$ is an open cover of $X \cup \{\Omega\}$. Choose $V \in \mathscr{O}$ such that $\omega \in V$. Say $V = \{\Omega\} \cup (X \backslash F)$, where F is closed and compact in X. Let $\mathscr{O}'$ be a finite subset of $\mathscr{O}$ such that $F \subseteq \bigcup \mathscr{O}'$. Then $\mathscr{O}' \cup \{V\}$ is a finite subset of $\mathscr{O}$ which covers $X \cup \{\Omega\}$. $\square$

Proposition 12.34. *If X is compact, then Ω is isolated in $X \cup \{\Omega\}$.*

Proof. $\{\Omega\} \cup (X \backslash X)$ is open. $\square$

Proposition 12.35. *If X is locally compact, then X is Hausdorff iff $X \cup \{\Omega\}$ is Hausdorff.*

Proof. $\Rightarrow$: Assume that X is Hausdorff, and suppose that x, y are distinct members of $X \cup \{\Omega\}$. If $x, y \in X$, then the conclusion is clear. Suppose that $x = \Omega$ and $y \neq \omega$. Let U be open with $y \in U$ and $\overline{U}$ compact. Then $\{\Omega\} \cup (X \backslash \overline{U})$ and U are disjoint neighborhoods of Ω, y respectively.

$\Leftarrow$: Assume that $X \cup \{\Omega\}$ is Hausdorff and x, y are distinct members of X. Let V, W be disjoint open neighborhoods of x, y respectively, in $X \cup \{\Omega\}$. Then $V \backslash \{\Omega\}$ and $W \backslash \{\Omega\}$ are disjoint open neighborhoods of x, y respectively, in X. $\square$

Proposition 12.36. *If X is locally compact but not compact, then* $\mathrm{is}(X) = \mathrm{is}(X \cup \{\Omega\})$.

Proof. $\subseteq$ is clear. Now suppose that $x \in \mathrm{is}(X \cup \{\Omega\})$. If $x = \Omega$, then X is compact, contradiction. So $x \neq \Omega$. Let V be open in $X \cup \{\Omega\}$ with $V = \{x\}$. Then V must be open in X. $\square$

Proposition 12.37. *If X is locally compact but not compact, and $Y = X \cup \{\Omega\}$, then for any α,*

(i) if $\Omega \in Y^{(\alpha+1)}$, then $\mathrm{is}(Y^{(\alpha)}) = \mathrm{is}(X^{(\alpha)})$ *and $Y^{(\alpha+1)} = X^{(\alpha+1)} \cup \{\Omega\}$;*
(ii) if $\Omega \in Y^{(\alpha)} \backslash Y^{(\alpha+1)}$, then $Y^{(\alpha+1)} = X^{(\alpha+1)}$;
(iii) if $\Omega \notin Y^{(\alpha)}$, then $Y^{(\alpha+1)} = X^{(\alpha+1)}$.

Proof. By induction on α. For $\alpha = 0$ we have $\Omega \notin \mathrm{is}(Y)$, and $\mathrm{is}(Y^{(0)}) = \mathrm{is}(X^{(0)})$ by Proposition 12.36. Then

$$Y^{(1)} = Y \backslash \mathrm{is}(Y^{(0)}) = (X \cup \{\Omega\}) \backslash \mathrm{is}(X^{(0)}) = (X \backslash \mathrm{is}(X^{0}) \cup \{\Omega\} = X^{(1)} \cup \{\Omega\}.$$

This proves (i) for $\alpha = 0$. (ii) and (iii) hold vacuously for $\alpha = 0$.

Now assume (i)–(iii) for α. For (i) for $\alpha+1$, assume that $\Omega \in Y^{(\alpha+2)}$. So $\Omega \in Y^{(\alpha+1)}$, and hence $Y^{(\alpha+1)} = X^{(\alpha+1)} \cup \{\Omega\}$ by (i) for α. It follows that $\mathrm{is}(Y^{(\alpha+1)}) = \mathrm{is}(X^{(\alpha+1)})$. Also,

$$\begin{aligned} Y^{(\alpha+2)} &= Y^{(\alpha+1)} \backslash \mathrm{is}(Y^{(\alpha+1)}) \\ &= (X^{(\alpha+1)} \cup \{\Omega\}) \backslash \mathrm{is}(X^{(\alpha+1)} \\ &= (X^{(\alpha+1)} \backslash \mathrm{is}(X^{(\alpha+1)}) \cup \{\Omega\} \\ &= X^{(\alpha+2)} \cup \{\Omega\}. \end{aligned}$$

For (ii) for $\alpha+1$, assume that $\Omega \in Y^{(\alpha+1)} \backslash Y^{(\alpha+2)}$. Thus $\Omega \in \mathrm{is}(Y^{(\alpha+1)})$. Hence, using (i) for α,

$$\begin{aligned} Y^{(\alpha+2)} &= Y^{(\alpha+1)} \backslash \mathrm{is}(Y^{(\alpha+1)}) \\ &= (X^{(\alpha+1)} \cup \{\Omega\}) \backslash \mathrm{is}(X^{(\alpha+1)} \cup \{\Omega\}) \\ &= X^{(\alpha+1)} \backslash \mathrm{is}(X^{(\alpha+1)}) = X^{(\alpha+2)}. \end{aligned}$$

Finally, for (iii) for $\alpha+1$, assume that $\Omega \notin Y^{(\alpha+1)}$. By (ii) and (iii) for α, $Y^{(\alpha+1)} = X^{(\alpha+1)}$. Hence $Y^{(\alpha+2)} = Y^{(\alpha+1)} \backslash \mathrm{is}(Y^{(\alpha+1)}) = X^{(\alpha+1)} \backslash \mathrm{is}(X^{(\alpha+1)}) = X^{(\alpha+2)}$.

Now suppose inductively that α is limit. For (i), suppose that $\Omega \in Y^{(\alpha+1)}$. Then $\Omega \in Y^{(\alpha)}$, hence $\Omega \in Y^{(\beta+1)}$ for each $\beta < \alpha$, and so by the inductive hypothesis, $Y^{(\beta+1)} = X^{(\beta+1)} \cup \{\Omega\}$. Hence $Y^{(\alpha)} = X^{(\alpha)} \cup \{\Omega\}$. So $\mathrm{is}(Y^{(\alpha)}) = \mathrm{is}(X^{(\alpha)})$ and $Y^{(\alpha+1)} = Y^{(\alpha)} \backslash \mathrm{is}(Y^{(\alpha)}) = (X^{(\alpha)} \cup \{\Omega\}) \backslash \mathrm{is}(X^{(\alpha)} \cup \{\Omega\}) = X^{(\alpha+1)} \cup \{\Omega\}$.

For (ii), suppose that $\Omega \in Y^{(\alpha)} \backslash Y^{(\alpha+1)}$. Then clearly $Y^{(\alpha)} = X^{(\alpha)} \cup \{\Omega\}$, and so $Y^{(\alpha+1)} = Y^{(\alpha)} \backslash \mathrm{is}(Y^{(\alpha)}) = (X^{(\alpha)} \cup \{\Omega\}) \backslash \mathrm{is}(X^{(\alpha)} \cup \{\Omega\}) = X^{(\alpha+1)}$.

For (iii), suppose that $\Omega \notin Y^{(\alpha)}$. Then there is a $\beta < \alpha$ such that $\Omega \notin Y^{(\beta)}$, and so $Y^{(\beta)} = X^{(\beta)}$ by the inductive hypothesis. So $Y^{(\alpha)} = X^{(\alpha)}$ and hence $Y^{(\alpha+1)} = X^{(\alpha+1)}$.

□

Theorem 12.38. (Lemma 1) *Let θ be a sequence of infinite cardinals such that there is a θ-poset. Then there is a locally compact Hausdorff scattered space X such that $CS(X) = \theta$.*

Theorem 12.39. (Theorem 1) *Let η be an ordinal less than ω_2, and let $\theta = \langle \kappa_\alpha : \alpha < \eta \rangle$ be a sequence of cardinals, each κ_α either ω or ω_1. Then there is a θ-poset.*

Theorem 12.40. (Theorem 2, Juhász, Weiss) *Let η be an ordinal less than ω_2, and let $\theta = \langle \kappa_\alpha : \alpha < \eta \rangle$ be a sequence of cardinals. Assume $\forall \alpha < \eta[\omega \le \kappa_\alpha \le 2^\omega$ and $\kappa_\alpha \le \omega_1$ for all $\alpha < \eta$ such that $\mathrm{cf}(\alpha) = \omega_1]$. Then there is a locally compact scattered space X such that $CS(X) = \theta$.*

The following is from Baker 1972.

For any ordinal ξ, $\Gamma(\xi)$ is the set $\xi + 1$ with the order topology. If X is a scattered space and α is the ordinal such that $X^{(\alpha)}$ is finite and nonempty, then the *characteristic* of X is the pair $(\alpha < |X^{(\alpha)}|)$.

Theorem 12.41. (Theorem 1) *If X is a compact scattered space with characteristic (λ, n) and every point of X has a neighborhood base consisting of a (possibly transfinite) decreasing sequence of sets, then there is a continuous mapping of X onto $\Gamma(\omega^\lambda \cdot n)$.*

Theorem 12.42. (Theorem 2) it X is homeomorphic to $\Gamma(\omega^\lambda \cdot n)$ iff X has characteristic (λ, n) and every point of X has a neighborhood base consisting of a decreasing sequence $\langle U_\alpha : \alpha < \beta\rangle$ of sets such that if $\gamma < \alpha$ is limit, then $\bigcap_{\beta<\gamma} U_\beta \backslash U_\gamma| \leq 1$.

The following is from Baumgartner, Shelah 1987.

For A a superatomic BA and $\alpha < \mathrm{rk}(A)$, let $\mathrm{wd}_\alpha(A)$ the the size of the set of atoms of $A/I_\alpha(A)$. The sequence $\langle \mathrm{wd}_\alpha(A) : \alpha < \mathrm{rk}(A)\rangle$ is the *cardinal sequence* of A. For κ and infinite cardinal, A is

$$\begin{aligned}
\kappa - \textit{thin-thick} \quad &\text{iff} \quad \mathrm{rk}(A) = \kappa + 1 \text{ and } \forall\alpha < \kappa[\mathrm{wd}_\alpha(A) = \kappa] \text{ and } \mathrm{wd}_\kappa(A) = \kappa^+;\\
\kappa - \textit{thin-very thick} \quad &\text{iff} \quad \mathrm{rk}(A) = \kappa + 1 \text{ and } \forall\alpha < \kappa[\mathrm{wd}_\alpha(A) = \kappa] \text{ and } \mathrm{wd}_\kappa(A) \geq \kappa^{++};\\
\kappa - \textit{thin-tall} \quad &\text{iff} \quad \mathrm{rk}(A) = \kappa^+ \text{ and } \forall\alpha < \kappa^+[\mathrm{wd}_\alpha(A) = \kappa];\\
\kappa - \textit{thin-very tall} \quad &\text{iff} \quad \mathrm{rk}(A) = \kappa^{++} \text{ and } \forall\alpha < \kappa^{++}[\mathrm{wd}_\alpha(A) = \kappa].
\end{aligned}$$

Theorem 12.43. (Theorem 2.1) *The following is relatively consistent: ZFC+MA+2^ω large + there are no ω_1-thin-very thick BAs.*

Theorem 12.44. (Corollary 3.3) *If it is consistent that a strong inaccessible exists, then it is consistent that there is no ω_1-thin-thick BA.*

Theorem 12.45. (Theorem 7.1) *It is relatively consistent that there is an ω_1-thin-very tall BA.*

The following is from Bekkali 2001.

Theorem 12.46. (Theorem 3.1) *For T a pseudotree the following are equivalent:*
(i) treealg(T) *is superatomic.*
(ii) η and ${}^{<\omega}2$ do not embed in T.

The following is from Juhasz, Weiss 2006.

Theorem 12.47. (Theorem 5) *A sequence $\langle \kappa_\alpha : \alpha < \omega_1\rangle$ is the cardinal sequence of a superatomic BA iff $\forall \xi, \eta[\xi < \eta \to \kappa_\eta \leq \kappa_\xi^\omega]$.*

The following is from Just 1985.

A superatomic BA is *thin-tall* iff it has countable width but uncountable height. It is *thin very-tall* iff in addition it has height $\geq \omega_2 + 1$. It is *thin-thick* iff it has height $\omega_1 + 2$, width $\leq \omega_1$, with ω_2 atoms at the next to last level.

Theorem 12.48. (Theorem 2.11) *It is relatively consistent that there is no thin-thick subalgebra of $\mathscr{P}(\omega)$.*

Theorem 12.49. (Theorem 2.13) *It is relatively consistent that there is no thin very-tall BA.*

The following is from Martinez 1995.

Theorem 12.50. *Suppose that in M, $\theta = \langle \kappa_\alpha : \alpha < \omega_1 \rangle$ is a sequence of infinite cardinals Then, there is a partial order P in M which preserves cardinals and such that changes cardinal exponentiation and whenever G is P-generic over M, in M[G] there is a superatomic BA B such that θ is the cardinal sequence of B.*

Theorem 12.51. *Suppose that in M, κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ and $\theta = \langle \kappa_\alpha : \alpha < \kappa^+ \rangle$ a cardinal sequence such that $\kappa_\alpha \geq \kappa$ for every $\alpha < \kappa^+$ and $\kappa_\alpha = \kappa$ for every $\alpha < \kappa^+$ $\mathrm{cf}(\alpha) < \kappa$. Then, there is a partial order in M which preserves cardinals and such that P changes cardinal exponentiation an whenever G is P-generic over M, in M[G] there is a superatomic BA B such that θ is the cardinal sequence of B.*

The following is from Martinez 1999.

If A is superatomic, the *width* of A is the supremum of the cardinality of the set of atoms of $A/I_\alpha(A)$. The *height* of A is the least ordinal α such that A/I_α is finite.

Theorem 12.52. *Assume that in M, κ is a cardinal such that $\kappa^{<\kappa} = \kappa$, and η is an ordinal with $0 < \eta < \kappa^{++}$. Then there is a generic extension preserving cardinals such that in the extension there is a superatomic BA of height η and width κ.*

The following is from Pierce 1959.

Let κ be an infinite cardinal. A poset P is *κ-compact* iff P has a zero element 0 and $\forall M \subseteq P[|M| \leq \kappa$ and $\forall F \in [M]^{<\omega} \exists p \in P\backslash\{0\}[p \leq f]] \to \exists p \in P\backslash\{0\}[p \leq M]$. A BA A is *κ-compact* iff it has a dense κ-compact subset.

Examples of κ-compact BAs are given, with connections with distributive laws.

13. Tail algebras

The following is from Bekkali, Pouzet, Zhani 2007.

An *incidence structure* is a triple $R \stackrel{\text{def}}{=} (I, \rho, J)$ such that $\rho \subseteq I \times J$. We define $\mathscr{R}_R = \{\rho[\{i\}] : i \in I\}$. $\mathfrak{B}(R)$ is the subalgebra of $\mathscr{P}(J)$ generated by $\mathscr{R}_R$.

Proposition 13.1. *For any BA A, $A \cong \mathfrak{B}(R)$ with $R = (A, \in, \mathrm{Ult}(A))$.*

If P is a poset, define $F_{<\omega}(P) = \{\{p : \exists q \in X[q \le p]\} : X \in [P]^{<\omega}\}$; $F_{<\omega}(P)$ is ordered by inclusion.

Proposition 13.2. (Proposition 2.10) *For any poset P, the free BA on P is isomorphic to* $\mathrm{tailalg}(F_{<\omega}(P))$.

The following is from Brown 2015.

If T is a pseudo-tree algebra and $C \subseteq T$ is an initial chain, then a subset $R \subseteq T$ is a *set of approximate immediate successors of* C iff the following conditions hold:

(1) $C < r$ for all $r \in R$.

(2) For all $s > C$ there is an $r \in R$ such that $r \le s$.

Then we define

$$\varepsilon_C = \min\{|R| : R \text{ is a set of approximate immediate successors of } C\}.$$

Theorem 13.3. (Theorem 2) *For any pseudotree T, the character of* $\mathrm{Treealg}(T)$ *is* $\sup\{\varepsilon_C : C$ *is an initial chain of* $T\}$.

The following is from Eda 1975.

Proposition 13.4. *If T is a tree, then $\{T \uparrow t : t \in T\}$ is a base for a topology on T.*

Proof. Suppose that $s \in (T \uparrow t_1) \cap (T \uparrow t_2)$. Then $s \in (T \uparrow s) \subseteq (T \uparrow t_1) \cap (T \uparrow t_2)$. If $t \in T$, then $t \in (T \uparrow t)$. $\square$

$\mathrm{treealg}'(T)$ is the complete BA of all regular open sets in this topology. A tree is *ever splitting* iff every node has at least two immediate successors.

Proposition 13.5. (Lemma 2) *A complete BA is isomorphic to* $\mathrm{Treealg}'(T)$ *for some tree T iff it contains a dense ever splitting tree.*

For any BA A, $sat(A)$ is the least cardinal λ such that every pairwise disjoint subset of A has size $\le \lambda$.

Proposition 13.6. (Theorem 3) *If* $\mathrm{Treealg}'(T)$ *is atomless and* $sat(\mathrm{Treealg}'(T)) \le \kappa$, *then the $(\kappa, 2)$-distributive law fails in* $\mathrm{Treealg}'(T)$.

The following is from Pierce 1973.

Theorem 13.7. *If A is a countable BA, then A has an ordered basis of one of the following types: $\omega^\lambda \cdot n + 1$, λ a countable ordinal; $\sum_{r\in\eta} \omega^{\alpha_r} + 1$, each α_r a countable ordinal; a sum of these two types.*

14. The Tarski invariants

We sketch the treatment from Koppelberg 1989. For any BA A we define

$$\begin{aligned} E(A) &= \{x \in A : \exists y, z \in A[x = y + z,\ A \restriction y \text{ is atomless},\ A \restriction z \text{ is atomic}]\};\\ T_0(A) &= \{0\};\\ T_{i+1}(A) &= \{a \in A : a/T_i(A) \in E(A/T_i(A))\}. \end{aligned}$$

Now we define the invariants:

$$\begin{aligned} \mathrm{Inv} &= \{(-1,0,0),(\omega,0,0)\}\\ &\qquad \cup \{(k,l,m) : k \in \omega,\ l \in 2,\ m \in (\omega+1),\ l+m \neq 0\};\\ \mathrm{inv}(A) &= (-1,0,0) \quad \text{iff } |A| = 1;\\ \mathrm{inv}(A) &= (\omega,0,0) \quad \text{iff } \forall i \in \omega[|A/T_i(A)| > 1];\\ \mathrm{inv}(A) &= (k,l,m) \quad \text{iff } k \in \omega \text{ and the following hold:}\\ &\qquad (a)\ |A/T_k(A)| > 1 = |A/T_{k+1}(A)|;\\ &\qquad (b)\ A/T_k(A) \text{ is atomic and } l = 0;\\ &\qquad (c)\ A/T_k(A) \text{ is not atomic and } l = 1;\\ &\qquad (d)\ m = \min\{\omega, |\mathrm{At}(A/T_k(A))|\}. \end{aligned}$$

Theorem 14.1. *For each* $(k,l,m) \in \mathrm{Inv}$ *there is a BA* A *such that* $\mathrm{inv}(A) = (k,l,m)$.

Theorem 14.2. $A \equiv B$ *iff* $\mathrm{inv}(A) = \mathrm{inv}(B)$.

Theorem 14.3. *For each* $(k,l,m) \in \mathrm{Inv}$ *there is a set* T_{klm} *of sentences such that for any BA* A, $A \models T_{klm}$ *iff* $\mathrm{inv}(A) = (k,l,m)$.

The following is from Bonnet, Rubin 1991.

For each $\theta \in \mathrm{Inv}$ let M_θ be the class of all BAs with invariant θ; M_θ^κ is the class of $A \in M_\theta$ of size $\leq \kappa$.

Theorem 14.4. (Theorem 2) *For every* $\theta \in \mathrm{Inv}\backslash\{(\omega,0,0)\}$. *every sequence in* $(M_\theta^\omega, \preceq)$ *has an increasing subsequence.*

Theorem 14.5. (Theorem 3) *(i) There is a family of size* 2^ω *of pairwise* $\preceq$*-incomparable members of* $M^\omega_{(\omega,0,0)}$.

(ii) There is a system $\langle A_r : r \in \mathbb{R}\rangle$ *of members of* $M^\omega_{(\omega,0,0)}$ *such that* $r \leq s$ *iff* $A_r \preceq A_s$.

15. Ultrafilters

A general theorem in Arhangelski 1969a has the following consequence:

Theorem 15.1. *If* $\mathrm{Ult}(A)$ *is uncountable and every ultrafilter on* A *is countably generated, then* $|\mathrm{Ult}(A)| = 2^\omega$.

The following is from Balcar, Simon 1980.

Let κ, τ be cardinal numbers, and $M \subseteq [\kappa]^\kappa$. We say that M is *strongly* τ*-decomposable* iff there is an almost disjoint $\mathscr{A} \subseteq [\kappa]^\kappa$ and a partition $\langle A_\alpha : \alpha < \tau\rangle$ of $\mathscr{A}$ such that $\forall a \in M \forall \alpha < \tau \exists b \in A_\alpha[|b \cap a| = \kappa]$.

Theorem 15.2. (Theorem) *If* κ *is regular and uncountable, then every uniform ultrafilter on* κ *is strongly* $\mathfrak{b}_\kappa$*-indecomposable.*

Theorem 15.3. (Proposition 8) *The collection of all stationary subsets of* κ *is strongly* $\mathfrak{b}_\kappa$*-decomposable.*

The following is from Bartoszynski, Shelah 2004.

If $f \in {}^\omega\omega$ and U is an ultrafilter on ω, then we define $f(U) = \{X \subseteq \omega : f^{-1}[X] \in U\}$.

An ultrafilter U on ω is *Hausdorff* iff for any $f, g \in {}^\omega\omega$, if $f(U) = g(U)$ then $\exists X \in U[f \restriction X = g \restriction X]$.

A function $f \in {}^\omega\omega$ is *finite-to-one* iff $\forall n \in \omega[f^{-1}[\{n\}]$ is finite].

An ultrafilter U on ω is *weakly Hausdorff* iff for any finite-to-one $f, g \in {}^\omega\omega$, if $f(U) = g(U)$ then $\exists X \in U[f \restriction X = g \restriction X]$.

Theorem 15.4. *If* $f(U) = U$, *then* $\{n : f(n) = n\} \in U$.

Theorem 15.5. (Theorem 12) *There is an ultrafilter which is not weakly Hausdorff.*

The following is from Banaschewski 1955.

Theorem 15.6. (Satz 2) *For every infinite cardinal* κ, *there are exactly* 2^{2^κ} *uniform ultrafilters on* κ.

Theorem 15.7. (Satz 3) *For infinite cardinals* κ, λ *with* $2^\lambda \le \kappa$, *there are at most* 2^λ *ultrafilters* F *on* κ *such that* λ *is the smallest size of a member of* F.

Lemma 15.8. (Lemma 3) *The automorphisms of the space* $\mathrm{Ult}(\mathscr{P}(\kappa))$ *are the permutations of* $\mathrm{Ult}(\mathscr{P}(\kappa))$ *induced by order isomorphisms of the set of filters on* $\mathscr{P}(\kappa)$.

Lemma 15.9. (Satz 4) *Let* κ *be an infinite cardinal. Let* f *be an automorphism of* $\mathscr{P}(\kappa)$.

(i) There is a permutation g *of* κ *such that* $f(\{\alpha\}) = \{g(\alpha)\}$ *for all* $\alpha \in \kappa$.

(ii) For any $A \subseteq \kappa$, $f(A)) = \{g(\alpha) : \alpha \in A\}$.

Proof. (i) is clear. For (ii), suppose that $A \subseteq \kappa$. If $\alpha \in A$, then $\{\alpha\} \subseteq A$, hence $f(\{\alpha\}) \subseteq f(A)$. So $\{g(\alpha)\} \subseteq f(A)$ and so $g(\alpha) \in f(A)$. On the other hand, suppose that $g(\alpha) \in f(A)$. Thus $f(\{\alpha\}) = \{g(\alpha)\} \subseteq f(A)$, hence $\{\alpha\} \subseteq A$ and so $\alpha \in A$. $\square$

Theorem 15.10. (Satz 5) *The conjugacy classes of* $\mathrm{Aut}(\mathrm{Ult}(\mathscr{P}(\kappa)))$ *are exactly the classes of order-isomorphic ultrafilters.*

The following is from Blass 1973.

If E is an ultrafilter on ω and $f : \omega \to \omega$, then $f(E) = \{x \subseteq \omega : f^{-1}[x] \in E\}$. Clearly $f(E)$ is an ultrafilter on ω.

Proposition 15.11. *If E is an ultrafilter on ω, $f, g : \omega \to \omega$, and $f \restriction x = g \restriction x$ for some $x \in E$, then $f(E) = g(E)$.*

Proof. Assume the hypotheses, and suppose that $y \in f(E)$. Then $f^{-1}[y] \in E$. Hence $x \cap f^{-1}[y] \in E$. Now $x \cap f^{-1}[y]\} = \{m \in \omega : m \in x \text{ and } f(m) \in y\} \subseteq \{m \in \omega : g(m) \in y\} = g^{-1}[y] \in E$; so $y \in g(E)$. $\square$

We say that $D \leq_{RK} E$ iff there is an $f : \omega \to \omega$ such that $D = f(E)$.

We say that D is *isomorphic* to E iff there is a permutation f of ω such that $D = F(E)$.

Proposition 15.12. *If $D = f(E)$ and f is one-one on a member of E, then $D \cong E$.*

Proof. Assume the hypotheses. Say $f \restriction x$ is one-one, with $x \in E$. If x is finite, then E is principal; say $\{m\} \in E$. Then $\{f(m)\} \in D$ and so $D \cong E$. Suppose that x is infinite. Write $x = y \cup z$ with $y \cap z = \emptyset$ and y, z infinite. Say $y \in E$. Then $\omega \backslash y$ and $\omega \backslash f[y]$ are infinite. Let g be a permutation of ω which agrees with $f \restriction y$. Then $D = f(E) = g(E)$ by Proposition 15.11, and $E \cong g(E)$ by definition. $\square$

Proposition 15.13. *If $D \leq_{RK} E \leq_{RK} F$, then $D \leq_{RK} F$.*

Proof. Let $f, g : \omega \to \omega$ be such that $D = f(E)$ and $E = g(F)$. Then

$$F = \{x \subseteq \omega : g^{-1}[x] \in E\} = \{x \subseteq \omega : f^{-1}[g^{-1}[E]] \in D\} = \{x \subseteq \omega : (g \circ f)^{-1}[D]\} \quad \square$$

Theorem 15.14. (Theorem 1) *If $f(D) = D$, then $\{m \in \omega : f(m) = m\} \in D$.*

Proof. Assume that $f(D) = D$. Let $T = \{n \in \omega : f(n) < n\}$. Suppose that $T \in D$; we want to get a contradiction.

(1) $\forall m \in \omega \exists n \in \omega [f^n(m) \notin T]$.

For, assume that $m \in \omega$ and $\forall n \in \omega [f^n(m) \in T]$. Then $\cdots < f^2(m) < f(m) < m$, contradiction.

For each $n \in \omega$ let $T_n = \{m \in \omega : n \text{ is minimum such that } f^n(m) \notin T\}$.

(2) $T = \bigcup_{0<n\in\omega} T_n$.

In fact, clearly $T_n \subseteq T$ for $n > 0$. Now let $m \in T$. By (1) there is an $n \in \omega$ such that $f^n(m) \notin T$; hence $m \in T_p$ for some p.

(3) $\forall n \in \omega[T_{n+1} = f^{-1}[T_n]]$.

In fact, $f^{-1}[T_n] = \{m \in \omega : f(m) \in T_n\} = T_{n+1}$.

(4) $\bigcup_{n\in\omega} T_{2n} \in D$ iff $\bigcup_{n\in\omega} T_{2n+2} \in \omega$.

In fact, suppose that $X = \bigcup_{n\in\omega} T_{2n} \in D$. Then

$$f^{-1}[X] = \bigcup_{n\in\omega} f^{-1}[T_{2n}] = \bigcup_{n\in\omega} T_{2n+1} \in \omega.$$

On the other hand, suppose that $\bigcup_{n\in\omega} T_{2n+1} \in D$. Then $f^{-1}[X] \in D$, and so $X \in D$.

Now by (2), $\bigcup_{n\in\omega} T_{2n} = \omega \backslash \bigcup_{n\in\omega} T_{2n+1}$. So (4) is a contradiction.

It follows that $T \notin D$.

Now let $R = \{m \in \omega : m < f(m)\}$. Suppose that $R \in D$. For each $n \in \omega$ let $R_n = \{m \in \omega : n$ is minimum such that $f^n(m) \notin R\}$. Then $f^{-1}[R_n] = \{m \in \omega : f(m) \in R_n\} = R_{n+1}$. If $\bigcup_{n\in\omega} R_{2n} \in D$, then $f^{-1}[\bigcup_{n\in\omega} R_{2n}] = \bigcup_{n\in\omega} R_{2n+1} \in D$. Similarly, $\bigcup_{n\in\omega} R_{2n+1} \in D$ implies that $\bigcup_{n\in\omega} R_{2n} \in D$. Hence $\bigcup_{n\in\omega} R_{2n} \notin D$ and $\bigcup_{n\in\omega} R_{2n+1} \notin D$. So $Y \stackrel{\text{def}}{=} \omega \backslash \bigcup_{n\in\omega} R_n \in D$. With m the least element of Y we have $m < f(m) < f^2(m) < \cdots$. Let $Z = \{m, f^2(m), f^4(m), \ldots\}$ and $W = \{f(m), f^3(m), \ldots\}$. Then $f^{-1}[Z] = W$ and so $Z \in D$ iff $W \in D$. Since $Z \cup W = Y$, this is a contradiction. $\square$

Corollary 15.15. (Corollary 1) *If $f(D) \cong D$, then f is one-one on some member of D.*

Proof. Assume that $f(D) \cong D$; say $g : \omega \to \omega$ and $g(f(D)) = D$. Then by Theorem 15.14, $x \stackrel{\text{def}}{=} \{m \in \omega : g(f(m)) = m\} \in D$. Clearly f is one-one on x. $\square$

Corollary 15.16. *If $D \leq_{RK} E \leq_{RK} D$, then $D \cong E$.*

Proof. Assume that $D \leq_{RK} E \leq_{RK} D$. Say $D = f(E)$ and $E = g(D)$. Then $D = f(g(D))$, so by Theorem 15.14, $x \stackrel{\text{def}}{=} \{m \in \omega : f(g(m)) = m\} \in D$. Hence g is one-one on $x \in D$, so $D \cong E$ by Proposition 15.12. $\square$

Proposition 15.17. *All principal ultrafilters are isomorphic.*

Proof. Suppose that $\{m\} \in D$ and $\{n\} \in E$. Let $f; \omega \to \omega$ be the function with constant value n. Then $x \in f(D)$ iff $f^{-1}[x] \in D$ iff $n \in x$, so $f(D) = E$. Thus $E \leq_{RK} D$. Similarly, $D \leq_{RK} E$. So $D \cong E$ by Corollary 15.16.

Proposition 15.18. *If D is principal and E is arbitrary, the $D \leq_{RK} E$.*

Proof. Assume that D is principal and E is arbitrary, Say $\{m\} \in D$. Let $f : \omega \to \omega$ have constant value m. Then for any $x \subseteq \omega$, $x \in f(E)$ iff $f^{-1}[x] \in E$ iff $m \in x$. Thus $f(E) = D$. $\square$

Proposition 15.19. *If D is an ultrafilter and $D \leq_{RK} E$ for every ultrafilter E, then D is principal.*

Proof. Assume that D is an ultrafilter and $D \leq_{RK} E$ for every ultrafilter E. Let $\{m\} \in E$, and let $f : \omega \to \omega$ be such that $D = f(E)$. Then for any $x \subseteq \omega$, $x \in D$ iff $f^{-1}[x] \in E$ iff $m \in f^{-1}[x]$ iff $f(m) \in x$. So D is principal, with $\{f(m)\} \in D$. □

Proposition 15.20. *D is minimal among nonprincipal ultrafilters iff for every $f : \omega \to \omega$ there is an $x \in D$ such that $f \restriction x$ is constant or one-one.*

Proof. $\Rightarrow$: Suppose that D is minimal among nonprincipal ultrafilters, and $f : \omega \to \omega$. Then $f(D) \leq_{RK} D$, so $f(D)$ is principal or $f(D) = D$. If $f(D)$ is principal, say $\{m\} \in f(D)$. Then $f^{-1}[\{m\}] \in D$. Thus $f \restriction f^{-1}[\{m]$ is constant. If $f(D) \cong D$, then f is one-one on a member of D by Corollary 15.15.

$\Leftarrow$: Assume the indicated condition, and suppose that $E \leq_{RK} D$. Say $f : \omega \to \omega$ and $E = f(D)$.

Case 1. f has constant value m on $x \in D$. Thus $x \subseteq f^{-1}[\{m\}]$, so $f^{-1}[\{m\}] \in D$. Now for any $y \subseteq \omega$, $y \in E$ iff $f^{-1}[\{m\}] \cap f^{-1}[y] \in D$ iff $f(m) \in y$; so $E = \{f(m)\}$.

Case 2. f is one-one on $x \in D$. Then $D \cong E$ by Proposition 15.12. □

A *P-point* is a nonprincipal ultrafilter D such that for every $f : \omega \to \omega$ there is an $x \in D$ such that $f \restriction x$ is constant or finite-to-one ($\forall m \in x[\{n \in x : f(n) = f(m)\}$ is finite]).

Proposition 15.21. *Every minimal nonprincipal ultrafilter is a P-point.*

Proof. By Proposition 15.20. □

Proposition 15.22. *If D is a P-point, $E \leq_{RK} D$, and E is nonprincipal, then E is a P-point.*

Proof. Suppose that D is a P-point, $E \leq_{RK} D$, and E is nonprincipal. Let $f : \omega \to \omega$. Since $E \leq_{RK} D$, let $g : \omega \to \omega$ be such that $E = g(D)$. Since D is a P-point, we have two cases.

Case 1. There is an $x \in D$ such that $(f \circ g) \restriction x$ is constant, say with value m. Then $\{n : f(n) = m\} \in E$ iff $g^{-1}[\{n : f(n) = m\}] \in D$ iff $\{p : f(g(p)) = m\} \in D$; since $x \subseteq \{p : f(g(p)) = m\}$, we have $\{p : f(g(p)) = m\} \in D$, and hence $\{n : f(n) = m\} \in E$, as desired.

Case 2. There is an $x \in D$ such that $(f \circ g) \restriction x$ is finite-to-one. Then $x \subseteq g^{-1}[g[x]]$, so $g^{-1}[g[x]] \in D$, and hence $g[x] \in E$. Now $\forall n \in x[\{m \in x : f(g(m)) = f(g(n))\}$ is finite]. Hence for all $p \in g[x]$, $\{m \in x : f(g(m)) = f(p)\}$ is finite, so $\{q \in g[x] : f(q) = f(p)\}$ is finite, as desired. □

Proposition 15.23. *Ultrafilters are directed upwards under $\leq_{RK}$.*

Proposition 15.24. (Theorem 2) (MA) *There are 2^{2^ω} isomorphism classes of minimal ultrafilters under $\leq_{RK}$, and also of P-points.*

Let $\overline{A}$ be a first-order structure and $f : \omega \to \omega$ with $D = f(E)$. Then there is an elementary embedding f^* of ${}^\omega\overline{A}/D$ into ${}^\omega\overline{A}/E$ such that for any $x \in {}^\omega A$ we have $f^*(x/D) = (x \circ f)/E$. In fact,

$$\begin{aligned}
x/D = y/D \quad &\text{iff} \quad \{m \in \omega : x(m) = y(m)\} \in D\\
&\text{iff} \quad f^{-1}[\{m \in \omega : x(m) = y(m)\}] \in E\\
&\text{iff} \quad \{m \in \omega : x(f(m)) = y(f(m))\} \in E\\
&\text{iff} \quad (x \circ f)/E = (y \circ f)/E.
\end{aligned}$$

Thus f^* is well-defined and one-one. For the elementary embedding property, let $x \in {}^\omega({}^\omega A)$, $\pi : {}^\omega A \to {}^\omega A/D$ natural, $\pi' : {}^\omega A \to {}^\omega A/E$ natural. Then

$$\begin{aligned}
{}^\omega\overline{A}/D \models \varphi[\pi \circ x] \quad &\text{iff} \quad \{i \in \omega : \overline{A} \models \varphi[\mathrm{pr}_i \circ x]\} \in D\\
&\text{iff} \quad f^{-1}[\{i \in \omega : \overline{A} \models \varphi[\mathrm{pr}_i \circ x]\}] \in E\\
&\text{iff} \quad \{i \in \omega : \overline{A} \models \varphi[\mathrm{pr}_{f(i)} \circ x]\} \in E.
\end{aligned}$$

Now let $x'_k(i) = x_k(f(i))$ for any $i, k \in \omega$. Then for any i, k, $(\mathrm{pr}_{f(i)} \circ x)(k) = x_k(f(i)) = x'_k(i) = (\mathrm{pr}_i \circ x')_k$. Hence by the above,

$$\begin{aligned}
{}^\omega\overline{A}/D \models \varphi[\pi \circ x] \quad &\text{iff} \quad \{i \in \omega : \overline{A} \models \varphi[\mathrm{pr}_i \circ x']\\
&\text{iff} \quad {}^\omega\overline{A}/E \models \varphi[\pi' \circ x'].
\end{aligned}$$

Now for any $i, k \in \omega$, $(x_k \circ f)(i) = x_k(f(i)) = x'_k(i)$; so $x_k \circ f = x'_k$. Hence $(f^* \circ \pi \circ x)(k) = f^*(x_k/D) = (x_k \circ f)/E = x'_k/E$. So $f^* \circ \pi \circ x = \pi' \circ x'$. Hence by the above we get ${}^\omega\overline{A}/D \models \varphi[\pi \circ x]$ iff ${}^\omega\overline{A}/E \models \varphi[f^* \circ \pi \circ x]$.

Theorem 15.25. (Theorem 8) (MA) *There is an order-isomorphism of $\mathbb{R}$ into the set of equivalence classes of P-points.*

The following is from Blass 1981.

Theorem 15.26. (Corollary 2c) (CH) *There is an initial segment of the Rudin-Keisler order of order type ω_1.*

Theorem 15.27. (Corollary 2d) (CH) *There is an initial segment of the Rudin-Keisler order which is a tree of height ω_1 in which each node has 2^{ω_1} immediate successors and each increasing ω-sequence of nodes has a unique least upper bound.*

The following is from Brown, Dobrinen 2016.

For posets P, Q, a *cofinal map* from P to Q is a function $f : P \to Q$ such that $\forall X \subseteq P[X$ cofinal in $P \to f[X]$ is cofinal in $Q]$. If there is a cofinal map from P to Q, then we say that Q is Tukey reducible to P, and we write $Q \leq_T P$. P and Q are *Tukey equivalent* iff $P \leq_T Q \leq_T P$. For an ultrafilter U on a BA A, the *Tukey type* of U is the Tukey type of the poset $(U, \geq)$.

Proposition 15.28. *An ultrafilter U on a BA A has maximal Tukey type among all ultrafilters on A iff $U \equiv_I ([|A|]^{<\omega}, \subseteq)$.*

For any BA A, its *Tukey spectrum* is the set $\mathrm{TS}(A)$ of all Tukey types of ultrafilters on A, ordered by $\leq_T$.

Proposition 15.29. *For each infinite cardinal κ there is a maximal Tukey type of ultrafilters on $\mathscr{P}(\kappa)$.*

Theorem 15.30. (Theorem 2.1) *If A has an independent subset of size $|A|$, then there is a maximal Tukey type of ultrafilters on A.*

The following is from Dow 1984

For any infinite cardinal κ, a function $f : [\kappa]^{<\omega} \to \mathscr{P}(\kappa)$ is *multiplicative* iff $\forall H \in [\kappa]^{<\omega}[f(H) = \{f(\{\alpha\}) : \alpha \in H\}]$. A filter P on κ is κ^+*-good* iff $\forall g : [\kappa]^{<\omega} \to P \exists f : [\kappa]^{<\omega} \to P \forall H \in [\kappa]^{<\omega}[f(H) \subseteq g(H)]$.

If (S, L) is a linear order, then (C, D) is a (κ, λ)*-gap* in (S, L) iff $L(C, D)$, C is an increasing chain of type κ, D is a decreasing chain of type λ, and there is no $x \in S$ with $L(C, x)$ and $L(x, D)$.

For any infinite cardinal κ and any $\gamma < \kappa$, let $\overline{\gamma} \in {}^\kappa\kappa$ be the function with constant value γ. For each $P \in U(\kappa)$, let $\lambda(\alpha, P) = \min\{\mu : ({}^\kappa\kappa, S)/P$ has a (ω_α, μ)-gap of the form $([\{\overline{\gamma}], \gamma < \omega_\alpha\}, \{[f_\delta] : \delta < \mu\})$ for each regular $\omega_\alpha < \mu\}$.

Theorem 15.31. (Proposition 1.4) *Let $P \in U(\kappa)$ be countably incomplete and κ^+-good. Suppose that (S, L) has chains of each finite length. Then for each regular $\omega_\alpha \leq \kappa$, $\lambda(\alpha, P)$ is the unique regular cardinal μ such that ${}^\kappa\kappa, S)/P$ has an (ω_α, μ)-gap.*

Theorem 15.32. (Theorem 2.1) *There is an ω-incomplete α^+-good ultrafilter p on α so that $\lambda(\beta, p) = \mathrm{cf}(2^\alpha)$ for every $\omega_\beta \leq \alpha$.*

Theorem 15.33. (Theorem 2.2) *For each regular κ with $\omega_1 \leq \kappa \leq 2^\omega$ there is a uniform ultrafilter p on ω such that $\lambda(p) = \kappa$.*

Jipsen, Rose 1990 characterizes saturated atomic BAs in terms of an ultraproduct of finite BAs.

The following is from Hajnal, Juhasz 1972.

If $\mathscr{A}$ is a collection of ultrafilters on a set X, then a function $f : \mathscr{A} \to \mathscr{P}(X)$ is a *disjoint representation* iff $\forall U \in [\mathscr{A}[f(U) \in u]$ and $\forall U, V \in \mathscr{A}[U \neq V \to f(U) \cap f(V) = \emptyset]$.

Theorem 15.34. (GCH) *If κ is an infinite cardinal, then there is a set $\mathscr{A}$ of ultrafilters on κ such that $|\mathscr{A}| = 2^{2^\kappa}$ and no uncountable subset of $\mathscr{A}$ has a disjoint representation.*

The following is from Lacava 1983.

An atomic BA A is *principal* iff for every subalgebra B of A such that $|B| < |\mathrm{at}(A)|$ and for every ultrafilter F of B there is a principal ultrafilter G of A such that $F = G \cap B$.

Proposition 15.35. *An atomic BA A is principal iff for every $F \subseteq A$ such that F has fip and $|F| < |\mathrm{at}(A)|$ there is a nonzero $a \in A$ such that $\forall b \in F[a \leq b]$.*

Proof. $\Rightarrow$: Let $B = \langle F \rangle$ and let F' be an ultrafilter on B containing F. Let G be a principal ultrafilter on A such that $F' = G \cap B$. Say G is generated by an atom a. If $b \in F$, then $b \in F'$ and so $b \in G$, hence $a \leq b$.

$\Leftarrow$: assume the indicated condition, and suppose that B is a subalgebra of A such that $|B| < |\mathrm{at}(A)|$ and F is an ultrafilter on B. Choose a nonzero $a \in A$ such that $\forall b \in F[a \leq b]$. We may assume that a is an atom. Let G be the principal ultrafilter on A determined by a. Clearly $F = G \cap B$. $\square$

An atomic BA A is *uniform* iff for every $a \in A$, if $A \restriction a$ is infinite, then $|A \restriction a| = |\mathrm{at}(A)|$.

Proposition 15.36. *Every principal BA is uniform.*

Proof. Suppose that A is principal, but also suppose that $A \restriction a$ is infinite and $|A \restriction a| < |\mathrm{at}(A)|$. Let $X = \{-x : x \leq a\} \cup \{a\}$. Then $|X| < |\mathrm{at}(A)|$ and X has fip. By Proposition 1 choose a nonzero $b \in A$ such that $\forall y \in X[b \leq y]$. Then $b \leq a$, so $-b \in X$ and $b \leq -b$, hence $b = 0$, contradiction. $\square$

Proposition 15.37. *Every saturated atomic BA is principal.* $\square$

16. Subalgebras

The following is from Duntsch 1985.

Theorem 16.1. (Proposition 3.3) $\mathrm{Sub}(A \times A)$ *is simple.*

Theorem 16.2. (Proposition 3.4) *If A has an independent subset of size $|A|$, then* $\mathrm{Sub}(A)$ *is simple.*

Theorem 16.3. (Proposition 3.5) $\sum(A \oplus B)$ *is simple.*

Theorem 16.4. (Proposition 3.8) *For each $n \in \omega$, the congruences of* $\mathrm{Sub}(\mathrm{Finco}(\aleph_n))$ *form a chain of type $n+3$.*

Theorem 16.5. (Proposition 3.8) *For each $\alpha \geq \omega$, the congruences of* $\mathrm{Sub}(\mathrm{Finco}(\aleph_\alpha))$ *form a chain of type $\alpha+2$.*

Theorem 16.6. (Proposition 3.9) *If $|A| = \kappa$ is regular, and for all $a \in A$, $|A \restriction a| < \kappa$ or $|A \restriction -a| < \kappa$, then* $\mathrm{Sub}(A)$ *is not simple.*

Theorem 16.7. (Proposition 3.11) *For any infinite ordinal α,* $\mathrm{Sub}(\mathrm{intalg}(\alpha))$ *is not simple iff α is a regular cardinal.*

17. Model theory of BAs

We list most papers on this subject, without a detailed description of contents.

Ershov 1964 *Decidability of the elementary theory of relatively complemented lattices and the theory of filters.* (Russian) Alg. i Log. Sem 3, no. 3, 17-38.

Mead 1975 **Prime models and model companions for the theories of Boolean algebras.** PhD thesis, University of Iowa, 1975.

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Mijajlovic 1979 *Saturated Boolean algebras with ultrafilters.* Publ. Inst. Math. (Beograd) 26, 175-197.

Olin 1976 *Homomorphisms of elementary types of Boolean algebras.* Algebra Universalis 6, 259-260.

Tarski 1949 *Arithmetical classes and types of Boolean algebras.* Bull. Amer. Math. Soc. 55, 64 (abstract)

Waszkiewicz 1974 $\forall n$*-theories of Boolean algebras.* Colloq. Math. 30, 171-175.

The following is from Pinus 1991.

Theorem 17.1. *The Löwenheim number of* $(\mathrm{BA}, \oplus, \leq)$ *is the least cardinal* κ *such that for any* $n \in \omega$ *and any BAs* $A_0, \ldots, A_{n-1}$ *there exists a cardinal* $\kappa' \leq \kappa$ *and BAs* $A'_0, \ldots, A'_{n-1}$ *of size less than* κ' *such that the the structures* $(BA, \oplus, \leq, A_0, \ldots, A_{n-1})$ *and* $(\{A \in BA : |A| < \kappa'\}, \oplus, \leq, A'_0, \ldots, A'_{n-1})$ *are elementarily equivalent.*

The Löwenheim number of $(\mathrm{BA}, \oplus, \leq)$ *is equal to the Löwenheim number of full second-order logic.*

18. $[\lambda]^\kappa$

The following is from Abe [86]. We define $D_\kappa\lambda = \{\{x, y\} : x.y \in [\lambda]^{<\kappa} \text{ and } x \subset y\}$.

Papers considered

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Incomplete:
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