Invariants of countable Boolean algebras and Ketonen's theorem

These are notes which are intended to give details in Ershov's system of invariants of countable Boolean algebras, and in Pierce's treatment of Ketonen's theorem. In the first part, concerning the invariants, we also use the treatment of Pinus. We assume an elementary knowledge of Boolean algebras; see e.g. the book of Koppelberg.

I. Invariants of countable Boolean algebras

1. Elementary theory

A distributive lattice with 0 $(A, +, \cdot, 0)$ is *relatively complemented* iff $\forall a, b \in A[a \leq b \rightarrow \exists c \in A[a+c = b \text{ and } a \cdot c = 0]]$. ABID is the class of all relatively complemented distributive lattices with 0. "ABID" abbreviates "abstract ideal".

Proposition 1.1. *If A is a ABID and $a, b \in A$ with $a \leq b$, then there is a unique $c \in A$ such that $a + c = b$ and $a \cdot c = 0]]$.*

Proof. Suppose that $c, d \in A$ with $a + c = b$, $a \cdot c = 0$, $a + d = b$, $a \cdot d = 0$. Then $c = c \cdot (a + c) = c \cdot b = c \cdot (a + d) = c \cdot a + c \cdot d = c \cdot d \leq d$. Similarly $d \leq c$. □

If A is a ABID and $a, b \in A$, then $a \cdot b \leq a$, and we let $a \backslash b$ be the unique c such that $a \cdot b + c = a$ and $a \cdot b \cdot c = 0$. Note that $\forall b \in A[0 \backslash b = 0]$. If $A = (A, +, \cdot, 0)$ is a ABID, then we define $A' = (A, +, \cdot, 0, \backslash)$.

Proposition 1.2. *If A and B are ABIDs and f is a homomorphism from A to B, then $\forall a, b \in A[f(a \backslash b) = f(a) \backslash f(b)]$.*

Proof.

$$f(a) \cdot f(b) + f(a \backslash b) = f(a \cdot b + a \backslash b) = f(a) \quad \text{and}$$
$$f(a) \cdot f(b) \cdot f(a \backslash b) = f(a \cdot b \cdot (a \backslash b)) = f(0) = 0.$$ □

Thus if f is a homomorphism from a ABID A into a ABID B, then f is a homomorphism from A' into B'.

Proposition 1.3. *If A is a ABID, B is a BA, and $(A, +_A, \cdot_A)$ is a subalgebra of $(B, +_B, \cdot_B)$, then $\forall a, b \in A[a \backslash b = a \cdot (-b)]$.*

Proof. Assume the hypotheses, and suppose that $a, b \in A$. Then

$$\begin{aligned} a \backslash b &= (a \backslash b) \cdot (a \cdot b + a \backslash b) = (a \backslash b) \cdot a = (a \backslash b) \cdot (a \cdot b + a \cdot (-b)) \\ &= (a \backslash b) \cdot a \cdot (-b) \leq a \cdot (-b) = a \cdot (-b) \cdot a \\ &= a \cdot (-b) \cdot (a \cdot b + a \backslash b) = a \cdot (-b) \cdot (a \backslash b) \leq a \backslash b. \end{aligned}$$ □

Proposition 1.4. *If A is a ABID, then the following are equivalent:*

(i) $\exists e \in A \forall a \in A[a \leq e]$.

(ii) There is an $e \in A$ *such that* $(A, +, \cdot, -, 0, e)$ *is a BA, where* $\forall a \in A[-a = e \backslash a]$.

Proof. (i)$\Rightarrow$(ii): Assume (i), with e as indicated. Define $-$ as in (ii). Then $a + -a = a + (e\backslash a) = e \cdot a + (e\backslash a) = e$ and $a \cdot -a = a \cdot (e\backslash a) = e \cdot a \cdot (e\backslash a) = 0$.

(ii)$\Rightarrow$(i): clear. $\square$

Proposition 1.5. *Every finitely generated ABID is a BA.*

Proof. If A is a ABID generated by a finite set F, then every element of A is $\leq \sum F$. $\square$

Proposition 1.6. *Every BA* A *is a maximal ideal in another BA* B*, where* $\forall a, b \in A[a +_A b = a +_B b$ *and* $a \cdot_A b = c \cdot_B b]$*, while* $\forall a \in A[-_A a = 1_A \cdot_B -_B a]$.

Proof. Define $f(a) = (0, a)$ for all $a \in A$. Then f is an injection of A into $2 \times A$. Let X be such that $A \cap X = \emptyset$ and $|X| = |A|$. Let g be a bijection from X onto $\{(1, a) : a \in A\}$. Define $B = A \cup X$ and define operations on B so that $f \cup g$ is an isomorphism of B with $2 \times A$. Then for all $a, b \in A$,

$$\begin{aligned}(f \cup g)(a +_B b) &= (f \cup g)(a) + (f \cup g)(b) = f(a) + f(b) \\ &= (0, a) + (0, b) = (0, a +_A b) = (f \cup g)(a +_A b);\end{aligned}$$

hence $a +_B b = a +_A b$. Similarly $a \cdot_B b = a \cdot_A b$. Further, if $a \in A$, then $(f \cup g)(1_A \cdot_B -_B a) = (0, 1_A) \cdot -(0, a) = (0, -_A a) = (f \cup g)(-_A a)$ and hence $1_A \cdot_B -_B a = -_A a$. Next, suppose that $a, b \in B$ and $a \leq_B b \in A$. Then $a \cdot_B b = a$, so $(f \cup g)(a) = (f \cup g)(a \cdot_B b) = (f \cup g)(a) \cdot (0, b)$; hence $(f \cup g)(a) = (0, x)$ for some x, so $a \in A$. It follows that A is an ideal in B. For any $b \in B$, if $b \notin A$ then $b \in X$, hence $(f \cup g)(-_B b) = -(f \cup g)(b) = -g(b) = -(1, b) = (0, -b)$; hence $-b \in A$. So A is a maximal ideal in B. $\square$

An ideal in a ABID A is a subset $I \subseteq A$ such that $0 \in I$, I is closed under $+$, and $\forall a \in A \forall b \in I[a \leq b \in I \rightarrow a \in I]$. I is *prime* iff $\forall a, b \in A[a \cdot b \in I \rightarrow a \in I$ or $b \in I]$.

Proposition 1.7. *If* I *is an ideal in a ABID* A*, then* $(I, +, \cdot, 0)$ *is a ABID.*

Proof. Given $a, b \in I$ with $a \leq b$, we have $b \backslash a \in I$. $\square$

Proposition 1.8. *If* A *is a ABID,* $a, b \in A$*, and* $a \neq b$*, then there is a prime ideal* I *containing one of* a, b *but not the other.*

Proof. Wlog $a \not\leq b$. Then $\{x \in A : x \leq b\}$ is a member of $\mathscr{A} \stackrel{\text{def}}{=} \{I : I$ is an ideal on A with $b \in I$ and $\forall u \geq a[u \notin I]\}$. By Zorn's lemma let I be a maximal member of $\mathscr{A}$. Suppose that $x \cdot y \in I$, but $x, y \notin I$. Then the ideal generated by $I \cup \{x\}$ is not in $\mathscr{A}$, so there exist $x' \in I$ and $u \geq a$ such that $u \leq x + x'$. Similarly there are $y' \in I$ and $v \geq a$ such that $v \leq y + y'$. Then $a \leq u \cdot v \leq (x + x') \cdot (y + y') = x \cdot y + x \cdot y' + x' \cdot y + x' \cdot y' \in I$, contradiction. $\square$

Proposition 1.9. *Any ABID A is isomorphic to a ABID B of the form*

$$B = (B, \cup, \cap, \emptyset).$$

Moreover, for any $a, b \in B$, $a \backslash_B b$ is the usual set-theoretic relative complement $a \backslash b$.

B consists of subsets of some set X, and each $I \in X$ is a member of some $b \in B$.

Proof. Let A be any ABID. Let X be the set of all proper prime ideals in A. For each $a \in A$ let $f(a) = \{I \in X : a \notin I\}$. Then

$$\begin{array}{rcl} I \in f(a+b) & \text{iff} & a + b \notin I \\ & \text{iff} & a \notin I \text{ or } b \notin I \\ & \text{iff} & I \in f(a) \cup f(b); \\ I \in f(a \cdot b) & \text{iff} & a \cdot b \notin I \\ & \text{iff} & a \notin I \text{ and } b \notin I; \\ I \in f(0) & \text{iff} & 0 \notin I \\ & \text{iff} & F. \end{array}$$

Moreover,

$(*)$ $\forall a, b \in A \forall I \in X [a \backslash b \notin I$ iff $a \notin I$ and $b \in I]$.

In fact, suppose that $a, b \in A$ and $I \in X$. First suppose that $a \backslash b \notin I$. Clearly then $a \notin I$. Now $(a \backslash b) \cdot b = 0 \in I$ implies that $b \in I$. Second, suppose that $a \notin I$, $b \in I$, and $a \backslash b \in I$. Then $a = a \cdot b + (a \backslash b) \in I$, contradiction. So $(*)$ holds. By $(*)$, $I \in f(a \backslash b)$ iff $I \in f(a) \backslash f(b)$.

Now f is one-one. For, suppose that $a, b \in A$ and $a \neq b$. By Proposition 1.8, wlog there is a prime ideal I such that $a \in I$ and $b \notin I$. Hence $I \notin f(a)$ but $I \in f(b)$.

If $I \in X$, then I is proper, so there is an $a \in A \backslash I$. Then $I \in f(a)$. □

Proposition 1.10. *If A is a ABID which is not a BA, then A is a maximal ideal in some BA B, with $\forall a, b \in A [a +_A b = a +_B b$, $a \cdot_A b = a \cdot_B b$, and $a \backslash_A b = a \cdot_B -_B b]$. Moreover, $B = A \cup \{-x : x \in A\}$ and $A \cap \{-x : x \in A\} = \emptyset$.*

Proof. By Proposition 1.9 we may assume that $A = (A, \cup, \cap, \emptyset)$, such that $\forall a, b \in A [a \backslash b$ is the set-theoretic relative complement$]$. Moreover, A consists of subsets of some set X, and each $x \in X$ is a member of some $a \in A$. Now let B be the Boolean algebra of subsets of X generated by A.

1.10(1) $B = A \cup \{X \backslash a : a \in A\}$.

In fact, it suffices to note that the set on the right is closed under $\cdot$. For $a, b \in A$ we have $a \cdot b \in A$, $a \cap (X \backslash b) = a \backslash b \in A$. Finally, $(X \backslash a) \cap (X \backslash b) = X \backslash (a \cup b)$. So 1.10(1) holds.

1.10(2) $A \cap \{X \backslash a : a \in A\} = \emptyset$.

In fact, if $a \in A$ and $X \backslash a \in A$ then $X \in A$, contradiction.

1.10(3) $\forall a, b \in B [b \in A$ and $a \subseteq b \rightarrow a \in A]$.

For, suppose that $a, b \in B$, $b \in A$, $a \subseteq b$, and $a \notin A$. By 1.10(1) there is a $c \in A$ such that $a = X \backslash c$. Then $X \backslash c = (X \backslash c) \cap b = b \backslash c \in A$, contradicting 1.10(2).

Now A is an ideal in B. For, obviously $\emptyset \in A$. If $a, b \in A$, then clearly $a \cup b \in A$. If $a, b \in B$, $b \in A$, and $a \subseteq b$, and $a \notin A$, then by (1), $X \backslash a \in A$, and hence $X = b \cup (X \backslash a) \in A$, contradiction. So A is an ideal in B.

By 1.10(1), A is maximal. □

Elementary arithmetic in a ABID can be carried out using the part of Propositions 1.6 and 1.10 that says that $a \backslash b = a \cdot -b$ in the associated BA.

For any ABID A and any $a, b \in A$, let $a \triangle b = (a \backslash b) + (b \backslash a)$. The proof of the following proposition illustrates the use of Propositions 1.6 and 1.10.

Proposition 1.11. *$a \triangle (b \triangle c) = (a \triangle b) \triangle c$.*

Proof.

$$\begin{aligned}
a \triangle (b \triangle c) &= (a \backslash ((b \backslash c) + (c \backslash b))) + (((b \backslash c) + (c \backslash b)) \backslash a) \\
&= a \cdot -(b \cdot -c + c \cdot -b) + (b \cdot -c + c \cdot -b) \cdot -a \\
&= a \cdot (-b + c) \cdot (-c + b) + b \cdot -c \cdot -a + c \cdot -b \cdot -a \\
&= a \cdot -b \cdot -c + a \cdot c \cdot b + b \cdot -c \cdot -a + c \cdot -b \cdot -a \\
&= a \cdot -b \cdot -c + a \cdot b \cdot c + -a \cdot b \cdot -c + -a \cdot -b \cdot c; \\
(a \triangle b) \triangle c &= (a \cdot -b + b \cdot -a) \cdot -c + c \cdot -(a \cdot -b + b \cdot -a) \\
&= a \cdot -b \cdot -c + b \cdot -a \cdot -c + c \cdot (-a + b) \cdot (-b + a) \\
&= a \cdot -b \cdot -c + b \cdot -a \cdot -c + c \cdot -a \cdot -b + c \cdot b \cdot a \\
&= a \cdot -b \cdot -c + -a \cdot b \cdot -c + -a \cdot -b \cdot c + a \cdot b \cdot c.
\end{aligned}$$

□

Proposition 1.12. *If R is a congruence relation on a ABID A, $a, b, a', b' \in A$, aRa', and bRb', then $(a \backslash b) R (a' \backslash b')$.*

Proof. Assume the hypotheses. For each $x \in A$ let $[x]$ be the equivalence class of x under R. Define $f(x) = [x]$ for all $x \in A$. Then f is a homomorphism. Hence by Proposition 1.2, $f(a \backslash b) = f(a) \backslash f(b) = f(a') \backslash f(b') = f(a' \backslash b')$ and the result follows. □

Proposition 1.13. *Let A be a ABID.*

(i) If R is a congruence on A, then $I_R \stackrel{\text{def}}{=} \{a \in A : aR0\}$ is an ideal in A.

(ii) If I is an ideal on A, then $R_I \stackrel{\text{def}}{=} \{(a, b) : (a \backslash b) + (b \backslash a) \in I\}$ is a congruence on A.

(iii) $R_{I_R} = R$ for any congruence R on A.

(iv) $I_{R_I} = I$ for any ideal I on A.

Proof. (i): Assume that R is a congruence on A. Since $0R0$, I_R is nonempty. Suppose that $a \leq b \in I_R$. Thus $bR0$. Hence $a = (a \cdot b) R (a \cdot 0) = 0$, so $a \in I_R$. Suppose that $a, b \in I_R$. Then $aR0$ and $bR0$, so $(a + b) R (0 + 0) = 0$ and $a + b \in I_R$.

(ii): Assume that I is an ideal on A. Suppose that aR_Ib and cR_Id. Then

$$\begin{aligned}(a+c)\backslash(b+d) &= (a\backslash(b+d)) + (c\backslash(b+d)) \\ &= (a\backslash b)\cdot(a\backslash d) + (c\backslash b)\cdot(c\backslash d) \\ &\leq (a\backslash b) + (c\backslash d) \in I,\end{aligned}$$

and similarly $(b+d)\backslash(a+c) \in I$, so $(a+c)R_I(b+d)$. Also,

$$\begin{aligned}(a\cdot c)\backslash(b\cdot d) &= ((a\cdot c)\backslash b) + ((a\cdot c)\backslash d) \\ &= ((a\backslash b)\cdot(c\backslash b)) + ((a\backslash d)\cdot(c\backslash d)) \\ &\leq (a\backslash b) + (c\backslash d) \in I,\end{aligned}$$

and similarly $(b\cdot d)\backslash(a\cdot c) \in I$, so $(a\cdot c)R_I(b\cdot d)$.

This shows that R_I is a congruence on A.

(iii): assume that R is a congruence on A. Suppose that $aR_{I_R}b$. Thus $((a\backslash b)+(b\backslash a)) \in I_R$, so $((a\backslash b)+(b\backslash a))R0$. Now

$$\begin{aligned}[(a\backslash b)+(b\backslash a)]\backslash a &= [(a\backslash b)\backslash a] + [(b\backslash a)\backslash a] \\ &= (b\backslash a).\end{aligned}$$

Thus $(b\backslash a) = ([(a\backslash b)+(b\backslash a)]\backslash a)R(0\backslash a) = 0$, so $a+b = ((b\backslash a)+a)Ra$. Similarly $(a+b)Rb$, so aRb.

Conversely, suppose that aRb. Then $0 = (a\backslash a)R(a\backslash b)$, so $(a\backslash b) \in I_R$. Similarly $(b\backslash a) \in I_R$, so $[(a\backslash b)+(b\backslash a)] \in I_R$. Thus $aR_{I_R}b$.

(iv): assume that I is an ideal on A. Suppose that $a \in I_{R_I}$. Thus aR_I0, so $a = [(a\backslash 0)+(0\backslash a)] \in I$.

Conversely suppose that $a \in I$. Again, $a = [(a\backslash 0)+(0\backslash a)]$, so aR_I0, hence $a \in I_{R_I}$. □

A *filter* on a ABID A is a subset F of A such that F is nonempty, F is closed under $\cdot$, and $\forall a, b \in A[b \geq a \in F \to b \in F]$. If A is a ABID and $a \in A$ we let $a^\perp = \{b \in A : a\cdot b = 0\}$. For F a filter, $F^\perp = \bigcup_{a\in F} a^\perp$.

Proposition 1.14. *$a^\perp$ is an ideal of A.* □

Proposition 1.15. *If F is a filter on a ABID A, then $F^\perp$ is an ideal of A.*

Proof. Fix $x \in F$. Then $0 \in x^\perp$, so $0 \in F^\perp$. Suppose that $u, v \in F^\perp$. Choose $a, b \in F$ such that $u \in a^\perp$ and $v \in b^\perp$. Thus $a\cdot u = 0 = b\cdot v$. Now $a\cdot b \in F$ and $(u+v)\cdot a\cdot b = 0$, so $u+v \in (a\cdot b)^\perp$ and so $u+v \in F^\perp$. Finally, suppose that $a \leq b \in F^\perp$. Say $y \in F$ and $b \in y^\perp$. Then $b\cdot y = 0$, and hence $a\cdot y = 0$. So $a \in y^\perp$ and hence $a \in F^\perp$. □

Proposition 1.16. *If A is a ABID and F is a filter on A, then $R_{F^\perp} = \{(a,b) : \exists c \in F[((a\backslash b)+(b\backslash a))\cdot c = 0]\}$.* □

Proposition 1.17. *If A is a ABID and $a \in A$, then $A \restriction a$ is a BA.* $\square$

Proposition 1.18. *If A is a ABID and $a \in A$, define $f(x) = (x \cdot a, x \backslash a)$. Then f is an isomorphism of A onto $(A \restriction a) \times a^{\perp}$.*

Proof. If $x \in A$, then $(x \backslash a) \cdot a = 0$, so $x \backslash a \in a^{\perp}$. Clearly f preserves the operations. If $x \leq a$ and $y \in a^{\perp}$, then $y = y \cdot a + y \backslash a = y \backslash a$ and so $f(x + y) = (x, y)$. Thus f maps onto $(A \restriction a) \times a^{\perp}$. Suppose that $f(x) = f(y)$. Thus $x \cdot a = y \cdot a$ and $x \backslash a = y \backslash a$. So $x = x \cdot a + x \backslash a = y \cdot a + y \backslash a = y$. So f is one-one. $\square$

Let A be a ABID, $a \in A$, and I an ideal of A. Then we let $[a]_I$ be the equivalence class of a under the ideal I; thus $[a]_I = \{a \triangle b : b \in I\}$.

Proposition 1.19. *Suppose that X generates a ABID A and $f : X \to B$ with B a ABID. Then f extends to a homomorphism $f^+ : A \to B$ iff*

$$\forall m \in \omega \backslash \{0\} \forall n \in \omega \forall a \in {}^m X \forall b \in {}^n X \left[\left(\prod_{i<m} a_i \right) \backslash \left(\sum_{j<n} b_j \right) = 0 \to \right.$$
$$\left. \left(\prod_{i<m} f(a_i) \right) \backslash \left(\sum_{j<n} f(b_j) \right) = 0 \right].$$

Proof. $\Rightarrow$: Clear. $\Leftarrow$: assume the indicated condition. Now $A = \bigcup \{\langle F \rangle : F \in [X]^{<\omega}\}$, so it suffices to find for each $F \in [X]^{<\omega}$ a homomorpism $g_F : \langle F \rangle \to B$ extending f. Now $\langle F \rangle$ is a BA, with unit $\sum F$. Moreover, $-x$ for any $x \in \langle F \rangle$ is $(\sum F) \backslash x$. Now $f \restriction F$ maps into the BA $B \restriction (\sum f[F])$. We claim that

$$(*) \quad \forall m, n \in \omega \forall a \in {}^m X \forall b \in {}^n X \left[\prod_{i<m} a_i \cdot \prod_{j<n} (-b_j) = 0 \to \prod_{i<m} f(a_i) \cdot \prod_{j<n} (-f(b_j)) = 0 \right].$$

Here $m = 0$ is possible, with $\prod_{i<0} a_i = \sum F$; similarly $\prod_{i<0} f(a_i)) = \sum f[F]$. To prove $(*)$ it suffices to note that $\prod_{j<n}(-b_j) = \prod_{j<n}((\sum F) \backslash b_j) = (\sum F) \backslash \sum_{j<n} b_j$, and similarly $\prod_{j<n}(-f(b_j)) = (\sum f[F]) \backslash \sum_{j<n} f(b_j)$. So $(*)$ follows from the indicated condition.

Now by $(*)$ there is a homomorphism g_F from the BA $\langle F \rangle$ into the BA $B \restriction (\sum f[F])$ which extends f. Clearly g_F is as desired. $\square$

Proposition 1.20. *If A, B are ABIDs, $L \subseteq A$ is a chain which generates A, and $f : L \to B$, then f extends to a homomorphism from A into B iff f is increasing.*

Proof. By Proposition 1.19. $\square$

Proposition 1.21. *If A, B are ABIDs, $L \subseteq A$ is a chain which generates A, and $f : L \to B$, then f extends to an isomorphism from A into B iff $\forall a, a' \in L[a \leq a'$ iff $f(a) \leq f(a')]$.*

Proof. By Proposition 1.19. $\square$

Proposition 1.22. *If L is a linear order which generates ABIDs A, B, then there is an isomorphism of A onto B which is the identity on L.* $\square$

2. Examples

For any linear order L which has a least element, $\mathrm{Intalg}(L)$ is the Boolean algebra of subsets of L generated by the half-open intervals $[a, b)$ with $a < b$ in L. This notion is carefully analysed in the Boolean algebra handbook. Now we define the *interval algebra* over L, as a ABID, to be

$$\mathrm{Intalg}_d(L) = \{[a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) : m \in \omega, a_0 < b_0 < a_1 < \cdots < b_{m-1}\}$$

Here we allow $m = 0$; so $\emptyset \in \mathrm{Intalg}_d(L)$. Note that ∞ is not adjoined, so $\mathrm{Intalg}_d(L)$ is in general not a BA. Thus, for example, $\mathrm{Intalg}(\omega)$ and $\mathrm{Intalg}_d(\omega)$ are different as sets.

Proposition 2.1. $\mathrm{Intalg}_d(L)$ *is closed under* $\cup, \cap, \setminus$. *Hence* $\mathrm{Intalg}_d(L)$ *is a ABID.*

Proof. For closure under $\cap$ it suffices to prove that

$$(*) \qquad [c, d) \cap ([a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1})) \in \mathrm{Intalg}_d(L)$$

whenever $c < d$ and $a_0 < b_0 < a_1 < \cdots < b_{m-1}$. To prove this, for each $i < m$ let $s_i = \max(c, a_i)$ and $t_i = \min(d, b_i)$. Thus $[c, d) \cap [a_i, b_i) = [s_i, t_i)$, which is nonempty iff $s_i < t_i$. Let $M = \{i < m : s_i < t_i\}$. If $i, j \in M$ and $i < j$, then $s_i < t_i \leq b_i < a_j \leq s_j$. If $M = \emptyset$, then $(*)$ is empty. If $M \neq \emptyset$ and we enumerate M in increasing order as $k(0) < \cdots < k(n-1)$, then $(*)$ is $[s_{k(0)}, t_{k(0)}) \cup \ldots \cup [s_{k(n-1)}, t_{k(n-1)})$. This shows closure under $\cap$.

For closure under $\cup$ it suffices to show that

$$(**) \qquad [c, d) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in \mathrm{Intalg}_d(L)$$

whenever $c < d$ and $a_0 < b_0 < a_1 < \cdots < b_{m-1}$. To prove this we consider several cases.

Case 1. $b_{m-1} < c$. Then $(**)$ is $[a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \cup [c, d)$.

Case 2. $b_{m-1} = c$. Then $(**)$ is $[a_0, b_0) \cup \ldots \cup [a_{m-2}, b_{m-2}) \cup [a_{m-1}, d)$.

Case 3. $c < a_0$.

Subcase 3.1. $d < a_0$. Then $(**)$ is clear.

Subcase 3.2. $i < m$, $a_i \leq d \leq b_i$. Then $(**)$ is $[c, b_i) \cup [a_{i+1}, b_{i+1}) \cup \cdots \cup [a_{m-1}, b_{m-1})$.

Subcase 3.3. $i < m-1$, $b_i < d < a_{i+1}$. Then $(**)$ is $[c, d) \cup [a_{i+1}, b_{i+1}) \cup \cdots \cup [a_{m-1}, b_{m-1})$.

Subcase 3.4. $b_{m-1} < d$. Then $(**)$ is $[c, d)$.

Case 4. $a_0 \leq c < b_{m-1} < d$.

Subcase 4.1. $a_i \leq c \leq b_i$. Then $(**)$ is $[a_0, b_0) \cup \ldots \cup [a_i, d)$.

Subcase 4.2. $i < m-1$ and $b_i < c < a_{i+1}$. Then $(**)$ is $[a_0, b_0) \cup \ldots \cup [a_i, b_i) \cup [c, d)$.

Case 5. $a_0 \leq c < b_{m-1}$ and $d \leq b_{m-1}$. Let

$$x = [c, b_{m-1}) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}),$$
$$y = [a_0, d) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}).$$

Note that $[c, b_{m-1}) \cap [a_0, d) = [c, d)$. Hence $x \cap y$ is equal to $(**)$. Hence, because $\mathrm{Intalg}_d(L)$ is closed under $\cap$, it suffices to show that $x, y \in \mathrm{Intalg}_d(L)$. First we consider x.

Subcase 5.1. $a_i \leq c < b_i$. Then x is $[a_0, b_0) \cup \ldots \cup [a_i, b_{m-1})$.

Subcase 5.2. $i < m-1$ and $b_i < c \leq a_{i+1}$. Then x is $[a_0, b_0) \cup \ldots \cup [a_i, b_i) \cup [c, b_{m-1})$.

Thus $x \in \mathrm{Intalg}_d(L)$. Now we consider y.

Subcase 5.3. $a_i \leq d \leq b_i$. Then y is $[a_0, b_i) \cup [a_{i+1}, b_{i+1}) \cup \cdots \cup [a_{m-1}, b_{m-1})$.

Subcase 5.4. $i < m-1$ and $b_i < d < a_{i+1}$. Then y is $[a_0, d) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1})$.

Thus $y \in \mathrm{Intalg}_d(L)$.

This completes the proof of closure under $\cup$.

To show closure under $\backslash$, note that $\forall x \in A[(0 \backslash x) = 0]$, and if $m \neq 0$ then

$$\begin{aligned}&([a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1})) \backslash ([c_0, d_0) \cup \ldots \cup [c_{n-1}, d_{n-1})) \\ &= \bigcup_{i<m} ([a_i, b_i) \backslash ([c_0, d_0) \cup \ldots \cup [c_{n-1}, d_{n-1}))),\end{aligned}$$

and for all $i < m$,

$$[a_i, b_i) \backslash ([c_0, d_0) \cup \ldots \cup [c_{n-1}, d_{n-1})) = \bigcap_{j<n} ([a_i, b_i) \backslash [c_j, d_j)).$$

Hence by the above it suffices to note that for $i < m$ and $j < n$ the element $[a_i, b_i) \backslash [c_j, d_j)$ is in $\mathrm{Intalg}_d(L)$. We show this by cases.

Case 1. $b_i \leq c_j$. Then $[a_i, b_i) \backslash [c_j, d_j) = [a_i, b_i)$.

Case 2. $a_i \leq c_j < b_i$.

Subcase 2.1. $b_i \leq d_j$. Then $[a_i, b_i) \backslash [c_j, d_j) = [a_i, c_j)$.

Subcase 2.2. $d_j < b_i$. Then $[a_i, b_i) \backslash [c_j, d_j) = [a_i, c_j) \cup [d_j, b_i)$.

Case 3. $c_j < a_i$.

Subcase 3.1. $b_i \leq d_j$. Then $[a_i, b_i) \backslash [c_j, d_j) = \emptyset$.

Subcase 3.2. $a_i < d_j < b_i$. Then $[a_i, b_i) \backslash [c_j, d_j) = [d_j, b_i)$.

Subcase 3.3. $d_j \leq a_i$. Then $[a_i, b_i) \backslash [c_j, d_j) = [a_i, b_i)$. □

3. Vaught's theorem

Let A and B be ABIDs. A *V-correspondence* between A and B is a relation $R \subseteq A \times B$ such that the following conditions hold:

(V1) $\forall a \in A[(a, 0) \in R$ iff $a = 0]$.

(V2) $\forall b \in B[(0, b) \in R$ iff $b = 0]$.

(V3) $\forall (a, b) \in R \forall c \in A \exists d_0, d_1 \in B[(a \cdot c, b \cdot d_0), (a \backslash c, b \backslash d_0), (a + c, b + d_1), (c \backslash a, d_1 \backslash b) \in R]$.

(V4) $\forall (a, b) \in R \forall d \in B \exists c_0, c_1 \in A[(a \cdot c_0, b \cdot d), (a \backslash c_0, b \backslash d), (a + c_1, b + d), (c_1 \backslash a, d \backslash b) \in R]$.

Proposition 3.1. *If R is a V-correspondence between countably infinite ABIDs A, B, then A and B are isomorphic.*

Proof. Assume the hypotheses. Let $c_0 = 0, c_1, \ldots$ enumerate all elements of A and $d_0 = 0, d_1, \ldots$ all elements of B. We now define finite linearly ordered sets $L_0 \subseteq L_1 \subseteq \cdots \subseteq A$ and strictly increasing functions $g_0 : L_0 \to B, g_1 : L_1 \to B, \ldots$ such that for each $n \in \omega$ the following conditions hold:

3.1(1) If $L_n = \{a_0, \ldots, a_k\}$ with $a_0 < \cdots < a_k$ and $\forall i \leq k[b_i = g_n(a_i)]$, then $(a_k, b_k) \in R$ and $\forall i < k[(a_{i+1} \backslash a_i, b_{i+1} \backslash b_i) \in R$.

3.1(2) $g_{n+1} \restriction L_n = g_n$.

3.1(3) $c_0, \ldots, c_n \in \langle L_n \rangle$, where $\langle L_n \rangle$ is the subalgebra of $(A, +, \cdot, \backslash, 0)$ generated by L_n.

3.1(4) $d_0, \ldots, d_n \in \langle g_n[L_n] \rangle$.

We define $L_0 = \{0\}$ and $g_0 = \{(0, 0)\}$. Clearly 3.1(1), 3.1(3) and 3.1(4) hold for $n = 0$. Now suppose that L_m and g_m have been defined for all $m \leq n$ so that 3.1(1)–3.1(4) hold. Say $L_n = \{a_0, \ldots, a_k\}$ with $a_0 < \cdots < a_k$ and $\forall i \leq k[b_i = g_n(a_i)]$. For $i \leq k$ let $a'_{2i} = a_i$. For $i < k$ let $a'_{2i+1} = c_{n+1} \cdot a_{i+1} + a_i$. Let $a'_{2k+1} = c_{n+1} + a_k$.

3.1(5) $a'_0 \leq \cdots \leq a'_{2k+1}$.

In fact, for $i < k$ we have $a'_{2i} = a_i \leq a'_{2i+1} \leq a_{i+1} = a'_{2i+2}$, and $a'_{2k} = a_k \leq a'_{2k+1}$.

3.1(6) $\forall i < k[a'_{2i+1} \backslash a'_{2i} = c_{n+1} \cdot (a_{i+1} \backslash a_i)]$.

In fact, if $i < k$ then $a'_{2i+1} \backslash a'_{2i} = (c_{n+1} \cdot a_{i+1} + a_i) \backslash a_i = (c_{n+1} \cdot a_{i+1}) \backslash a_i = c_{n+1} \cdot (a_{i+1} \backslash a_i)$.

3.1(7) $\forall i < k[a'_{2i+2} \backslash a'_{2i+1} = (a_{i+1} \backslash a_i) \backslash c_{n+1}]$.

In fact, if $i < k$ then $a'_{2i+2} \backslash a'_{2i+1} = a_{i+1} \backslash ((c_{n+1} \cdot a_{i+1}) + a_i) = (a_{i+1} \backslash (c_{n+1} \cdot a_{i+1})) \cdot (a_{i+1} \backslash a_i) = (a_{i+1} \backslash c_{n+1}) \cdot (a_{i+1} \backslash a_i) = (a_{i+1} \backslash a_i) \backslash c_{n+1}$.

3.1(8) $a'_{2k+1} \backslash a'_{2k} = c_{n+1} \backslash a_k$.

For, $a'_{2k+1} \backslash a'_{2k} = (c_{n+1} + a_k) \backslash a_k = c_{n+1} \backslash a_k$.

Now by 3.1(1), if $i < k$, we have $(a_{i+1} \backslash a_i, b_{i+1} \backslash b_i) \in R$. Hence by (V3) there exist $d'_i \in B$ for $i < k$ such that

$$((a_{i+1} \backslash a_i) \cdot c_{n+1}, (b_{i+1} \backslash b_i) \cdot d'_i) \in R \quad \text{and}$$
$$((a_{i+1} \backslash a_i) \backslash c_{n+1}, (b_{i+1} \backslash b_i) \backslash d'_i) \in R.$$

Also, $(a_k, b_k) \in R$, so there is a $d'_k \in B$ such that

$$(a_k + c_{n+1}, b_k + d'_k) \in R \text{ and } (c_{n+1} \backslash a_k, d'_k \backslash b_k) \in R.$$

Now for all $i \leq k$ let $b'_{2i} = b_i$. For all $i < k$ let $b'_{2i+1} = b_i + ((b_{i+1} \backslash b_i) \cdot d'_i)$, and let $b'_{2k+1} = b_k + d'_k$.

3.1(9) $b'_0 \leq \cdots \leq b'_{2k+1}$.

To prove this, first note that for any $i \leq k$ we have $b_i = g_n(a_i) \leq g_n(a_{i+1}) = b_{i+1}$. Hence if $i < k$, then $b'_{2i} = b_i \leq b_i + ((b_{i+1} \backslash b_i) \cdot d'_i = b'_{2i+1} \leq b_{i+1} = b'_{2i+2}$. Also, $b'_{2k-1} = b'_{2(k-1)+1} = b_{k-1} + ((b_k \backslash b_{k-1}) \cdot d'_{k-1}) \leq b_k = b'_{2k} \leq b'_{2k+1}$.

3.1(10) $\forall i < k[b'_{2i+1}\backslash b'_{2i} = (b_{i+1}\backslash b_i) \cdot d'_i]$.

In fact, if $i < k$ then $b'_{2i+1}\backslash b'_{2i} = ((b_{i+1}\backslash b_i) \cdot d'_i)\backslash b_i = ((b_{i+1}\backslash b_i)\backslash b_i) \cdot (d'_i\backslash b_i) = (b_{i+1}\backslash b_i) \cdot (d'_i\backslash b_i) = (b_{i+1}\backslash b_i) \cdot d'_i$.

3.1(11) $\forall i < k[b'_{2i+2}\backslash b'_{2i+1} = (b_{i+1}\backslash b_i)\backslash d'_i]$.

For, if $i < k$, then

$$\begin{aligned} b'_{2i+2}\backslash b'_{2i+1} &= b_{i+1}\backslash(b_i + ((b_{i+1}\backslash b_i) \cdot d'_i)) \\ &= (b_{i+1}\backslash b_i) \cdot (b_{i+1}\backslash((b_{i+1}\backslash b_i) \cdot d'_i)) \\ &= (b_{i+1}\backslash b_i) \cdot ((b_{i+1}\backslash(b_{i+1}\backslash b_i)) + (b_{i+1}\backslash d'_i)) \\ &= (b_{i+1}\backslash b_i) \cdot (b_{i+1} \cdot b_i + (b_{i+1}\backslash d'_i) \\ &= (b_{i+1}\backslash b_i)\backslash d'_i. \end{aligned}$$

3.1(12) $b'_{2k+1}\backslash b'_{2k} = d'_k\backslash b_k$.

For, $b'_{2k+1}\backslash b'_{2k} = (b_k + d'_k)\backslash b_k = d'_k\backslash b_k$.

3.1(13) $\forall j < 2k+1[(a'_{j+1}\backslash a'_j, b'_{j+1}\backslash b'_j) \in R]$.

In fact, for $i < k$ we have

$$\begin{aligned} a'_{2i+1}\backslash a'_{2i} &= c_{n+1} \cdot (a_{i+1}\backslash a_i) \quad \text{by 3.1(6)}, \\ b'_{2i+1}\backslash b'_{2i} &= (b_{i+1}\backslash b_i) \cdot d'_i \quad \text{by 3.1(10)}. \end{aligned}$$

So $(a'_{2i+1}\backslash a'_{2i}, b'_{2i+1}\backslash b'_{2i}) \in R$ by the definition of d'_i.

Also, for $i < k$ we have

$$\begin{aligned} a'_{2i+2}\backslash a'_{2i+1} &= (a_{i+1}\backslash a_i)\backslash c_{n+1} \quad \text{by 3.1(7)}; \\ b'_{2i+2}\backslash b'_{2i+1} &= (b_{i+1}\backslash b_i)\backslash d'_i \quad \text{by 3.1(11)}. \end{aligned}$$

So $(a'_{2i+2}\backslash a'_{2i+1}, b'_{2i+2}\backslash b'_{2i+1}) \in R$ by the definition of d'_i.

Finally,

$$\begin{aligned} a'_{2k+1}\backslash a'_{2k} &= c_{n+1}\backslash a_k \quad \text{by 3.1(8)}; \\ b'_{2k+1}\backslash b'_{2k} &= d'_k\backslash b_k \quad \text{by 3.1(12)}. \end{aligned}$$

So $(a'_{2k+1}\backslash a'_{2k}, b'_{2k+1}\backslash b'_{2k}) \in R$ by the definition of d'_k.

This proves 3.1(13).

3.1(14) $(a'_{2k+1}, b'_{2k+1}) \in R$.

For, $a'_{2k+1} = (a'_{2k+1}\backslash a'_{2k}) + (a'_{2k+1} \cdot a'_{2k}) = (c_{n+1}\backslash a_k) + a_k = c_{n+1} + a_k$ by 3.1(8), and $b'_{2k+1} = b_k + d'_k$. Hence $(a'_{2k+1}, b'_{2k+1}) \in R$ by the definition of d'_k.

3.1(15) $\forall j < 2k+1[a'_j = a'_{j+1} \leftrightarrow b'_j = b'_{j+1}]$.

In fact, suppose that $j < 2k+1$ and $a'_j = a'_{j+1}$. Then $a'_{j+1} \backslash a'_j = 0$, and so by 3.1(13) and (V2), $b'_{j+1} \backslash b'_j = 0$. So $b'_{j+1} = b'_{j+1} \cdot b'_j = b'_j$. The other direction is similar, using 3.1(13) and (V1).

3.1(16) $\forall i < k[a'_{2i} < a'_{2i+1}$ or $a'_{2i+1} < a'_{2i+2}]$.

This follows from 3.1(1) and 3.1(5), since $a'_{2i} = a_i < a_{i+1} = a'_{2i+2}$.

Now we let $a''_0 < \cdots < a''_l$ be $a'_0, \ldots, a'_{2k+1}$ in increasing order, and set $L'_{n+1} = \{a''_0, \ldots, a''_l\}$. Further, if $a''_i = a'_j$ we let $b''_i = b'_j$. This is unambigous by 3.1(15) and 3.1(16). Let $g'_{n+1}(a''_i) = b''_i$. Now $a''_l = a'_{2k+1}$, so

3.1(17) $(a''_l, b''_l) \in R$

by 3.1(14). Suppose that $j < l$. Say $a''_j = a'_i$ with $a'_i < a'_{i+1}$. Then $a''_{j+1} = a'_{i+1}$, so

3.1(18) $(a''_{j+1} \backslash a''_j, b''_{j+1} \backslash b''_j) \in R$

by 3.1(13).

Now if $i \leq k$ then $a'_{2i} = a_i$ and $b'_{2i} = b_i$. For $a''_j = a'_{2i}$ we have $g'_{n+1}(a''_j) = b''_j = b'_{2i} = b_i = g_n(a_i) = g_n(a'_{2i}) = g_n(a''_j)$.

3.1(19) $\forall i \leq k[i > 0 \rightarrow a_i = (a_i \backslash a_{i-1}) + (a_{i-1} \backslash a_{i-2}) + \cdots + (a_1 \backslash a_0)]$.

We prove 3.1(19) by induction on i. It is true for $i = 1$ since $a_0 = 0$. Assume it for $i < k$. Then $a_{i+1} = (a_{i+1} \cdot a_i) + (a_{i+1} \backslash a_i) = (a_{i+1} \backslash a_i) + a_i = (a_{i+1} \backslash a_i) + (a_i \backslash a_{i-1}) + \cdots + (a_1 \backslash a_0)$. So 3.1(19) holds.

Now for $k = 0$ we have $a_0 = 0$, $a'_0 = 0$, $a'_1 = c_1$, $c_1 = a'_1 \in L_1$. If $k > 0$, then

$$\begin{aligned} c_{n+1} &= c_{n+1} \cdot a_k + c_{n+1} \backslash a_k = \sum_{i<k} (c_{n+1} \cdot (a_{i+1} \backslash a_i) + c_{n+1} \backslash a_k \quad \text{by 3.1(19)} \\ &= \sum_{i<k} (a'_{2i+1} \backslash a'_{2i}) + a'_{2k+1} \backslash a'_{2k} \quad \text{by 3.1(6) and 3.1(8)} \\ &\in L'_{n+1}. \end{aligned}$$

Now for $i \leq l$ let $b'''_{2i} = b''_i$ and for $i < l$ let $b'''_{2i+1} = (d_{n+1} \cdot b''_{i+1}) + b''_i$. Let $b'''_{2l+1} = d_{n+1} + b''_l$.

3.1(20) $b'''_0 \leq \cdots \leq b'''_{2l+1}$.

In fact, for $i < l$ we have $b'''_{2i} = b''_i \leq b'''_{2i+1} \leq b''_{i+1} = b'''_{2i+2}$, and $b'''_{2l} = b''_l \leq b'''_{2l+1}$.

3.1(21) $\forall i < l[b'''_{2i+1} \backslash b'''_{2i} = d_{n+1} \cdot (b''_{i+1} \backslash b''_i)]$.

In fact, if $i < l$ then $b'''_{2i+1} \backslash b'''_{2i} = (d_{n+1} \cdot b''_{i+1} + b''_i) \backslash b''_i = (d_{n+1} \cdot b''_{i+1}) \backslash b''_i = d_{n+1} \cdot (b''_{i+1} \backslash b''_i)$.

3.1(22) $\forall i < l[b'''_{2i+2} \backslash b'''_{2i+1} = (b''_{i+1} \backslash b''_i) \backslash d_{n+1}]$.

In fact, if $i < l$ then $b'''_{2i+2} \backslash b'''_{2i+1} = b''_{i+1} \backslash ((d_{n+1} \cdot b''_{i+1}) + b''_i) = (b''_{i+1} \backslash (d_{n+1} \cdot b''_{i+1})) \cdot (b''_{i+1} \backslash b''_i) = (b''_{i+1} \backslash d_{n+1}) \cdot (b''_{i+1} \backslash b''_i) = (b''_{i+1} \backslash b''_i) \backslash d_{n+1}$.

3.1(23) $b'''_{2l+1} \backslash b'''_{2l} = d_{n+1} \backslash b''_l$.

For, $b'''_{2l+1} \backslash b'''_{2l} = (d_{n+1} + b''_l) \backslash b''_l = d_{n+1} \backslash b''_l$.

Now by 3.1(18), if $i < l$, we have $(a''_{i+1}\backslash a''_i, b''_{i+1}\backslash b''_i) \in R$. Hence by (V4) there exist $e_i \in A$ for $i < l$ such that

$$((a''_{i+1}\backslash a''_i) \cdot e_i, (b''_{i+1}\backslash b''_i) \cdot d_{n+1}) \in R \quad \text{and}$$
$$((a''_{i+1}\backslash a''_i)\backslash e_i, (b''_{i+1}\backslash b''_i)\backslash d_{n+1}) \in R.$$

Also, by 3.1(17) $(a''_l, b''_l) \in R$, so by (V4) there is a $e_l \in A$ such that

$$(a''_l + e_l, b''_l + d_{n+1}), (e_l\backslash a''_l, d_{n+1}\backslash b''_l) \in R.$$

Now for all $i \leq l$ let $a'''_{2i} = a''_i$. For all $i < l$ let $a'''_{2i+1} = a''_i + ((a''_{i+1}\backslash a''_i) \cdot e_i)$, and let $a'''_{2l+1} = a''_l + e_l$.

3.1(24) $a'''_0 \leq \cdots \leq a'''_{2l+1}$.

For, first note that for any $i \leq l$ we have $a''_i = g'^{-1}_{n+1}(b''_i) \leq g'^{-1}_{n+1}(b''_{i+1}) = a''_{i+1}$. Hence if $i < l$, then $a'''_{2i} = a''_i \leq a''_i + ((a''_{i+1}\backslash a''_i) \cdot e_i = a'''_{2i+1} \leq a''_{i+1} = a'''_{2i+2}$. Also, $a'''_{2l-1} = a'''_{2(l-1)+1} = a''_{l-1} + (a''_l\backslash a''_{l-1}) \cdot e_{l-1} \leq a''_l = a'''_{2l} \leq a'''_{2l+1}$.

3.1(25) $\forall i < l[a'''_{2i+1}\backslash a'''_{2i} = (a''_{i+1}\backslash a''_i) \cdot e_i]$.

In fact, if $i < l$ then $a'''_{2i+1}\backslash a'''_{2i} = ((a''_{i+1}\backslash a''_i)\cdot e_i)\backslash a''_i = ((a''_{i+1}\backslash a''_i)\backslash a''_i)\cdot(e_i\backslash a''_i) = (a''_{i+1}\backslash a''_i)\cdot(e_i\backslash a''_i) = (a''_{i+1}\backslash a''_i) \cdot e_i$.

3.1(26) $\forall i < l[a'''_{2i+2}\backslash a'''_{2i+1} = (a''_{i+1}\backslash a''_i)\backslash e_i]$.

For, if $i < l$, then $a'''_{2i+2}\backslash a'''_{2i+1} = a''_{i+1}\backslash(a''_i + ((a''_{i+1}\backslash a''_i)\cdot e_i)) = (a''_{i+1}\backslash a''_i)\cdot(a''_{i+1}\backslash((a''_{i+1}\backslash a''_i)\cdot e_i)) = (a''_{i+1}\backslash a''_i) \cdot (a''_{i+1}\backslash e_i) = (a''_{i+1}\backslash a''_i)\backslash e_i]$.

3.1(27) $a'''_{2l+1}\backslash a'''_{2l} = e_l\backslash a''_l$.

For, $a'''_{2l+1}\backslash a'''_{2l} = (a''_l + e_l)\backslash a''_l = e_l\backslash a''_l$.

3.1(28) $\forall j < 2l+1[(a'''_{j+1}\backslash a'''_j, b'''_{j+1}\backslash b'''_j) \in R]$.

In fact, for $i < l$ we have

$$a'''_{2i+1}\backslash a'''_{2i} = (a''_{i+1}\backslash a''_i) \cdot e_i \quad \text{by 3.1(25)},$$
$$b'''_{2i+1}\backslash b'''_{2i} = d_{n+1} \cdot (b''_{i+1}\backslash b''_i) \quad \text{by 3.1(21)}.$$

So $(a'''_{2i+1}\backslash a'''_{2i}, b'''_{2i+1}\backslash b'''_{2i}) \in R$ by the definition of e_i.

Also, for $i < l$ we have

$$a'''_{2i+2}\backslash a'''_{2i+1} = (a''_{i+1}\backslash a''_i)\backslash e_i \quad \text{by 3.1(26)};$$
$$b'''_{2i+2}\backslash b'''_{2i+1} = (b''_{i+1}\backslash b''_i)\backslash d_{n+1} \quad \text{by 3.1(22)}.$$

So $(a'''_{2i+2}\backslash a'''_{2i+1}, b'''_{2i+2}\backslash b'''_{2i+1}) \in R$ by the definition of e_i.

Finally,

$$a'''_{2l+1}\backslash a'''_{2l} = e_l\backslash a''_l \quad \text{by 3.1(27)};$$
$$b'''_{2l+1}\backslash b'''_{2l} = d_{n+1}\backslash b_l \quad \text{by 3.1(23)}.$$

So $(a'''_{2l+1}\backslash a'''_{2l}, b'''_{2l+1}\backslash b'''_{2l}) \in R$ by the definition of e_l.

This proves 3.1(28).

3.1(29) $(a'''_{2l+1}, b'''_{2l+1}) \in R$.

For, $a'''_{2l+1} = (a'''_{2l+1}\backslash a'''_{2l}) + (a'''_{2l+1} \cdot a'''_{2l}) = (e_l\backslash a''_l) + a''_l = e_l + a''_l$ by 3.1(27), and $b'''_{2l+1} = b''_l + d_{n+1}$. Hence $(a'''_{2l+1}, b'''_{2l+1}) \in R$ by the definition of e_l.

3.1(30) $\forall j < 2l+1[a'''_j = a'''_{j+1} \leftrightarrow b'''_j = b'''_{j+1}]$.

In fact, suppose that $j < 2l+1$ and $a'''_j = a'''_{j+1}$. Then $a'''_{j+1}\backslash a'''_j = 0$, and so by 3.1(28) and (V2), $b'''_{j+1}\backslash b'''_j = 0$. So $b'''_{j+1} = b'''_{j+1} \cdot b'''_j = b'''_j$. The other direction is similar, using 3.1(28) and (V1).

3.1(31) $\forall i < l[a'''_{2i} < a'''_{2i+1}$ or $a'''_{2i+1} < a'''_{2i+2}]$.

This follows since $a'''_{2i} = a''_i < a''_{i+1} = a'''_{2i+2}$, using the definition of a'' following 3.1(16).

Now we let $a^{iv}_0 < \cdots < a^{iv}_s$ be $a'''_0, \ldots, a'''_{2l+1}$ in increasing order, and set $L_{n+1} = \{a^{iv}_0, \ldots, a^{iv}_s\}$. Further, if $a^{iv}_i = a'''_j$ we let $b^{iv}_i = b'''_j$. This is unambigous by 3.1(30). Let $g_{n+1}(a^{iv}_i) = b^{iv}_i$. Now $a^{iv}_s = a'''_{2l+1}$, so $(a^{iv}_s, b^{iv}_s) \in R$ by 3.1(29). Suppose that $j < s$. Say $a^{iv}_j = a'''_i$ with $a'''_i < a'''_{i+1}$. Then $a^{iv}_{j+1} = a'''_{i+1}$, so $(a^{iv}_{j+1}\backslash a^{iv}_j, b^{iv}_{j+1}\backslash b^{iv}_j) \in R$ by 3.1(28).

3.1(32) $\forall i \leq l[i > 0 \rightarrow b''_i = (b''_i\backslash b''_{i-1}) + (b''_{i-1}\backslash b''_{i-2}) + \cdots + (b''_1\backslash b''_0)]$.

We prove 3.1(32) by induction on i. It is true for $i = 1$ since $b''_0 = 0$. Assume it for $i < l$. Then $b''_{i+1} = (b''_{i+1} \cdot b''_i) + (b''_{i+1}\backslash b''_i) = (b''_{i+1}\backslash b''_i) + b''_i = (b''_{i+1}\backslash b''_i) + (b''_i\backslash b''_{i-1}) + \cdots + (b''_1\backslash b''_0)$. So 3.1(32) holds.

Now $s > 0$, and

$$\begin{aligned} d_{n+1} &= d_{n+1} \cdot b''_l + d_{n+1}\backslash b''_l = \sum_{i<l}(d_{n+1} \cdot (b''_{i+1}\backslash b''_i)) + d_{n+1}\backslash b''_l \quad \text{by 3.1(32)} \\ &= \sum_{i<l}(b'''_{2i+1}\backslash b'''_{2i}) + b'''_{2l+1}\backslash b'''_{2l} \quad \text{by 3.1(21) and 3.1(23)} \\ &\in L_{n+1}. \end{aligned}$$

3.1(33) $L_n \subseteq L_{n+1}$.

In fact, let $i \leq k$, so that a_i is a typical element of L_n. Then $a'_{2i} = a_i$. Say $j < s$ and $a''_j = a'_{2i}$. Then $a'''_{2j} = a''_j$. There is a $u < s$ such that $a^{iv}_u = a'''_{2j}$. Then $a^{iv}_u \in L_{n+1}$, proving 3.1(33).

3.1(34) $g_n \subseteq g_{n+1}$

For, let i, j, u be as in the proof of 3.1(33). Then $g_{n+1}(a_i) = g_{n+1}(a^{iv}_u) = b^{iv}_u$. Now $a^{iv}_u = a'''_{2j}$, so $b^{iv}_u = b'''_{2j} = b''_j$. Since $a''_j = a'_{2i}$, we have $b''_j = b'_{2i} = b_i = g_n(a_i)$. This proves 3.1(34).

We have now verified 3.1(1)–3.1(4) for $n+1$. It follows that $\bigcup_{n\in\omega} g_n$ is strictly increasing from a generating chain of A to a generating chain of B. By Proposition 1.22 this union extends to an isomorphism from A onto B. □

Let A and B be BAs. A *weak V-relation between A and B* is a relation $R \subseteq A \times B$ such that the following conditions hold:

(V5) $(0, b) \in R$ iff $b = 0$.

(V6) If $(a, b) \in R$ and $c \in A$, then there exist $d_0, d_1 \in B$ such that $(a \cdot c, b \cdot d_0), (a \backslash c, b \backslash d_0), (a + c, b + d_1), (c \backslash a, d_1 \backslash b) \in R$.

Proposition 3.2. *Suppose that A and B are BAs, A is countable, and R is a weak V-relation between A and B. Then there is a homomorphism $f : A \to B$ such that for every $a \in A$ there exist $m \in \omega \backslash \{0\}$ and $b \in {}^m A$ such that*

(i) $\forall i < m[(b_i, f(b_i)) \in R]$.

(ii) b is pairwise disjoint and $\sum_{i<m} b_i = a$.

Proof. Assume the hypotheses. Let $0 = c_0, c_1, \ldots$ enumerate the members of A without repetitions. Let $L_0 = \{0\}$ and $f_0 = \{(0, 0)\}$. Suppose that $L_0 \subseteq \ldots \subseteq L_n$, and $f_0, \ldots, f_n$ have been defined, so that each L_i is a finite linearly ordered subset of A and each f_i is an increasing function from L_i into B. Say $L_n = \{e_0, \ldots, e_k\}$, with $e_0 < \cdots < e_k$. For all $i < k$ let $f_n(e_i) = g_i$. We assume $\forall i \leq k[(e_i, g_i) \in R]$ and $\forall i < k[(e_{i+1} \backslash e_i, g_{i+1} \backslash g_i) \in R]$. For all $i \leq k$ let $e'_{2i} = e_i$. For all $i < k$ let $e'_{2i+1} = c_{n+1} \cdot e_{i+1} + e_i$, and let $e'_{2k+1} = c_{n+1} + e_k$. Then

3.2(1) $e'_0 \leq e'_1 \leq \cdots \leq e'_{2k+1}$.

To prove 3.2(1) we show by induction on j that $e'_0 \leq \ldots \leq e'_j$ for all $j \leq 2k + 1$. The case $j = 0$ is trivial. Now assume that $j < 2k + 1$ and $e'_0 \leq \ldots \leq e'_j$. For $j = 0$, $e'_1 = c_0 \cdot e_1 + e_0 = e_0 = e'_0$. If $j = 2i$ with $i \neq 0$, then $e'_j = e'_{2i} = e_i \leq e'_{2i+1} = e'_{j+1}$. If $j = 2i + 1$, then $e'_j = e'_{2i+1} = c_{n+1} \cdot e_{i+1} + e_i \leq e_{i+1} = e'_{2i+2} = e'_{j+1}$.

Also we have for all $i < k$

$$\begin{aligned}
e'_{2i+1} \backslash e'_{2i} &= ((c_{n+1} \cdot e_{i+1}) + e_i) \backslash e_i \\
&= c_{n+1} \cdot (e_{i+1} \backslash e_i); \\
e'_{2i+2} \backslash e'_{2i+1} &= e_{i+1} \backslash (c_{n+1} \cdot e_{i+1} + e_i) \\
&= e_{i+1} \cdot -(c_{n+1} \cdot e_{i+1} + e_i) \\
&= e_{i+1} \cdot (-c_{n+1} + (-e_{i+1})) \cdot (-e_i) \\
&= e_{i+1} \cdot (-c_{n+1}) \cdot (-e_i) \\
&= (e_{i+1} \backslash e_i) \backslash c_{n+1}.
\end{aligned}$$

Also,

$$e'_{2k+1} \backslash e'_{2k} = (c_{n+1} + e_k) \backslash e_k = c_{n+1} \backslash e_k.$$

Now by (V6), for each $i < k$, with a, b, c replaced by $e_{i+1} \backslash e_i$, $g_{i+1} \backslash g_i$, c_{n+1}, we obtain $h'_i \in B$ such that

$$((e_{i+1} \backslash e_i) \cdot c_{n+1}, (g_{i+1} \backslash g_i) \cdot h'_i), ((e_{i+1} \backslash e_i) \backslash c_{n+1}, (g_{i+1} \backslash g_i) \backslash h'_i) \in R$$

and with a, b, c replaced by e_k, g_k, c_{n+1} we get $h'_k \in B$ such that

$$(e_k + c_{n+1}, g_k + h'_k), (c_{n+1} \backslash e_k, h'_k \backslash g_k) \in R.$$

For all $i \leq k$ let $g'_{2i} = g_i$, and for all $i < k$ let $g'_{2i+1} = g_i + ((g_{i+1} \backslash g_i) \cdot h'_i)$. Also, let $g'_{2k+1} = g_k + h'_k$. Then

3.2(2) $g'_0 \leq g'_1 \leq \cdots \leq g'_{2k+1}$.

In fact, we prove 3.2(2) by showing that $g'_0 \leq \ldots \leq g'_j$ for all $j \leq 2k+1$ by induction on j. This is trivial for $j = 0$. Assume that it holds for $j < 2k+1$. If $j = 2i$ and $i < k$, then $g'_j = g'_{2i} = g_i \leq g'_{2i+1}$. Further, $g'_{2k} = g_k \leq g'_{2k+1}$. If $j = 2i+1 < 2k+1$, then $g'_j = g'_{2i+1} = g_i + ((g_{i+1} \backslash g_i) \cdot h'_i) \leq g_{i+1} = g'_{2i+2} = g'_{j+1}$.

Now for any $i < k$,

$$\begin{aligned}
g'_{2i+1} \backslash g'_{2i} &= (g_i + ((g_{i+1} \backslash g_i) \cdot h'_i)) \backslash g_i \\
&= (g_{i+1} \backslash g_i) \cdot h'_i; \\
g'_{2i+2} \backslash g'_{2i+1} &= g_{i+1} \backslash (g_i + ((g_{i+1} \backslash g_i) \cdot h'_i)) \\
&= g_{i+1} \cdot -(g_i + g_{i+1} \cdot (-g_i) \cdot h'_i) \\
&= g_{i+1} \cdot (-g_i) \cdot (-g_{i+1} + g_i + (-h'_i) \\
&= g_{i+1} \cdot (-g_i) \cdot (-h'_i) \\
&= (g_{i+1} \backslash g_i) \backslash h'_i.
\end{aligned}$$

Also,

$$g'_{2k+1} \backslash g'_{2k} = (g_k + h'_k) \backslash g_k = h'_k \backslash g_k.$$

It follows that $\forall i < 2k+1 [(e'_{i+1} \backslash e'_i, g'_{i+1} \backslash g'_i) \in R]$. Also, $(e'_{2k+1}, g'_{2k+1}) \in R$.

3.2(3) If $i < 2k+1$ and $e'_i = e'_{i+1}$, then $g'_i = g'_{i+1}$.

In fact, suppose that $e'_i = e'_{i+1}$. Then $e'_{i+1} \backslash e'_i = 0$. and so by (V5), $g'_{i+1} \backslash g'_i = 0$ and hence $g'_i = g'_{i+1}$.

Now let $L_{n+1} = \{e'_0, \ldots, e'_{2k+1}\}$ and $f_{n+1}(e'_i) = g'_i$ for all $i < 2k+1$.

3.2(4) $\forall i \leq k [i > 0 \rightarrow e_i = (e_i \backslash e_{i-1}) + (e_{i-1} \backslash e_{i-2}) + \cdots + (e_1 \backslash e_0)]$.

We prove 3.2(4) by induction on i. It is true for $i = 1$ since $e_0 = 0$. Assume it for $i < k$. Then $e_{i+1} = (e_{i+1} \cdot e_i) + (e_{i+1} \backslash e_i) = (e_{i+1} \backslash e_i) + e_i = (e_{i+1} \backslash e_i) + (e_i \backslash e_{i-1}) + \cdots + (e_1 \backslash e_0)$. So 3.2(4) holds.

Now

$$c_{n+1} = c_{n+1} \cdot e_k + (c_{n+1} \backslash e_k) = c_{n+1} \cdot \sum_{i=0}^{k-1} (e_{i+1} \backslash e_i) + (c_{n+1} \backslash e_k) = \sum_{i=0}^{k-1} (e'_{2i+1} \backslash e'_{2i}) + (e'_{2k+1} \backslash e'_{2k}).$$

This proves that c_{n+1} is in the subalgebra of A generated by L_{n+1} and also at the final stage gives (i) and (ii). $\square$

If A and B are BAs, a subset R of $A \times B$ is a *V-relation* between A and B iff the following conditions hold:

(V7) $1R1$
(V8) $0R0$
(V9) $\forall x[xR0 \to x = 0]$
(V10) $\forall y[0Ry \to y = 0]$
(V11) $\forall x, y, z[xR(y+z)$ and $y \cdot z = 0 \to \exists u, v[x = u + v$ and $u \cdot v = 0$ and uRy and $vRz]]$.
(V12) $\forall x, y, z[(x+y)Rz$ and $x \cdot y = 0 \to \exists u, v[z = u + v$ and $u \cdot v = 0$ and xRu and $yRv]]$.

Theorem 3.3. *If R is a V-relation between denumerable BAs A and B, then there is an isomorphism f from A onto B such that for any $x \in A$ there exist $n \in \omega$ and $y \in {}^nA$ such that $\forall i, j \in n[i \neq j \to y_i \cdot y_j = 0]$ and $x = \sum_{i<n} y_i$ and $y_i R f(y_i)$ for all $i < n$.*

Proof. Let $A = \{a_i : i \in \omega\}$ and $B = \{b_i : i \in \omega\}$. We now define $c \in {}^\omega A$ and $d \in {}^\omega B$ by recursion. Let $c_0 = 1$ and $d_0 = 1$. Now suppose that c_j and d_j have been defined for all $j \leq 2i$ so that the following conditions hold:

3.3(1) $\{(c_j, d_j) : j \leq 2i\}$ extends to an isomorphism f from $\langle\{c_j : j \leq 2i\}\rangle$ into B.

3.3(2) For each $x \in \langle\{c_j : j \leq 2i\}\rangle$ there exist $n \in \omega$ and $y \in {}^n\langle\{c_j : j \leq 2i\}\rangle$ such that $\forall i, j < n[i \neq j \to y_i \cdot y_j = 0]$ and $x = \sum_{k<n} y_k$ and $y_k R f(y_k)$ for all $k < n$.

Clearly 3.3(1) and 3.3(2) hold for $i = 0$. Now we let $c_{2i+1} = a_i$. Let the atoms of $\langle\{c_j : j \leq 2i\}\rangle$ be $e_0, \ldots, e_n$ without repetitions. Take any k with $k \leq n$. Then there exist $m \in \omega$ and $y \in {}^m\langle\{c_j : j \leq 2i\}\rangle$ such that $\forall i, j < m[i \neq j \to y_i \cdot y_j = 0]$ and $e_k = \sum_{l<m} y_l$ and $y_l R f(y_l)$ for all $l < m$. Since e_k is an atom, there is an $l < m$ such that $y_l = e_k$. So $e_k R f(e_k)$. Thus

3.3(3) $\forall k \leq n[e_k R f(e_k)]$.

Now $e_k = e_k \cdot a_i + e_k \cdot -a_i$, so by (V12) there exist u_k, v_k such that $u_k \cdot v_k = 0$, $f(e_k) = u_k + v_k$, $(e_k \cdot a_i)Ru_k$, and $(e_k \cdot -a_i)Rv_k$. Let $d_{2i+1} = \sum_{k \leq n} u_k$. To show that $f \cup \{(c_{2i+1}, d_{2i+1})\}$ extends to an isomorphism of $\langle\{c_j : j \leq 2i+1\}\rangle$ into B, by Sikorski's extension criterion it suffices to take any $x \in \langle\{c_j : j \leq 2i\}\rangle$ and show that $(x \cdot c_{2i+1} = 0 \leftrightarrow f(x) \cdot d_{2i+1} = 0)$ and $(x \cdot -c_{2i+1} = 0 \leftrightarrow f(x) \cdot -d_{2i+1} = 0)$. Write $x = \sum_{k \in M} e_k$.

Suppose that $x \cdot c_{2i+1} = 0$. Thus $x \cdot a_i = 0$, so $\forall k \in M[e_k \cdot a_i = 0]$. Since $\forall k \leq n[(e_k \cdot a_i)Ru_k]$, it follows by (V10) that $\forall k \in M[u_k = 0]$. Hence

$$f(x) \cdot d_{2i+1} = \sum_{k \notin M} (f(x) \cdot u_k) \leq \sum_{k \notin M} (f(x) \cdot f(e_k)) = \sum_{k \notin M} f(x \cdot e_k) = 0.$$

Conversely, suppose that $x \cdot c_{2i+1} \neq 0$. Say $k \in M$ and $e_k \cdot a_i \neq 0$. Since $(e_k \cdot a_i)Ru_k$, it follows by (V9) that $u_k \neq 0$. Then $f(x) \cdot d_{2i+1} \geq f(e_k) \cdot d_{2i+1} \geq u_k \neq 0$.

Suppose that $x \cdot -c_{2i+1} = 0$. Thus $x \cdot -a_i = 0$, so $\forall k \in M[e_k \cdot -a_i = 0]$. Since $\forall k \leq n[(e_k \cdot -a_i)Rv_k]$, it follows by (V10) that $\forall k \in M[v_k = 0]$, hence $\forall k \in M[u_k = f(e_k)]$,

hence $\forall k \in M[-u_k = f(-e_k)]$. Hence

$$f(x) \cdot -d_{2i+1} = f(x) \cdot \prod_{k \leq n} -u_k \leq f(x) \cdot \prod_{k \in M} -u_k = f(x) \cdot \prod_{k \in M} f(-e_k)$$
$$= f\left(\sum_{k \in M} e_k\right) \cdot \prod_{k \in M} f(-e_k) = 0.$$

Conversely suppose that $x \cdot -c_{2i+1} \neq 0$. Say $k \in M$ and $e_k \cdot -a_i \neq 0$. So by (V9), $v_k \neq 0$. Then $f(x) \cdot -d_{2i+1} \geq f(e_k) \cdot -d_{2i+1} \geq v_k \neq 0$.

Thus we have shown

3.3(4) $\{(c_j, d_j) : j \leq 2i+1\}$ extends to an isomorphism f' from $\langle\{c_j : j \leq 2i+1\}\rangle$ into B; and f' extends f.

Now we claim

3.3(5) For each $x \in \langle\{c_j : j \leq 2i+1\}\rangle$ there exist $m \in \omega$ and $y \in {}^m\langle\{c_j : j \leq 2i+1\}\rangle$ such that $\forall i, j < m[i \neq j \to y_i \cdot y_j = 0]$ and $x = \sum_{k<m} y_k$ and $y_k R f'(y_k)$ for all $k < m$.

For, suppose that $x \in \langle\{c_j : j \leq 2i+1\}\rangle$. Then there are $x', x'' \in \langle\{c_j : j \leq 2i\}\rangle$ such that $x = x' \cdot c_{2i+1} + x'' \cdot -c_{2i+1}$. Say $x' = \sum_{k \in M} e_k$ and $x'' = \sum_{k \in N} e_k$. Then

$$x = x' \cdot c_{2i+1} + x'' \cdot -c_{2i+1} = \sum_{k \in M \cap N} e_k + \sum_{k \in M \setminus N} (e_k \cdot c_{2i+1}) + \sum_{k \in N \setminus M} (e_k \cdot -c_{2i+1}).$$

For $k \in M \cap N$ we have $e_k R f(e_k) = f'(e_k)$. For $k \in M \setminus N$ we have

$$(e_k \cdot c_{2i+1}) = (e_k \cdot a_i) R u_k = f(e_k) \cdot d_{2i+1} = f(e_k) \cdot f'(c_{2i+1}) = f'(e_k \cdot c_{2i+1}).$$

For $k \in N \setminus M$ we have

$$(e_k \cdot -c_{2i+1}) = (e_k \cdot -a_i) R v_k = f(e_k) \cdot -d_{2i+1} = f(e_k) \cdot f'(-c_{2i+1}) = f'(e_k \cdot -c_{2i+1}).$$

This proves 3.3(5).

Now we let $d_{2i+2} = b_i$. Let the atoms of $\langle\{d_j : j \leq 2i+1\}\rangle$ be $e_0, \ldots, e_n$ without repetitions. Take any k with $k \leq n$. Then there exist $m \in \omega$ and $y \in {}^m\langle\{c_j : j \leq 2i+1\}\rangle$ such $\forall i, j \in n[i \neq j \to y_i \cdot y_j = 0]$ and $f'^{-1}(e_k) = \sum_{l<m} y_l$ and $y_l R f(y_l)$ for all $l < m$. Since e_k is an atom, there is an $l < m$ such that $y_l = f'^{-1}(e_k)$. So $f'^{-1}(e_k) R e_k$. Thus

3.3(6) $\forall k \leq n[f'^{-1}(e_k) R e_k]$.

Now $e_k = e_k \cdot b_i + e_k \cdot -b_i$, so by (V11) there exist u_k, v_k such that $u_k \cdot v_k = 0$, $f'^{-1}(e_k) = u_k + v_k$, $u_k R(e_k \cdot b_i)$, and $v_k R(e_k \cdot -b_i)$. Let $c_{2i+2} = \sum_{k \leq n} u_k$. To show that $f' \cup \{(c_{2i+2}, d_{2i+2})\}$ extends to an isomorphism of $\langle\{c_j : j \leq 2i+2\}\rangle$ into B, by Sikorski's extension criterion it suffices to take any $x \in \langle\{c_j : j \leq 2i+1\}\rangle$ and show that $(x \cdot c_{2i+2} = 0 \leftrightarrow f'(x) \cdot d_{2i+2} = 0)$ and $(x \cdot -c_{2i+2} = 0 \leftrightarrow f'(x) \cdot -d_{2i+2} = 0)$. Write $f'(x) = \sum_{k \in M} e_k$.

Suppose that $f'(x) \cdot d_{2i+2} = 0$. Thus $f'(x) \cdot b_i = 0$, so $\forall k \in M[e_k \cdot b_i = 0]$. Since $u_k R(e_k \cdot b_i)$, it follows from (V9) that $\forall k \in M[u_k = 0]$. Hence

$$x \cdot c_{2i+2} = \sum_{k \notin M} (x \cdot u_k) \leq \sum_{k \notin M} (x \cdot f'^{-1}(e_k)) = \sum_{k \notin M} f'^{-1}(f'(x) \cdot e_k) = 0.$$

Conversely suppose that $f'(x) \cdot d_{2i+2} \neq 0$. Say $k \in M$ and $e_k \cdot d_{2i+2} \neq 0$. So $e_k \cdot b_i \neq 0$. Hence by (V10), $u_k \neq 0$. Hence $x \cdot c_{2i+2} \geq \sum_{l \in M} f'^{-1}(e_l) \cdot c_{2i+2} \geq f'^{-1}(e_k) \cdot c_{2i+2} = u_k \neq 0$.

Suppose that $f'(x) \cdot -d_{2i+2} = 0$. Thus $f'(x) \cdot -b_i = 0$, so $\forall k \in M[e_k \cdot -b_i = 0]$. Hence by (V9), $\forall k \in M[v_k = 0]$. So $\forall k \in M[f'^{-1}(e_k) = u_k]$. Hence $f'(x) = \sum_{k \in M} e_k = \sum_{k \in M} f'(f'^{-1}(e_k)) = \sum_{k \in M} f'(u_k)$. Hence $x = \sum_{k \in M} u_k$. So

$$x \cdot -c_{2i+2} = x \cdot \prod_{i \leq n} -u_k = \sum_{k \in M} u_k \cdot \prod_{i \leq n} -u_k = 0.$$

Conversely suppose that $f'(x) \cdot -d_{2i+2} \neq 0$. Say $k \in M$ and $e_k \cdot -b_i \neq 0$. Then by (V10) $v_k \neq 0$. Now $x \cdot -c_{2i+2} \geq f'^{-1}(e_k) \cdot v_k = v_k \neq 0$.

Thus we have shown

3.3(7) $\{(c_j, d_j) : j \leq 2i+2\}$ extends to an isomorphism f'' from $\langle \{c_j : j \leq 2i+2\} \rangle$ into B; and f'' extends f'.

Now we claim

3.3(8) For each $x \in \langle \{c_j : j \leq 2i+2\} \rangle$ there exist $m \in \omega$ and $y \in {}^m\langle \{c_j : j \leq 2i+2\} \rangle$ such that $\forall i, j < m[i \neq j \to y_i \cdot y_j = 0]$ and $x = \sum_{k<m} y_k$ and $y_k R f(y_k)$ for all $k < m$.

For, suppose that $x \in \langle \{c_j : j \leq 2i+2\} \rangle$. Then $f''(x) \in \langle \{d_j : j \leq 2i+2\} \rangle$, so there exist $x', x'' \in \langle \{d_j : j \leq 2i+1\} \rangle$ such that $f''(x) = x' \cdot d_{2i+2} + x'' \cdot -d_{2i+2}$. Say $x' = \sum_{k \in M} e_k$ and $x'' = \sum_{k \in N} e_k$. Then

$$f''(x) = x' \cdot d_{2i+2} + x'' \cdot -d_{2i+2} = \sum_{k \in M \cap N} e_k + \sum_{k \in M \setminus N} (e_k \cdot d_{2i+2}) + \sum_{k \in N \setminus M} (e_k \cdot -d_{2i+2}).$$

For $k \in M \cap N$ we have $f'^{-1}(e_k) R e_k$. For $k \in M \setminus N$ we have

$$(e_k \cdot d_{2i+2}) = (e_k \cdot b_i) R^{-1} u_k = f'^{-1}(e_k) \cdot c_{2i+2} = f''^{-1}(e_k \cdot d_{2i+2}),$$

For $k \in N \setminus M$ we have

$$(e_k \cdot -d_{2i+2}) = (e_k \cdot -b_i) R^{-1} v_k = f'^{-1}(e_k) \cdot -c_{2i+2} = f''^{-1}(e_k \cdot -d_{2i+2}).$$

This proves 3.3(8). □

Proposition 3.4. *Suppose that R is a relation between countable BAs such that the following conditions hold:*

(i) If ARB with $|B| = 1$ then $|A| = 1$.

(ii) If ARB with $|A| = 1$, then $|B| = 1$.

(iii) If ARB and $A \cong A_0 \times A_1$ then there exist B_0, B_1 such that $B \cong B_0 \times B_1$, A_0RB_0, and A_1RB_1.

(iv) If ARB and $B \cong B_0 \times B_1$ then there exist A_0, A_1 such that $A \cong A_0 \times A_1$, A_0RB_0, and A_1RB_1.

Then ARB implies that $A \cong B$.

Proof. Assume the hypotheses, and suppose that ARB. Let $S = \{(a, b) : a \in A, b \in B$, and $(A \restriction a)R(B \restriction b)$. Then S is a V-relation between A and B, so $A \cong B$ by Theorem 3.3. □

4. Superatomic ABIDs

An element a of a ABID a is an *atom* iff $a \neq 0$ and $\forall b \in A[b \cdot a = 0$ or $b \cdot a = a]$. A is *atomless* iff it has no atoms, and *atomic* iff $\forall b \in A[b \neq 0 \to \exists a[a$ is an atom and $a \leq b]]$. A is *superatomic* iff every homomorphic image of A is atomic. A ABID is *trivial* iff it has only one element. Note that the trivial ABID is atomless, atomic, and superatomic. Now for any ABID A we define

$$\begin{aligned} I_0(A) &= \{0\}; \\ I_{\alpha+1}(A) &= \{a \in A : [a]_{I_\alpha(A)} \text{ is a finite sum of atoms of } A/I_\alpha(A)\} \\ I_\lambda(A) &= \bigcup_{\alpha<\lambda} I_\alpha(A) \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

Note that $I_\alpha \subseteq I_{\alpha+1}$, since $\forall a \in I_\alpha[[a]_{I_\alpha(A)} = 0]$ and 0 is a finite sum of atoms of $A/I_\alpha(A)$.

Proposition 4.1. *Let A be a ABID, and α and γ ordinals.*

(i) $I_\gamma(A/I_\alpha(A)) = \{[a]_{I_\alpha(A)} : a \in I_{\alpha+\gamma}(A)\}$.

(ii) There is an isomorphism f_γ from $(A/I_\alpha(A))/I_\gamma(A/I_\alpha(A))$ onto $A/I_{\alpha+\gamma}(A)$ such that $\forall a \in A[f_\gamma([[a]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A))}) = [a]_{I_{\alpha+\gamma}(A)}$.

Proof. Induction on γ. It is clear for $\gamma = 0$. Assume it for γ. To prove (i) for $\gamma + 1$, let $[a]_{I_\alpha(A)} \in I_{\gamma+1}(A/I_\alpha(A))$. Then exist an $m \in \omega$ and for each $i < m$ atoms $[[c_i]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A)}$ of $(A/I_\alpha(A))/I_\gamma(A/I_\alpha(A))$ such that

$$(*) \qquad [[a]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A))} = \sum_{i<m} [[c_i]_{I_\alpha(A)}]_{I_\gamma(A/I_\alpha(A))}.$$

By (ii) for γ, for each $i < m$, $[c_i]_{I_{\alpha+\gamma}(A)}$ is an atom of $A/I_{\alpha+\gamma}(A)$, and by applying f_γ to $(*)$ we have $[a]_{I_{\alpha+\gamma}(A)} = \sum_{i<m}[c_i]_{I_{\alpha+\gamma}(A)}$. Thus $a \in I_{\alpha+\gamma+1}(A)$. This proves that $I_{\gamma+1}(A/I_\alpha(A)) \subseteq \{[a]_{I_\alpha(A)} : a \in I_{\alpha+\gamma+1}(A)\}$. The other inclusion is proved similarly, so (i) holds for $\gamma + 1$. For (ii) for $\gamma + 1$,

$$\begin{aligned} &[[a]_{I_\alpha(A)}]_{I_{\gamma+1}(A/I_\alpha(A))} = [[a']_{I_\alpha(A)}]_{I_{\gamma+1}(A/I_\alpha(A)} \\ &\text{iff } [a]_{I_\alpha(A)} \triangle [a']_{I_\alpha(A)} \in I_{\gamma+1}(A/I_\alpha(A)) \end{aligned}$$

$$\begin{aligned}
&\text{iff } [a\triangle a']_{I_\alpha(A)} \in I_{\gamma+1}(A/I_\alpha(A))\\
&\text{iff } \exists b \in I_{\alpha+\gamma+1}(A)[[a\triangle a']_{I_\alpha(A)} = [b]_{I_\alpha(A)}] \quad \text{by (i) for } \gamma+1,\\
&\text{iff } \exists b \in I_{\alpha+\gamma+1}(A)[((a\triangle a')\triangle b \in I_\alpha(A)]\\
&\text{iff } a\triangle a' \in I_{\alpha+\gamma+1}(A) \text{ iff } [a]_{I_{\alpha+\gamma+1}(A)} = [a']_{I_{\alpha+\gamma+1}(A)}.
\end{aligned}$$

Now (ii) for $\gamma+1$ follows.

The case with γ limit is straightforward. □

Proposition 4.2. *Let A be a ABID and α and β ordinals with $\beta \leq \alpha$. For $a \in I_\alpha(A)$ we denote by $[a]'_{I_\alpha(A)}$ the equivalence class of a in $I_\alpha(A)$, while $[a]_{I_\alpha(A)}$ is the equivalence class of a in A. Then there is an isomorphism f from $I_\alpha(A)/I_\beta(A)$ into $A/I_\beta(A)$ such that $\forall a \in I_\alpha(A)\left[f([a]'_{I_\beta(A)}) = [a]_{I_\beta(A)}\right]$. Moreover, for any $a \in I_\alpha(A)$, $[a]'_{I_\beta(A)}$ is an atom of $I_\alpha(A)/I_\beta(A)$ iff $[a]_{I_\beta(A)}$ is an atom of $A/I_\beta(A)$.*

Proof. The first part of the proposition is clear. Now suppose that $a \in I_\alpha(A)$ and $[a]'_{I_\beta(A)}$ is an atom of $I_\alpha(A)/I_\beta(A)$. Suppose that $b \in A$ and $[b]_{I_\beta(A)} \leq [a]_{I_\beta(A)}$. Then $b\backslash a \in I_\beta(A)$. So $b = b\cdot a + b\backslash a \in I_\alpha(A)$. Hence $b \in I_\beta(A)$ or $(a\backslash b) \in I_\beta(A)$. So $[a]_{I_\beta(A)}$ is an atom of $A/I_\beta(A)$.

Conversely, suppose that $[a]_{I_\beta(A)}$ is an atom of $A/I_\beta(A)$. Suppose that $b \in I_\alpha(A)$ and $[b]'_{I_\beta(A)} \leq [a]'_{I_\beta(A)}$. Then $(b\backslash a) \in I_\beta(A)$, so $[b]_{I_\beta(A)} \leq [a]_{I_\beta(A)}$. So $b \in I_\beta(A)$ or $(a\backslash b) \in I_\beta(A)$. Hence $[a]'_{I_\beta(A)}$ is an atom of $I_\beta(A)/I_\beta(A)$. □

Proposition 4.3. $\forall\beta \leq \alpha[I_\beta(I_\alpha(A)) = I_\beta(A)]$.

Proof. Induction on β. It is clear for $\beta = 0$. Now assume it for β. Suppose that $x \in I_{\beta+1}(I_\alpha(A))$. Say

$$[x]_{I_\beta(I_\alpha(A))} = \sum_{i<m} [y_i]_{I_\beta(I_\alpha(A))}$$

with each $[y_i]_{I_\beta(I_\alpha(A))}$ an atom of $I_\alpha(A)/I_\beta(I_\alpha(A))$. Thus by the inductive hypothesis,

$$[x]_{I_\beta(A)} = \sum_{i<m} [y_i]_{I_\beta(A)},$$

and by Proposition 4.2, each $[y_i]_{I_\beta(A)}$ an atom of $A/I_\beta(A)$. Thus $x\triangle\sum_{i<m} y_i \in I_\beta(A)$, so $x \in I_{\beta+1}(A)$. The converse is similar.

The limit step is clear. □

The *atomic rank* of A is the least ordinal $\mathrm{ar}(A) = \alpha$ such that $I_\alpha(A) = I_{\alpha+1}(A)$.

Proposition 4.4. *If A is an atomic ABID and A is a maximal ideal in a BA B, then B is atomic.*

Proof. Suppose that $b \in B$ and $B \restriction b$ is nontrivial and atomless. Let u, v be nonzero and disjoint with $u+v \leq b$. Then $u, v \notin A$. In fact, suppose that $u \in A$. Let a be an atom of A with $a \leq u$. Then $a \leq b$, so there is a nonzero $w \in B$ with $w < a$. But $w \in A$ since A

is an ideal, contradicting a being an atom of A. Similarly $v \notin A$. say $u = -a$ and $v = -c$ with $a, c \in A$. Then $0 = u \cdot v = (-a) \cdot (-c)$ and hence $a + c = 1$, contradiction. $\square$

Proposition 4.5. *If A is a ABID, A is a maximal ideal in a BA B, and $f : B \to C$ is a surjective homomorphism with C nontrivial, then $f[A] = C$ or $f[A]$ is a maximal ideal in C.*

Proof. Assume the hypotheses. Clearly $f[A]$ is closed under $+$. If $c \leq f(a)$ with $c \in C$ and $a \in A$, choose $x \in B$ such that $c = f(x)$. Then $f(x \cdot a) = f(x) \cdot f(a) = c \cdot f(a) = c$. Since $x \cdot a \in A$, this shows that $c \in f[A]$. Thus $f[A]$ is an ideal of C. Now $\forall x \in B[x \in A$ or $-x \in A]$. If $c \in C$, choose $x \in B$ such that $f(x) = c$. Then $c \in f[A]$ or $-c \in f[A]$; the conclusion follows. $\square$

Proposition 4.6. *If A is a superatomic ABID and A is a maximal ideal in a BA B, then B is superatomic.*

Proof. Assume the hypotheses, and let $f : B \to C$ be a surjective homomorphism, with C nontrivial. By Proposition 4.5 we have two cases.

Case 1. $f[A] = C$. Then C is atomic, since A is superatomic.

Case 2. $f[A]$ is a maximal ideal in C. Then $f[A]$ is atomic since A is superatomic, and then C is atomic by Proposition 4.4. $\square$

Proposition 4.7. *If A is an atomless ABID which is not a BA, B is a BA, and $(A, +, \cdot)$ is a subalgebra of $(B, +, \cdot)$, then the subalgebra A' of B generated by A has at most one atom.*

Proof. First note that $A' = A \cup \{-c : c \in A\}$. For, if $a, b \in A$, then

$$\begin{aligned} a \cdot_B b &= a \cdot_A b \in A; \\ a \cdot_B (-_B b) &= a \backslash_A b \in A \quad \text{by Proposition 1.3;} \\ (-_B a) \cdot_B (-_B b) &= -_B (a +_A b). \end{aligned}$$

Now suppose that A' has two distinct atoms x, y. Then there exist $a, b \in A$ such that $x = -a$ and $y = -b$. Then $(-a) \cdot (-b) = 0$, so $a + b = 1$, contradiction. $\square$

Proposition 4.8. *For any ABID A the following conditions are equivalent:*

(i) A is superatomic.

(ii) No nontrivial homomorphic image of A is atomless.

(iii) Every subalgebra of A is atomic.

(iv) Every subalgebra of A has an atom.

(v) $\neg \exists a \in {}^{\mathbb{Q}}A \forall q, r \in \mathbb{Q}[q < r \to a_q < a_r]$.

(vi) $I_{\mathrm{ar}(A)}(A) = A$.

Proof. (i)$\Rightarrow$(ii): obvious.

(ii)$\Rightarrow$(i): Assume that (i) fails; so there is a homomorphic image B of A which is not atomic. Hence there is a $b \in B$ such that $B \restriction b$ is atomless and nontrivial. Clearly $B \restriction b$ is a homomorphic image of A, so (ii) fails.

(i)⇒(iii): Suppose that A is a superatomic ABID and B is a non-atomic subalgebra of A. Thus A is not a BA. Say $b \in B\backslash\{0\}$ with no atoms of B below b. So $B \restriction b$ is atomless. By Proposition 1.10, A is a maximal ideal in some BA C. By Proposition 4.6, C is superatomic. Hence $C \restriction b$ is superatomic. But $B \restriction b$ is an atomless subalgebra of $C \restriction b$, contradiction.

(iii)⇒(iv): obvious.

(iv)⇒(v): Assume that (iv) holds but (v) fails; so we get $a \in {}^{\mathbb{Q}}A$ such that $\forall q, r \in \mathbb{Q}[q < r \to a_q < a_r]$. Let B be the subalgebra of $(A, +, \cdot, \backslash, 0)$ generated by $\mathrm{rng}(a)$.

4.8(1) B consists of all finite sums of elements of $\{a_q : q \in \mathbb{Q}\} \cup \{a_q\backslash a_r : q, r \in \mathbb{Q}, r < q\}$.

In fact, let S be the set of all such finite sums. Obviously S contains $\mathrm{rng}(a)$ and is closed under $+$. To see that it is closed under $\cdot$ it suffices to note that

$$a_q \cdot a_r = a_{\min(q,r)};$$

$$\text{if } t < r, \text{ then} \quad a_q \cdot (a_r\backslash a_t) = \begin{cases} a_r\backslash a_t & \text{if } t < r \leq q, \\ a_q\backslash a_t & \text{if } t < q < r, \\ 0 & \text{if } q \leq t < r. \end{cases}$$

$$\text{if } r < q \text{ and } t < s, \text{ then}$$

$$(a_q\backslash a_r) \cdot (a_s\backslash a_t) = \begin{cases} a_{\min(q,s)}\backslash a_{\max(r,t)} & \text{if } \max(r,t) < \min(q,s), \\ 0 & \text{if } \min(q,s) \leq \max(r,t) \end{cases}.$$

For closure under $\backslash$, note that

$$a_q\backslash a_r = \begin{cases} a_q\backslash a_r & \text{if } r < q, \\ 0 & \text{if } q \leq r; \end{cases}$$

$$\text{if } t < r, \text{ then} \quad a_q\backslash(a_r\backslash a_t) = (a_q\backslash a_r) + a_{\min(q,t)},$$

$$\text{if } t < r, \text{ then} \quad (a_r\backslash a_t)\backslash a_q = \begin{cases} a_r\backslash a_{\max(q,t)} & \text{if } q < r, \\ 0 & \text{if } r \leq q. \end{cases}$$

$$\text{if } r < q \text{ and } t < s, \text{ then} \quad (a_q\backslash a_r)\backslash(a_s\backslash a_t) = (a_q\backslash a_{\max(r,s)}) + (a_{\min(q,t)}\backslash a_r).$$

Thus 4.8(1) holds.

Now let x be an atom of B. By (1) we may assume that one of the following two cases holds:

Case 1. $x = a_q$ for some $q \in \mathbb{Q}$. Let r be a rational less than q. Then $0 < a_r < a_q$, contradiction.

Case 2. $x = a_q\backslash a_r$ for some $q, r \in \mathbb{Q}$ with $r < q$. Choose $t \in \mathbb{Q}$ such that $r < t < q$. Then $0 < a_q\backslash a_t < a_q\backslash a_r$, contradiction. To see that $a_q\backslash a_t < a_q\backslash a_r$, note that $a_r < a_t$, hence $a_q\backslash a_t \leq a_q\backslash a_r$. Also, $(a_q\backslash a_r)\backslash(a_q\backslash a_t) = a_t\backslash a_r$, so $a_q\backslash a_t < a_q\backslash a_r$.

(v)⇒(i): Assume that (i) fails; so there is a homomorphism f of A onto a ABID B which is not atomic. Then there is a nonzero $b \in B$ such that $B \restriction b$ is atomless. For any $a \in A$ let $g(a) = f(a) \cap b$. Then g is a homomorphism from A onto $B \restriction b$.

We now construct $b \in {}^{\mathbb{Q}}(B \restriction b)$. Let $\langle q_i : i \in \omega\rangle$ be a one-one enumeration of $\mathbb{Q}$. We define $c \in {}^{\omega}(B \restriction b)\backslash\{0\}$ by recursion. Let $c_0 \in (B \restriction b)\backslash\{0\}$ be arbitrary. Now suppose that $c_0, \ldots, c_{m-1}$ have been constructed so that the following condition holds:

4.8(2) If p is the permutation of m such that $q_{p(0)} < \cdots < q_{p(m-1)}$, then $c_{p(0)} < \cdots < c_{p(m-1)}$.

To define c_m we consider three cases.

Case 1. $\forall i < m[q_m < q_i]$. Let c_m be such that $\forall i < m[c_m < c_i]$.

Case 2. $\forall i < m[q_i < q_m]$. Let c_m be such that $\forall i < m[c_i < c_m]$.

Case 3. $\exists i < m-1[q_{p(i)} < q_m < q_{p(i+1)}]$. Take d such that $0 < d < c_{p(i+1)} \backslash c_{p(i)}$ and let $c_m = c_{p_i} + d$.

Clearly in any case 4.8(2) holds for $m+1$.

Now for any $r \in \mathbb{Q}$ choose $i \in \omega$ such that $r = q_i$, and let $b_r = c_i$. Suppose that $r, s \in \mathbb{Q}$ and $r < s$. Say $r = q_i$ and $s = q_j$. Let $m = \max(i,j)+1$. Let p be the permutation of m such that $q_{p(0)} < \cdots < q_{p(m-1)}$. Say $i = p(u)$ and $j = p(v)$. Then $r < s$ implies that $q_i < q_j$, hence $q_{p(u)} < q_{p(v)}$, hence $u < v$, hence $c_{p(u)} < c_{p(v)}$, hence $c_i < c_j$, hence $b_r < b_s$.

Now for each $r \in \mathbb{Q}$ let $d_r \in A$ be such that $g(d_r) = b_r$. We now define e_{q_i} for each $i \in \omega$ by recursion. Let $e_{q_0} = d_{q_0}$. Suppose that $e_{q_0}, \ldots, e_{q_m}$ have been defined so that the following conditions hold:

4.8(3) $\forall i \leq m[g(e_{q_i}) = b_{q_i}]$.

4.8(4) $\forall i, j \leq m[b_{q_i} < b_{q_j} \to e_{q_i} < e_{q_j}]$.

Then we define

$$e_{q_{m+1}} = \Big(d_{q_{m+1}} \cdot \prod\{e_{q_i} : i \leq m, b_{q_{m+1}} < b_{q_i}\}\Big) + \sum\{e_{q_i} \backslash d_{q_{m+1}} : i \leq m, b_{q_i} < b_{q_{m+1}}\}.$$

Clearly 4.8(3) holds for $m+1$. Now suppose that $i \leq m$. If $b_{q_i} < b_{q_{m+1}}$, then $\forall j \leq m[b_{q_{m+1}} < b_{q_j} \to e_{q_i} < e_{q_j}]$, and so $e_{q_i} \leq e_{q_{m+1}}$. By 4.8(3), $e_{q_i} < e_{q_{m+1}}$. If $b_{q_{m+1}} < b_{q_i}$, then $\forall j \leq m[b_{i_j} < b_{q_{m+1}} \to e_{q_j} < e_{q_i}]$, and so $e_{q_{m+1}} \leq e_{q_i}$. By 4.8(3), $e_{q_{m+1}} < e_{q_i}$.

Thus 4.8(3) and 4.8(4) hold for all m, giving $\neg$(v).

(ii)$\Rightarrow$(vi): Assume (ii). If $I_{\mathrm{ar}(A)} \neq A$, then $A/I_{\mathrm{ar}(A)}$ is nontrivial and atomless, contradicting (ii).

(vi)$\Rightarrow$(iv): Assume that (iv) fails. Let B be a nontrivial atomless subalgebra of A. We claim

4.8(5) $B \cap I_\alpha(A) = \{0\}$ for all α.

Of course this will show that (vi) fails. We prove 4.8(5) by induction on α. It is obvious for $\alpha = 0$. Assume it for α, but suppose that $0 \neq b \in B \cap I_{\alpha+1}(A)$. Then there exist $c_0, \ldots, c_m$ such that each $[c_i]_{I_\alpha(A)}$ is an atom in $A/I_\alpha(A)$ and $[b]_{I_\alpha(A)} = [c_0]_{I_\alpha(A)} + \cdots + [c_m]_{I_\alpha(A)}$. There are disjoint nonzero $d_0, \ldots, d_{m+1} \in B$ with each $d_i < b$. For each $i \leq m+1$ there is a nonempty subset M_i of $m+1$ such that $[d_i]_{I_\alpha(A)} = \sum_{j \in M_i} [c_j]_{I_\alpha(A)}$. The sets M_i are pairwise disjoint, contradiction. The induction step is clear, so 4.8(5) holds. $\square$

Proposition 4.9. *If A is a ABID and α is an ordinal, then $I_\alpha(A)$ is superatomic.*

Proof. By Proposition 4.8(vi) it suffices to show that $I_{\alpha+1}(I_\alpha(A)) = I_\alpha(I_\alpha(A))$. The inclusion $\supseteq$ is clear. Suppose that $x \in I_{\alpha+1}(I_\alpha(A))$. Then in particular $x \in I_\alpha(A)$. By Proposition 4.3, $I_\alpha(I_\alpha(A)) = I_\alpha(A)$. □

If A is a superatomic ABID, then $\alpha_*(A)$ is the least ordinal β such that $A/I_\beta(A)$ has a greatest element. Thus $\alpha_*(A) \leq \mathrm{ar}(A)$. If $\alpha_*(A) < \mathrm{ar}(A)$, then $\mathrm{ar}(A)$ is a successor ordinal. If $\alpha_*(A) = \mathrm{ar}(A)$ and $\mathrm{ar}(A)$ is a successor ordinal $\beta+1$, then $A/I_\beta(A)$ has infinitely many atoms.

If $\alpha_*(A) = \mathrm{ar}(A)$, we set $\alpha^*(A) = \mathrm{ar}(A)$ and $n(A) = 0$. If $\alpha_*(A) < \mathrm{ar}(A)$, then $\alpha^*(A)$ is the β such that $\mathrm{ar}(A) = \beta + 1$; then $A/I_{\alpha^*(A)}(A)$ is a finite BA, and we let $n(A)$ be the number of atoms of $A/I_{\alpha^*(A)}(A)$. Note that in any case $\alpha_*(A) \leq \alpha^*(A)$. The *atomic type* of a superatomic ABID A is the triple $\tau_s(A) = (\alpha^*(A), \alpha_*(A), n(A))$. Note that τ_s applies only to superatomic ABIDs. If A is the trivial BA, then $\tau_s(A) = (0,0,0)$. If A is a finite BA with n atoms, then $\tau_s(A) = (0,0,n)$.

Proposition 4.10. *For any superatomic ABID A with $\tau_s(A) = (\beta, \alpha, n)$ one of the following holds:*

(i) $n > 0$, $\beta \geq \alpha$, and $\alpha < \mathrm{ar}(A) = \beta + 1$.

(ii) $\beta = \alpha$ and $n = 0$. □

Proposition 4.11. *If A and B are ABIDs and α is an ordinal, then:*

(i) $I_\alpha(A \times B) = I_\alpha(A) \times I_\alpha(B)$.

(ii) There is an isomorphism of $(A \times B)/I_\alpha(A \times B)$ onto $(A/I_\alpha(A)) \times (B/I_\alpha(B))$ such that $\forall a \in A \forall b \in B[f([(a,b)]_{I_\alpha(A\times B)}) = ([a]_{I_\alpha(A)}, [b]_{I_\alpha(B)})$.

Proof. Induction on α. It is clear for $\alpha = 0$. Now assume it for α. For (i) for $\alpha+1$, first suppose that $(a,b) \in I_{\alpha+1}(A \times B)$. Say

$$[(a,b)]_{I_\alpha(A\times B)} = [(c_0,d_0)]_{I_\alpha(A\times B)} + \cdots + [(c_{m-1},d_{m-1})]_{I_\alpha(A\times B)}$$

with each $[(c_i,d_i)]_{I_\alpha(A\times B)}$ an atom of $(A \times B)/I_\alpha(A \times B)$. We apply (ii) for α: for each $i < m$, $f([(c_i,d_i)]_{I_\alpha(A\times B)}) = ([c_i]_{I_\alpha(A)}, [d_i]_{I_\alpha(B)})$. It follows that we can write $m = M \cup N$ with M and N disjoint, $\forall i \in M[[d_i]_{I_\alpha(B)} = 0$ and $[c_i]_{I_\alpha(A)}$ is an atom of $A/I_\alpha(A)]$ and $\forall i \in N[[c_i]_{I_\alpha(A)} = 0$ and $[d_i]_{I_\alpha(B)}$ is an atom of $B/I_\alpha(B)]$. Hence $[a]_{I_\alpha(A)} = \sum_{i\in M}[c_i]_{I_\alpha(A)}$, and so $a \in I_{\alpha+1}(A)$. Similarly $b \in I_{\alpha+1}(B)$. This proves that $I_{\alpha+1}(A\times B) \subseteq I_{\alpha+1}(A)\times I_{\alpha+1}(B)$. The other inclusion is proved similarly, so (i) holds for $\alpha+1$. (ii) for $\alpha+1$ is clear.

The limit step is clear. □

Proposition 4.12. *For any ABIDs A, B, $\mathrm{ar}(A \times B) = \max(\mathrm{ar}(A), \mathrm{ar}(B))$.*

Proof. Let $\gamma = \mathrm{ar}(A \times B)$. Then by Proposition 4.11,

$$I_\gamma(A) \times I_\gamma(B) = I_\gamma(A \times B) = I_{\gamma+1}(A \times B) = I_{\gamma+1}(A) \times I_{\gamma+1}(B).$$

Hence $I_\gamma(A) = I_{\gamma+1}(A)$ and $I_\gamma(B) = I_{\gamma+1}(B)$. So $\max(\mathrm{ar}(A), \mathrm{ar}(B)) \leq \gamma$. On the other hand, let $\alpha = \mathrm{ar}(A)$ and $\beta = \mathrm{ar}(B)$. Let $\delta = \max(\alpha, \beta)$. Then by Proposition 4.11,

$$I_\delta(A \times B) = I_\delta(A) \times I_\delta(B) = I_{\delta+1}(A) \times I_{\delta+1}(B) = I_{\delta+1}(A \times B).$$

Hence the assertion of the proposition follows. □

Proposition 4.13. *If A and B are superatomic ABIDS, then so is $A \times B$.*

Proof. By Proposition 4.12, $\mathrm{ar}(A \times B) = \max(\mathrm{ar}(A), \mathrm{ar}(B)$. By Theorem 4.8, $I_{\mathrm{ar}(A)}(A) = A$ and $I_{\mathrm{ar}(B)}(B) = B$. Hence by Proposition 4.11,

$$I_{\mathrm{ar}(A\times B)}(A \times B) = I_{\mathrm{ar}(A\times B)}(A) \times I_{\mathrm{ar}(A\times B)}(B) = A \times B.$$

So by Theorem 4.8, $A \times B$ is superatomic. □

Proposition 4.14. *Suppose that A_1 and A_2 are superatomic ABIDs, with $\tau_s(A_1) = (\alpha_1, \beta_1, n_1)$ and $\tau_s(A_2) = (\alpha_2, \beta_2, n_2)$. Define $\gamma = \max(\alpha_1, \alpha_2)$, $\delta = \max(\beta_1, \beta_2)$ and*

$$m = \begin{cases} n_1 & \text{if } \alpha_2 < \alpha_1, \\ n_2 & \text{if } \alpha_1 < \alpha_2, \\ n_1 + n_2 & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

Then $\tau_s(A_1 \times A_2) = (\gamma, \delta, m)$.

Proof. First we show that

4.14(1) $\alpha_*(A_1 \times A_2) = \max(\alpha_*(A_1), \alpha_*(A_2))$.

For, let $\gamma = \alpha_*(A_1 \times A_2)$. Then $(A_1 \times A_2)/I_\gamma(A_1 \times A_2)$ has a greatest element. By Proposition 4.11, $(A_1/I_\gamma(A_1)) \times (A_2/I_\gamma(A_2))$ has a greatest element (x, y). Clearly x is the greatest element of $A_1/I_\gamma(A_1)$ and y is the greatest element of $A_2/I_\gamma(A_2)$. Hence $\max(\alpha_*(A_1), \alpha_*(A_2)) \leq \alpha_*(A_1 \times A_2)$. The other inequality is clear also. So 4.14(1) holds.

Now we consider some cases.

Case 1. $\alpha_*(A_1) < \mathrm{ar}(A_1)$ and $\alpha_*(A_2) < \mathrm{ar}(A_2)$. Then $\mathrm{ar}(A_1) = \alpha^*(A_1) + 1$ and $\mathrm{ar}(A_2) = \alpha^*(A_2) + 1$. Also, $\alpha_*(A_1 \times A_2) = \max(\alpha_*(A_1), \alpha_*(A_2)) < \max(\mathrm{ar}(A_1), \mathrm{ar}(A_2)) = \mathrm{ar}(A_1 \times A_2)$. Hence $\mathrm{ar}(A_1 \times A_2) = \alpha^*(A_1 \times A_2) + 1$ and $n(A_1 \times A_2)$ is the number of atoms in $(A_1 \times A_1)/I_{\alpha^*(A_1\times A_2)}(A_1 \times A_2)$. Also, $\max(\alpha^*(A_1), \alpha^*(A_2)) = \alpha^*(A_1 \times A_2)$, since $\mathrm{ar}(A_1 \times A_2) = \max(\mathrm{ar}(A_1), \mathrm{ar}(A_2))$.

Subcase 1.1. $\mathrm{ar}(A_1) < \mathrm{ar}(A_2)$. Then $(A_1 \times A_2)/I_{\alpha^*(A_2)}(A_1 \times A_2) \cong A_2/I_{\alpha^*(A_2)}(A_2)$. Hence $n(A_1 \times A_2) = n(A_2)$, as desired.

Subcase 1.2. $\mathrm{ar}(A_2) < \mathrm{ar}(A_1)$. This is symmetric to Subcase 1.1.

Subcase 1.3. $\mathrm{ar}(A_1) = \mathrm{ar}(A_2)$. Then

$$(A_1 \times A_2)/I_{\alpha^*(A_2)}(A_1 \times A_2) \cong (A_1/I_{\alpha^*(A_1)}(A_1)) \times (A_2/I_{\alpha^*(A_2)}(A_2)).$$

Hence $n(A_1 \times A_2) = n(A_1) \cdot n(A_2)$, as desired.

Case 2. $\alpha_*(A_1) = \mathrm{ar}(A_1)$ and $\mathrm{ar}(A_2) < \mathrm{ar}(A_1)$. Now $\alpha_*(A_2) \leq \mathrm{ar}(A_2) < \mathrm{ar}(A_1) = \alpha_*(A_1)$, so $\alpha_*(A_1 \times A_2) = \alpha_*(A_1) = \mathrm{ar}(A_1) = \max(\mathrm{ar}(A_1, \mathrm{ar}(A_2)) = \mathrm{ar}(A_1 \times A_2)$. Hence $n(A_1 \times A_2) = 0$. Also, $\alpha^*(A_1 \times A_2) = \mathrm{ar}(A_1 \times A_2) = \mathrm{ar}(A_1) = \alpha^*(A_1)$. Moreover, $\alpha^*(A_2) \leq \mathrm{ar}(A_2) < \mathrm{ar}(A_1) = \alpha^*(A_1)$. Hence $\alpha^*(A_1 \times A_2) = \max(\alpha^*(A_1), \alpha^*(A_2))$. Since $\alpha^*(A_2) < \alpha^*(A_1)$ and $n(A_1) = 0$, this is as desired.

Case 3. $\alpha_*(A_1) = \mathrm{ar}(A_1)$ and $\mathrm{ar}(A_2) = \mathrm{ar}(A_1)$. Then $\mathrm{ar}(A_1 \times A_2) = \mathrm{ar}(A_1)$ and $\alpha_*(A_1 \times A_2) = \alpha_*(A_1) = \mathrm{ar}(A_1 \times A_2)$. So $\alpha^*(A_1) = \mathrm{ar}(A_1)$. Since $\alpha_*(A_1 \times A_2) = \mathrm{ar}(A_1 \times A_2)$, we have $\alpha^*(A_1 \times A_2) = \mathrm{ar}(A_1 \times A_2) = \mathrm{ar}(A_1) = \alpha^*(A_1) = \max(\alpha^*(A_1), \alpha^*(A_2))$. Note that $n(A_1) = n(A_1 \times A_2) = 0$. Now if $\alpha_*(A_2) < \mathrm{ar}(A_2)$ then the condition on m holds. If $\alpha_*(A_2) = \mathrm{ar}(A_2)$, then $n(A_2) = 0$ and again the condition on m holds.

Case 4. $\alpha_*(A_1) = \mathrm{ar}(A_1)$ and $\mathrm{ar}(A_1) < \mathrm{ar}(A_2)$.

Subcase 4.1. $\alpha_*(A_2) < \mathrm{ar}(A_2)$. Say $\alpha^*(A_2) = \varepsilon$ with $\mathrm{ar}(A_2) = \varepsilon + 1$. Then $\mathrm{ar}(A_1) \leq \varepsilon$, so $(A_1 \times A_2)/I_\varepsilon(A_1 \times A_2) \cong A_2/I_\varepsilon(A_2)$. Hence $\alpha^*(A_1 \times A_2) = \varepsilon$. Since $\alpha^*(A_1) \leq \mathrm{ar}(A_1)$ we have $\alpha^*(A_1) \leq \varepsilon$. So $\alpha^*(A_1 \times A_2) = \max(\alpha^*(A_1), \alpha^*(A_2))$. Now $n(A_1) = 0$ and $n(A_1 \times A_2) = n(A_2)$, so the conditions on m hold.

Subcase 4.2. $\alpha_*(A_2) = \mathrm{ar}(A_2)$. Then $\alpha^*(A_2) = \mathrm{ar}(A_2)$ and the conditions in the proposition are clear.

Case 5. $\alpha_*(A_2) = \mathrm{ar}(A_2)$. This is symmetric to cases 2–4. □

Proposition 4.15. *If A is a superatomic BA, then $\tau_s(A) = (\alpha^*(A), 0, n)$ with $\mathrm{ar}(A) = \alpha^*(A) + 1$ and $n > 0$ is the finite number of atoms in $A/I_{\alpha^*(A)}(A)$.* □

Proposition 4.16. *If A is a ABID, $a \in A$, and α is an ordinal, then $I_\alpha(A) \cap (A \restriction a) = I_\alpha(A \restriction a)$ and there is an isomorphism f_α of $(A \restriction a)/I_\alpha(A \restriction a)$ onto $(A/I_\alpha(A)) \restriction [a]_{I_\alpha(A)}$ such that $f_\alpha([b]_{I_\alpha(A \restriction a)}) = [b]_{I_\alpha(A)}$ for all $b \leq a$.*

Proof. Induction on α. It is clear for $\alpha = 0$. Assume it is true for α, and suppose that $b \in I_{\alpha+1}(A) \cap (A \restriction a)$. Then there exist $c_0, \ldots, c_{m-1} \in A$ such that each $[c_i]_{I_\alpha(A)}$ is an atom of $A/I_\alpha(A)$ and $[b]_{I_\alpha(A)} = [c_0]_{I_\alpha(A)} + \cdots + [c_{m-1}]_{I_\alpha(A)}$. Since $b \leq a$, we may assume that each $c_i \leq a$. Then $f_\alpha^{-1}[c_i]_{I_\alpha(A)} = [c_i]_{I_\alpha(A \restriction a)}$ is an atom of $(A \restriction a)/I_\alpha(A \restriction a)$ and $[b]_{I_\alpha(A \restriction a)} = [c_0]_{I_\alpha(A \restriction a)} + \cdots + [c_{m-1}]_{I_\alpha(A \restriction a)}$. Hence $b \in I_{\alpha+1}(A \restriction a)$. Thus $I_{\alpha+1}(A) \cap (A \restriction a) \subseteq I_{\alpha+1}(A \restriction a)$. The other inclusion is proved similarly, so $I_{\alpha+1}(A) \cap (A \restriction a) = I_{\alpha+1}(A \restriction a)$.

Now for all $b, b' \leq a$,

$$\begin{aligned} [b]_{I_{\alpha+1}(A \restriction a)} = [b']_{I_{\alpha+1}(A \restriction a)} \quad & \text{iff} \quad b \triangle b' \in I_{\alpha+1}(A \restriction a) \\ & \text{iff} \quad b \triangle b' \in I_{\alpha+1}(A) \\ & \text{iff} \quad [b]_{I_{\alpha+1}(A)} = [b']_{I_{\alpha+1}(A)}. \end{aligned}$$

Hence $f_{\alpha+1}$ exists as indicated.

The limit case is clear. □

Proposition 4.17. *Suppose that A is a superatomic BA, $\tau_s(A) = (\alpha, 0, m)$, and β, n, γ, p are such that $\alpha = max(\beta, \gamma)$ and*

$$m = \begin{cases} p & \text{if } \beta < \gamma, \\ n & \text{if } \gamma < \beta, \\ p + n & \text{if } \beta = \gamma. \end{cases}$$

Then there is an $a \in A$ such that $\tau_s(A \restriction a) = (\beta, 0, n)$ and $\tau_s(A \restriction (-a)) = (\gamma, 0, p)$.

Proof. Assume the hypotheses.

Case 1. $\beta < \gamma$. Then $A/I_\beta(A)$ is infinite and superatomic, so we can choose $b_1, \dots, b_n \in A$ such that $[b_1]_{I_\beta(A)}, \dots, [b_n]_{I_\beta(A)}$ are distinct atoms of $A/I_\beta(A)$. We may assume that $b_1, \dots, b_n$ are pairwise disjoint. Let $a = b_1 + \cdots + b_n$. Then by Proposition 4.16, $(A \restriction a)/I_\beta(A \restriction a) \cong (A/I_\beta(A)) \restriction [a]_{I_\beta(A)}$, so $(A \restriction a)/I_\beta(A \restriction a)$ is finite with exactly n atoms. Hence $\tau_s(A \restriction a) = (\beta, 0, n)$. By Proposition 4.14, $\tau_s(A \restriction (-a)) = (\gamma, 0, m)$.

Case 2. $\gamma < \beta$. This is symmetric to Case 1.

Case 3. $\beta = \gamma$. Now $A/I_\alpha(A)$ has exactly m atoms. Let $[b_1]_{I_\beta(A)}, \dots, [b_n]_{I_\beta(A)}$ be n of them. We may assume that $b_1, \dots, b_n$ are pairwise disjoint. Let $a = b_1 + \cdots + b_n$. Clearly a is as desired. □

Proposition 4.18. *If two superatomic countable BAs have the same atomic type, then they are isomorphic.*

Proof. Let A and B be superatomic countable BAs each with type $(\alpha, 0, m)$. We may assume that $\alpha \neq 0$; then A and B are infinite. Define

$$aRb \quad \text{iff} \quad a \in A,\ b \in B,\ \tau_s(A \restriction a) = \tau_s(B \restriction b),\ \tau_s(A \restriction (-a)) = \tau_s(B \restriction (-b)).$$

We check the conditions (V1)–(V4) for a V-correspondence. (V1) and (V2) are clear. Now by symmetry it suffices to treat (V3). So, suppose that aRb and $c \in A$. Say $\tau_s(A \restriction a) = \tau_s(B \restriction b) = (\beta, 0, n)$, $\tau_s(A \restriction (a \cdot c)) = (\gamma, 0, p)$, $\tau_s(A \restriction (a \cdot -c)) = (\delta, 0, q)$, and $\tau_s(A \restriction (-a)) = \tau_s(B \restriction (-b)) = (\varepsilon, 0, r)$. By Proposition 4.14 we have $\beta = \max(\gamma, \delta)$, and

$$n = \begin{cases} p & \text{if } \delta < \gamma, \\ q & \text{if } \gamma < \delta, \\ p + q & \text{if } \gamma = \delta. \end{cases}$$

Hence by Proposition 4.17 there is a $d_0 \leq b$ such that $\tau_s(B \restriction d_0) = (\gamma, 0, p)$ and $\tau_s(B \restriction (b \cdot -d_0)) = (\delta, 0, q)$. Thus $\tau_s(A \restriction (a \cdot c)) = \tau_s(B \restriction d_0)$ and $\tau_s(A \restriction (a \cdot -c)) = \tau_s(B \restriction (b \cdot -d_0))$. Now $B \restriction (-d_0) \cong (B \restriction -b) \times (B \restriction (b \cdot -d_0))$, so by Proposition 4.14 we get

4.18(1) $\tau_s(B \restriction (-d_0)) = (\varphi, 0, t)$, where $\varphi = \max(\varepsilon, \delta)$ and

$$t = \begin{cases} r & \text{if } \delta < \varepsilon), \\ q & \text{if } \varepsilon < \delta, \\ r + q & \text{if } \delta = \varepsilon. \end{cases}$$

But also $A \restriction (-a + -c) \cong A \restriction (-a + a \cdot -c) \cong (A \restriction (-a)) \times (A \restriction (a \cdot -c)$ and $\tau_s(A \restriction (-a)) = \tau_s(B \restriction (-b))$ and $\tau_s(A \restriction (a \cdot -c)) = \tau_s(B \restriction (b \cdot -d_0))$, so by Proposition 4.14 and 4.18(1) we get $\tau_s(A \restriction (-a + -c)) = \tau_s(B \restriction (-d_0))$. Thus

4.18(2) $(a \cdot c)R(b \cdot d_0)$.

Now $A \restriction (-a + c) = A \restriction (-a + c \cdot a) \cong (A \restriction (-a)) \times (A \restriction (c \cdot a))$ and $B \restriction (-b + d_0) \cong (B \restriction (-b) \times (B \restriction d_0)$. Since $\tau_s(A \restriction (-a)) = \tau_s(B \restriction (-b))$ and $\tau_s(A \restriction (c \cdot a) = \tau_s(B \restriction d_0)$, it follows from Proposition 4.14 that $\tau_s(A \restriction (-a + c)) = \tau_s(-b + d_0)$. Thus

4.18(3) $(a \cdot -c)R(b \cdot -d_0)$.

Now let $\tau_s(A \restriction (-a \cdot -c) = (\varphi, 0, s)$ and $\tau_s(A \restriction (-a \cdot c) = (\psi, 0, t)$. Then by Proposition 4.14 we get $\varepsilon = \max(\varphi, \psi)$ and

$$r = \begin{cases} s & \text{if } \psi < \varphi, \\ t & \text{if } \varphi < \psi, \\ s+t & \text{if } \varphi = \psi. \end{cases}$$

Since $\tau_s(B \restriction (-b)) = \tau_s(A \restriction (-a))$, it follows from Proposition 4.17 that there is a $d_1 \leq -b$ such that $\tau_s(B \restriction d_1) = \tau_s(A \restriction (-a \cdot -c))$ and $\tau_s(B \restriction (-b \cdot -d_1)) = \tau_s(A \restriction (-a \cdot c))$ Now $A \restriction (a{+}c) = A \restriction (a{+}c{\cdot}{-}a) \cong (A \restriction a){\times}(A \restriction (c{\cdot}{-}a))$ and $B \restriction (b{+}{-}d_1) = B \restriction (b{+}{-}d_1{\cdot}{-}b) \cong (B \restriction b){\times}(B \restriction (-d_1{\cdot}{-}b)$. Since $\tau_s(A \restriction a) = \tau_s(B \restriction b)$ and $\tau_s(A \restriction (c{\cdot}{-}a)) = \tau_s(B \restriction (-d_1{\cdot}{-}b)$, it follows from Proposition 4.14 that $\tau_s(A \restriction (a+c)) = \tau_s(B \restriction (b+-d_1))$. Thus

4.18(4) $(a+c)R(b+-d_1)$.

Now $A \restriction (-c+a) = A \restriction (a+-c \cdot -a) \cong (A \restriction a) \times (A\langle(-c \cdot -a))$ and $B \restriction (d_1+b) = B \restriction (b+d_1 \cdot -b) \cong (B \restriction b) \times (B \restriction (d_1 \cdot -b)$. Since $\tau_s(A \restriction a) = \tau_s(B \restriction b)$ and $\tau_s(A \restriction (-c \cdot -a)) = \tau_s(d_1 \cdot -b)$, it follows from Proposition 4.14 that $\tau_s(A \restriction (-c+a)) = \tau_s(B \restriction (d_1+b))$. Thus

4.18(5) $(c \cdot -a)R(-d_1 \cdot -b)$.

Now 4.18(2)–4.18(5) verify (V3). □

If A is a countable superatomic BA, we define the *invariant* of A to be $\mathrm{INV}(A) = (0, \alpha^*(A), 0, n(A))$. INV_0 is the set of all invariants of countable superatomic BAs.

Theorem A. *(i)* $\mathrm{INV}_0 = \{(0, \alpha, 0, n) : \alpha$ *a countable ordinal,* n *a positive integer*$\} \cup \{(0,0,0,0)\}$.

(ii) For any countable superatomic BAs A, B, $A \cong B$ *iff* $\mathrm{INV}(A) = \mathrm{INV}(B)$. □

Proposition 4.19. *For every* $\alpha > 0$ *and* $m > 0$, $\tau_s(\mathrm{Intalg}(\omega^\alpha \cdot m)) = (\alpha, 0, m)$.

Proof. Let $A = \mathrm{Intalg}(\omega^\alpha \cdot m)$. We claim that for all $\beta < \alpha$,

4.19(1) $\mathrm{atoms}(A/I_\beta(A)) = \{[[\omega^\alpha \cdot i + \omega^\beta \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot (\xi+1))]_{I_\beta(A)} : i < n, \xi < \omega^\alpha\}$.

4.19(2) $A/I_\beta(A)$ is generated by the increasing sequence

$$\begin{aligned} &\{[[0, \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}^\frown \\ &[[0, \omega^\alpha + \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}^\frown \\ &\dots \\ &[[0, \omega^\alpha \cdot (n-1) + \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}. \end{aligned}$$

We prove these statements by induction on β. They are clear for $\beta = 0$. Assume them for β. Then they are clear for $\beta+1$. Now assume that γ is limit $< \omega^\alpha$ and 4.19(1) and 4.19(2) hold for all $\beta < \gamma$. Then for any $i < n$ and $\xi < \omega^\alpha$, $[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi+1))]_{I_\gamma(A)}$

is an atom of $A/I_\gamma(A)$. In fact, first note that each member of $I_\gamma(A)$ is a finite union of intervals $[\varphi,\psi)$ with $\psi-\varphi<\gamma$; hence $[[\omega^\alpha\cdot i+\omega^\gamma\cdot\xi,\omega^\alpha\cdot i+\omega^\gamma\cdot(\xi+1))]_{I_\gamma(A)}\neq 0$. Now suppose that $i<n$, $\xi<\omega^\alpha$, and $\delta<\omega^\gamma$. Choose $\beta<\gamma$ so that $\delta<\omega^\beta$. Say $\beta+\varepsilon=\gamma$. Then

$$\begin{aligned}
&[[\omega^\alpha\cdot i+\omega^\gamma\cdot\xi,\omega^\alpha\cdot i+\omega^\gamma\cdot\xi+\delta))]_{I_\gamma(A)}\\
&=[[\omega^\alpha\cdot i+\omega^\beta\cdot\omega^\varepsilon\cdot\xi,\omega^\alpha\cdot i+\omega^\beta\cdot\omega^\varepsilon\cdot\xi+\delta)]_{I_\gamma(A)}\\
&\leq[[\omega^\alpha\cdot i+\omega^\beta\cdot\omega^\varepsilon\cdot\xi,\omega^\alpha\cdot i+\omega^\beta\cdot\omega^\varepsilon\cdot\xi+\omega^\beta)]_{I_\gamma(A)}\\
&\leq[[\omega^\alpha\cdot i+\omega^\beta\cdot\omega^\varepsilon\cdot\xi,\omega^\alpha\cdot i+\omega^\beta\cdot(\omega^\varepsilon\cdot\xi+1))]_{I_\gamma(A)}=0
\end{aligned}$$

Now 4.19(1) and 4.19(2) follow for γ.

So 4.19(1) and 4.19(2) hold for all $\beta<\alpha$. It follows that $A/I_\alpha(A)$ is generated by

$$\{[[0,\omega^\alpha)]_{I_\alpha(A)},\ldots,[[\omega^\alpha\cdot(n-1),\infty)]_{I_\alpha(A)}\}.$$

So $A/I_\alpha(A)$ is finite with n atoms. Hence $\tau_s(A)=(\alpha,0,m)$. □

Corollary 4.20. *If A is a countable superatomic BA with $\tau_s(A)=(\alpha,0,m)$, then $A\cong$* $\mathrm{Intalg}(\omega^\alpha\cdot m)$. □

A *commutative monoid with* 0 is an algebra $(M,+,0)$ such that $+$ is commutative and associative, and $\forall x\in M[x+0=x=0+x]$. **BA** is the set of all BAs with universe contained in in some $\alpha\leq\omega$. For $A,B\in\mathbf{BA}$, $A+B$ is a member of **BA** isomorphic to $A\times B$. 0 is the one-element BA. Let M be the set $\omega_1\times(\omega\backslash\{0\})\cup\{0'\}$ with $+$ defined as follows on M:

$$(\alpha,m)+(\beta,n)=\begin{cases}(\alpha,m) & \text{if } \alpha>\beta,\\ (\beta,n) & \text{if } \alpha<\beta,\\ (\alpha,m+n) & \text{if } \alpha=\beta;\end{cases}$$
$$(\alpha,m)+0'=0'+(\alpha,m)=(\alpha,m);$$
$$0'+0'=0'$$

SBA is the submonoid of **BA** consisting of superatomic BAs with universe contained in ω.

Theorem 4.21. $(\mathbf{SBA},+,0)\cong(M,+,0')$. □

Proposition 4.22. *If α is a nonzero ordinal, A is a superatomic ABID, $\tau_s(A)=(\alpha,\alpha,0)$, $\beta<\alpha$, and $n\in\omega\backslash\{0\}$, then there is an $a\in A$ such that $\tau_s(A\restriction a)=(\beta,0,n)$ and $\tau_s(a^\perp)=(\alpha,\alpha,0)$.*

Proof. $A/I_\beta(A)$ is infinite and superatomic, so we can choose $b_1,\ldots,b_n\in A$ such that $[b_1]_{I_\beta(A)},\ldots,[b_n]_{I_\beta(A)}$ are distinct atoms of $A/I_\beta(A)$. We may assume that $b_1,\ldots,b_n$ are pairwise disjoint. Let $a=b_1+\cdots+b_n$. Then by Proposition 4.16, $(A\restriction a)/I_\beta(A\restriction$

$a) \cong (A/I_\beta(A)) \restriction [a]_{I_\beta(A)}$, so $(A \restriction a)/I_\beta(A \restriction a)$ is finite with exactly n atoms. Hence $\tau_s(A \restriction a) = (\beta, 0, n)$. Then $\tau_s(a^\perp) = (\alpha, \alpha, 0)$ by Propositions 1.18 and 4.14. □

Proposition 4.23. *For any countable ordinal α and any countable superatomic ABIDs A, B, if $\tau_s(A) = (\alpha, \alpha, 0) = \tau_s(B)$, then $A \cong B$.*

Proof. Assume the hypotheses. Let $R = \{(a, b) \in A \times B : a = b = 0$, or $a \neq 0 \neq b$, $A \restriction a \cong B \restriction b\}$. It suffices now to show that R is a V-correspondence between A and B. Clearly (V1) and (V2) hold. By symmetry, now, it suffices to check (V3). So suppose that aRb and $c \in A$. Let f be an isomorphism from $A \restriction a$ onto $B \restriction b$. Let $d_0 = f(a \cdot c)$. Then $A \restriction (a \cdot c) \cong B \restriction (b \cdot d_0)$ and $A \restriction (a \backslash c) \cong B \restriction (b \backslash d_0)$. So $(a \cdot c)R(b \cdot d_0)$ and $(a \backslash c)R(b \backslash d_0)$.

Next, $(c \backslash a) \in a^\perp$. By Proposition 1.18 we have $A \cong (A \restriction a) \times a^\perp$. Say $\tau_s(A \restriction b) = \tau_s(A \restriction a) = (\beta, 0, n)$. By Proposition 4.14 we have $\beta < \alpha$. Hence by Proposition 4.14, $\tau_s(b^\perp) = (\alpha, \alpha, 0)$. Say $\tau_s(c \backslash a) = (\beta', 0, n')$. Then $\beta' < \alpha$ by Proposition 4.14. By Proposition 4.21 let $d_1 \in b^\perp$ be such that $\tau_s(b^\perp \restriction d_1) = (\beta', 0, n')$ and $\tau_s(d_1^\perp) = (\alpha, \alpha, 0)$. Then

$$A \restriction (a + c) = A \restriction (a + (c \backslash a)) \cong (A \restriction a) \times A \restriction (c \backslash a) \cong (B \restriction b) \times (B \restriction d_1) \cong B \restriction (b + d_1).$$

Hence $(a + c)R(b + d_1)$. Also,

$$A \restriction (c \backslash a) \cong (B \restriction d_1) = B \restriction (d_1 \backslash b).$$

Hence $(c \backslash a)R(d_1 \backslash b)$. □

Proposition 4.24. *For every $\alpha > 0$, $\tau_s(\mathrm{Intalg}_d(\omega^\alpha)) = (\alpha, \alpha, 0)$.*

Proof. Let $A = \mathrm{Intalg}_d(\omega^\alpha)$. We prove the following statement by induction on β:

4.23(1) For all $\beta < \alpha$, $A/I_\beta(A)$ is generated by

$$\{[[0, \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}.$$

This is obvious for $\beta = 0$ and the induction steps are clear. From 4.23(1) it is clear that $A/I_\alpha(A) = \{0\}$. □

Corollary 4.25. *If α is a countable ordinal and A is a superatomic ABID with $\tau_s(A) = (\alpha, \alpha, 0)$, then $A \cong \mathrm{Intalg}_d(\omega^\alpha)$.* □

Proposition 4.26. *Suppose that A is a countable superatomic ABID, $\alpha \geq \beta > 0$ and $n \in \omega \backslash \{0\}$. Suppose that $\tau_s(A) = (\alpha, \beta, n)$. Then $A \cong \mathrm{Intalg}(\omega^\alpha \cdot n) \times \mathrm{Intalg}_d(\omega^\beta)$.*

Proof. Assume the hypotheses. Let $[a_0]_{I_\alpha(A)}, \ldots, [a_{n-1}]_{I_\alpha(A)}$ be the distinct atoms of $A/I_\alpha(A)$. We may assume that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i<n} a_i$. Then $\tau_s(A \restriction b) = (\alpha, 0, n)$. By Proposition 4.14, $\tau_s(b^\perp) = (\beta, \beta, 0)$. Now the desired result follows from Proposition 1.18 and Corollaries 4.20 and 4.25. □

Proposition 4.27. *If $\alpha < \beta$ and $n \in \omega$, then* $\mathrm{Intalg}(\omega^\alpha \cdot n) \times \mathrm{Intalg}_d(\omega^\beta) \cong \mathrm{Intalg}_d(\omega^\beta)$.

Proof. By Propositions 4.19 and 4.24,

$$\tau_s(\mathrm{Intalg}(\omega^\alpha \cdot n)) = (\alpha, 0, n) \quad \text{and} \quad \tau_s(\mathrm{Intalg}_d(\omega^\beta)) = (\beta, \beta, 0).$$

Then by Proposition 4.14, $\tau_s(\mathrm{Intalg}(\omega^\alpha \cdot n) \times \mathrm{Intalg}_d(\omega^\beta)) = (\beta, \beta, 0)$. Hence the desired conclusion follows from Corollary 4.25. □

Proposition 4.28. *If A and B are countable superatomic ABIDs such that $\tau_s(A) = \tau_s(B)$, then $A \cong B$.*

Proof. By Proposition 4.26. □

Proposition 4.29. *Suppose that A is a countable superatomic ABID, and let $\tau_s(A) = (\alpha, \beta, n)$. Also suppose given ordinals α_0, α_1 and natural numbers n_0, n_1 such that the following conditions hold:*

(i) $\alpha = \max(\alpha_0, \alpha_1)$.
(ii) $\beta \leq \alpha_1$.
(iii) $n_0 > 0$.
(iv) If $n_1 = 0$, then $\alpha_1 = \beta$.
(v) If $\alpha_1 < \alpha_0$, then $n_0 = n$.
(vi) If $\alpha_0 < \alpha_1$, then $n_1 = n$.
(vii) If $\alpha_0 = \alpha_1$, then $n_0 + n_1 = n$.

Then there is an $a \in A$ such that $\tau_s(A \restriction a) = (\alpha_0, 0, n_0)$ and $\tau_s(a^\perp) = (\alpha_1, \beta, n_1)$.

Proof. *Case 1.* $n = 0$. Then $\alpha = \beta$. By (iii), (v), (vii), $\alpha_0 < \alpha_1$. Hence the ABID $A/I_{\alpha_0}(A)$ has at least n_0 distinct atoms $[a_i]_{I_{\alpha_0}(A)}$ for $i < n_0$. We may assume that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i<n_0} a_i$. Then $\tau_s(A \restriction b) = (\alpha_0, 0, n_0)$. By Proposition 4.17, $\tau_s(b^\perp) = (\alpha, \alpha, 0)$, as desired.

Case 2. $n \neq 0$. Then $\mathrm{ar}(A) = \alpha + 1$, and $A/I_\alpha(A)$ is finite with n atoms.

Subcase 2.1. $\alpha_0 < \alpha$. Then $\alpha = \alpha_1$ and $n_1 = n$ by (vi). Now $A/I_{\alpha_0}(A)$ has infinitely many atoms. Let $[a_i]_{I_{\alpha_0}(A)}$ be an atom for $i < n_0$. We may assume that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i<n_0} a_i$. Then $\tau_s(A \restriction b) = (\alpha_0, 0, n_0)$. By Proposition 4.14, $\tau_s(b^\perp) = (\alpha_1, \beta, n_1)$.

Subcase 2.2. $\alpha_0 = \alpha$ and $n_0 < n$. By (v), $\alpha_1 = \alpha$. Hence by (vii), $n_1 = n - n_0$. Let $[a_i]_{I_{\alpha_0}(A)}$ be an atom for $i < n_0$. We may assume that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i<n_0} a_i$. Then $\tau_s(A \restriction b) = (\alpha_0, 0, n_0)$. By Proposition 4.14, $\tau_s(b^\perp) = (\alpha_1, \beta, n_1)$.

Subcase 2.3. $\alpha_0 = \alpha$ and $n_0 = n$.

Subsubcase 2.3.2. $\alpha_1 = \alpha$. Then $n_1 = 0$ by (vii), and $\alpha_1 = \beta$ by (iv). Let $[a_i]_{I_{\alpha_0}(A)}$ be an atom for $i < n_0$. We may assume that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i<n_0} a_i$. Then $\tau_s(A \restriction b) = (\alpha_0, 0, n_0)$. By Proposition 4.14, $\tau_s(b^\perp) = (\alpha_1, \beta, n_1)$.

Subcase 3.2. $\alpha_1 < \alpha$.

Subsubcase 3.2.1. $n_1 = 0$. So $\alpha_1 = \beta$ by (iv). Let f be an isomorphism of A onto $\mathrm{Intalg}(\omega^\alpha \cdot n) \times \mathrm{Intalg}_d(\omega^\beta)$. Let $b = f^{-1}(1, 0)$. Then $\tau(b) = (\alpha, 0, n)$ and $\tau(b^\perp) = (\beta, \beta, 0)$.

Subsubcase 3.2.2. $n_1 \neq 0$. Now $\beta \leq \alpha_1$, so $A/I_{\alpha_1}(A)$ is a BA. It has infinitely many atoms since $\alpha_1 < \alpha$. Let $[a_i]_{I_{\alpha_1}(A)}$ be distinct atoms for $i < n_1$. We may assume

that $a_i \cdot a_j = 0$ for $i \neq j$. Let $b = \sum_{i<n_1} a_i$. Then $\tau(A \restriction b) = (\alpha_1, 0, n_1)$. By Proposition 4.14, $\tau(b^\perp) = (\alpha, \beta, n)$. Let g be an isomorphism from $b^\perp$ onto $\mathrm{Intalg}(\omega^\alpha \cdot n) \times \mathrm{Intalg}_d(\omega^\beta)$. Let $c = g^{-1}(1,0)$. Then $\tau(A \restriction c) = (\alpha, 0, n)$. For any $x \in c^\perp$ let $f(x) = (x \cdot b, x \backslash b)$. Then $f(x) \in (A \restriction b) \times \{y \in b^\perp : y \cdot g^{-1}(1,0) = 0\}$. Clearly f is one-one and is a homomorpism. If $y \in (A \restriction b)$ and $z \in b^\perp$ with $z \cdot g^{-1}(1,0) = 0$, then $y + z \in c^\perp$, and $f(y+z) = (y, z)$. So f is an isomorphism, and hence $\tau(c^\perp) = (\alpha_1, \beta, n_1)$, as desired. □

Proposition 4.30. *If J is an ideal in a ABID A and α is an ordinal, then*

(i) $\forall \alpha [J \cap I_\alpha(A) = I_\alpha(J)]$.

(ii) $\{([a]_{I_\alpha(J)}, [a]_{I_\alpha(A)}) : a \in J\}$ *is an isomorphism of* $J/I_\alpha(J)$ *onto an ideal of* $A/I_\alpha(A)$.

Proof. Induction on α. It is clear for $\alpha = 0$. Assume it for α. First we prove (i) for $\alpha + 1$. Suppose that $x \in J \cap I_{\alpha+1}(A)$. Then we can write $[x]_{I_\alpha(A)} = \sum_{i<m} [y_i]_{I_\alpha(A)}$, where each $[y]_{I_\alpha(A)}$ is an atom of $A/I_\alpha(A)$. Let $y_i' = y_i \cdot x$ for all $i < m$. Then for each $i < m$, $y_i \triangle y_i' = y_i \backslash x \in I_\alpha(A)$, so $[y_i]_{I_\alpha(A)} = [y_i']_{I_\alpha(A)}$. Note that $\forall i < m [y_i' \in J]$. By (ii) for α, $[y_i']_{I_\alpha(J)}$ is an atom of $J/I_\alpha(J)$. Now $x \triangle \sum_{i<m} y_i' \in J \cap I_\alpha(A)$, so by (i) for α, $x \triangle \sum_{i<m} y_i' \in I_\alpha(J)$. It follows that $x \in I_{\alpha+1}(J)$.

Conversely, suppose that $x \in I_{\alpha+1}(J)$. Then we can write

$$[x]_{I_\alpha(J)} = \sum_{i<m} ([y_i]_{I_\alpha(J)},$$

with each $[y_i]_{I_\alpha(J)}$ an atom of $J/I_\alpha(J)$. Then by (ii) for α, each $[y_i]_{I_\alpha(A)}$ is an atom of $A/I_\alpha(A)$. Also, $x \triangle \sum_{i<m} y_i \in I_\alpha(J) \subseteq I_\alpha(A)$. So $x \in I_{\alpha+1}(A)$. This proves (i) for $\alpha + 1$.

To prove (ii) for $\alpha + 1$, suppose that $a, a' \in J$. Then

$$\begin{aligned} [a]_{I_\alpha(J)} = [a']_{I_\alpha(J)} \quad &\text{iff} \quad a \triangle a' \in I_\alpha(J) \\ &\text{iff} \quad a \triangle a' \in I_\alpha(A). \end{aligned}$$

Then (ii) follows.

The limit case is clear. □

Corollary 4.31. *For any ordinals α, β,* $\mathrm{Intalg}_d(\omega^\alpha) \times \mathrm{Intalg}_d(\omega^\beta) \cong \mathrm{Intalg}_d(\omega^{\max(\alpha,\beta)})$.

Proof. By Proposition 4.14. □

Corollary 4.32. *Let A be a countable superatomic ABID, $\tau_s(A) = (\alpha, \alpha, 0)$, $\beta < \alpha$, $m \in \omega \backslash \{0\}$. Then there is an $a \in A$ such that $\tau_s(A \restriction a) = (\beta, 0, m)$ and $\tau_s(a^\perp) = (\alpha, \alpha, 0)$.*

Proof. Replace $A, \alpha, \beta, n, \alpha_0, \alpha_1, n_0, n_1$ by $A, \alpha, \alpha, 0, \beta, \alpha, m, 0$ in Proposition 4.29. □

5. Countable BAs in general

Proposition 5.1. *If A is a BA, J is a superatomic ideal of A, and A/J is atomless, then* $\forall \alpha [I_\alpha(A) \subseteq J]$.

Proof. Suppose not, and let α be minimum such that $I_\alpha(A) \not\subseteq J$. Clearly then α is a successor ordinal $\beta+1$. Say $x \in I_\alpha(A)\backslash J$. Write $[x]_{I_\beta(A)} = \sum_{i<m}[y_i]_{I_\beta(A)}$ with each $[y_i]_{I_\beta(A)}$ an atom of $A/I_\beta(A)$. Wlog each $y_i \leq x$. Now $x \triangle \sum_{i<m} y_i \in I_\beta(A)$, so $x \backslash \sum_{i<m} y_i \in J$. It follows that there is an $i<m$ such that $y_i \notin J$. Choose $u \notin J$ with $[u]_J < [y_i]_J$. Let $u' = u \cdot y_i$. Since $u \backslash y_i \in J$, it follows that $u' \notin J$. Hence $u' \notin I_\beta(A)$, so $[u']_{I_\beta(A)} = [y_i]_{I_\beta(A)}$. Hence $y_i \backslash u' \in I_\beta(A)$, and so $y_i \backslash u' \in J$, contradiction. $\square$

Proposition 5.2. *If A is a BA, J is a superatomic ideal of A, and A/J is atomless, then $I_{\mathrm{ar}(A)}(A) = J$.*

Proof. By Proposition 4.30, $I_{\mathrm{ar}(A)+1}(J) = J \cap I_{\mathrm{ar}(A)+1}(A) = J \cap I_{\mathrm{ar}(A)}(A) = I_{\mathrm{ar}(A)}(J)$. Hence $\mathrm{ar}(J) \leq \mathrm{ar}(A)$ and $I_{\mathrm{ar}(A)}(J) = J \subseteq I_{\mathrm{ar}(A)}(A)$. By Proposition 4.30, $I_{\mathrm{ar}(A)}(A) \subseteq J$. So $I_{\mathrm{ar}(A)}(A) = J$. $\square$

For any ABID A define the *atomic type* of A to be $\tau(A) \stackrel{\mathrm{def}}{=} \tau_s(I_{\mathrm{ar}(A)}(A))$. The *atomic type* of an element $a \in A$ is $\tau(a) \stackrel{\mathrm{def}}{=} \tau_s((A \restriction a) \cap I_{\mathrm{ar}(A)}(A))$. If A is superatomic, then $I_{\mathrm{ar}(A)}(A) = A$, and hence $\tau(A) = \tau_s(A)$. A function $f : A \to \alpha$ is *additive* iff $f(0) = 0$ and for all $a, b \in A$, $f(a+b) = \max\{f(a), f(b)\}$. The middle entry of $\tau(A)$ is denoted by $\sigma'(A)$, and the middle entry of $\tau(a)$ by $\sigma'(a)$. These entries are also called *special*. Thus $\sigma'(A)$ is the least β such that $I_{\mathrm{ar}(A)}(A)/I_\beta(I_{\mathrm{ar}(A)}(A))$ has a greatest element, and for each $a \in A$, $\sigma'(a)$ is the least β such that $((A \restriction a) \cap I_{\mathrm{ar}(A)}(A))/I_\beta((A \restriction a) \cap I_{\mathrm{ar}(A)}(A))$ has a greatest element. By convention, $\sigma'(0) = 0$. A BA A is *special* iff it is atomic with atomic type $(\mathrm{ar}(A), \mathrm{ar}(A), 0)$.

Proposition 5.3. *Suppose that A is a ABID, $\tau(A) = (\alpha, \beta, n)$, $a \in A$, $\sigma'(a) = \gamma$, $\sigma'(a^\perp) = \delta$, and (α_0, γ, m) and (α_1, δ, k) are such that*

(i) $\alpha_0 \geq \gamma$, $\alpha_1 \geq \delta$, and $\alpha = \max(\alpha_0, \alpha_1)$.
(ii) If $\alpha_1 < \alpha_0$, then $m = n$.
(iii) If $\alpha_0 < \alpha_1$, then $k = n$.
(iv) If $\alpha_0 = \alpha_1$, then $m + k = n$.
(v) If $m = 0$ then $\alpha_0 = \gamma$.
(vi) If $k = 0$ then $\alpha_1 = \delta$.

Then there is a $b \in A$ such that $[a]_{I_{\mathrm{ar}(A)}(A)} = [b]_{I_{\mathrm{ar}(A)}(A)}$, $\tau(A \restriction b) = (\alpha_0, \gamma, m)$, and $\tau(b^\perp) = (\alpha_1, \delta, k)$.

Proof. Assume the hypotheses. We may assume that $|A| > 1$. Let $A' = I_{\mathrm{ar}(A)}(A)$. Let f be an isomorphism from $(A \restriction a) \cap A'$ onto $\mathrm{Intalg}(\omega^\varphi \cdot p) \times \mathrm{Intalg}_d(\omega^\gamma)$, and let g be an isomorphism from $a^\perp \cap A'$ onto $\mathrm{Intalg}(\omega^\xi \cdot q) \times \mathrm{Intalg}_d(\omega^\delta)$. Then

$$A' \cong \mathrm{Intalg}(\omega^\varphi \cdot p) \times \mathrm{Intalg}(\omega^\xi \cdot q) \times \mathrm{Intalg}_d(\omega^\gamma) \times \mathrm{Intalg}_d(\omega^\delta);$$

$$\mathrm{Intalg}(\omega^\varphi \cdot p) \times \mathrm{Intalg}(\omega^\xi \cdot q) \cong \mathrm{Intalg}(\omega^\alpha \cdot n);$$

$$\mathrm{Intalg}_d(\omega^\gamma) \times \mathrm{Intalg}_d(\omega^\delta) \cong \mathrm{Intalg}_d(\omega^\beta).$$

Let $d = f^{-1}(1,0) + g^{-1}(1,0)$. Then $A' \restriction d \cong \mathrm{Intalg}(\omega^\alpha \cdot n)$.

Case 1. $n \neq 0$.

Subcase 1.1. $m \neq 0 \neq k$. We apply Proposition 4.29 with β, A, n_0, n_1 replaced by $0, A' \restriction d, m, k$ to get an element $c \in A' \restriction d$ such that $\tau_s(A' \restriction c) = (\alpha_0, 0, m)$ and $\tau_s(c^\perp \cap (A' \restriction d) = (\alpha_1, 0, k)$. Let $b = c + a\backslash f(1,0)$. Now $c \leq d = f^{-1}(1,0) + g^{-1}(1,0)$ and $a \cdot g^{-1}(1,0) = 0$, so $c \cdot (a\backslash f(1,0)) = 0$ and $\tau(a\backslash f(1,0)) = (\gamma, \gamma, 0)$. Hence $\tau(A \restriction b) = (\alpha_0, \gamma, m)$. Now we claim

5.3(1) $b^\perp \cap A' = \{x + y : x, y \in A', x \leq d\backslash c, y \cdot (a + g^{-1}(1,0)) = 0\}$.

In fact if $x \leq d\backslash c$, then $x \cdot c = 0$. Also, $d\backslash f(1,0) = g(1,0) \leq a^\perp$, so $x \cdot (a\backslash f(1,0)) \leq (d\backslash c) \cdot (a\backslash f(1,0)) \leq d \cdot (a\backslash f(1,0)) = 0$. Also, if $y \cdot (a + g^{-1}(1,0)) = 0$, then $y \cdot d = y \cdot (f^{-1}(1,0) + g^{-1}(1,0)) \leq y \cdot (a + g^{-1}(1,0)) = 0$. Hence $y \cdot c = 0$. Clearly $y \cdot (a\backslash f(1,0)) = 0$, so $y \cdot b = 0$. Hence the right side of 5.3(1) is contained in the left side. Now suppose that $z \in b^\perp \cap A'$. Then $z = z \cdot (d\backslash c) + z\backslash(d\backslash c)$. So to show that z is in the right side of 5.3(1) it suffices to show that $y \cdot (a + g^{-1}(1,0)) = 0$, where $y = z\backslash(d\backslash c)$. Now $y = (z\backslash d) + z \cdot c$. But $z \cdot c = 0$ since $z \cdot b = 0$. We have $a \cdot (z\backslash d) = z \cdot (a\backslash d) \leq f(1,0)\backslash d = 0$. Next, $g^{-1}(1,0) \cdot (z\backslash d) \leq g^{-1}(1,0)\backslash d = 0$. All of this proves 5.3(1).

Now $y \cdot (a + g^{-1}(1,0)) = 0$ iff $y \in a^\perp\backslash g(1,0)$. Now if $y \in a^\perp\backslash g(1,0)$, then $y \cdot (d\backslash c) = y \cdot (f^{-1}(1,0) + g^{-1}(1,0))\backslash c = y \cdot f^{-1}(0,1)\backslash c = 0$. Thus with x and y as in (1) we have $x \cdot y = 0$. Hence from (1) we get $b^\perp \cap A' \cong (A' \restriction (d\backslash c) \times (a^\perp\backslash g(1,0))$ Now $\tau_s(A' \restriction (d\backslash c) = (\alpha_1, 0, k)$ and $\tau_s(a^\perp\backslash g(1,0)) = (\delta, \delta, 0)$. Hence $\tau(b^\perp) = (\alpha_1, \delta, k)$.

Finally, $a\backslash f(1,0) \leq b$ and so $a\backslash b \leq f(1,0)$; hence $a\backslash b \in A'$. Also, $b\backslash a = c\backslash a \leq c$; hence $b\backslash a \in A'$. Thus $a \triangle b \in A'$.

Subcase 1.2. $m = 0$. Then by (v), $\alpha_0 = \gamma$. Let $b = a\backslash f^{-1}(1,0)$. By (iii) and (iv), $n = k$. Since $m = 0$ and $n \neq 0$, by (ii) we have $\alpha_0 \leq \alpha_1$; hence $\alpha = \alpha_1$. Then $\tau(A \restriction b) = (\gamma, \gamma, 0)$. Now $b^\perp = a^\perp + f(1,0)$, so $\tau(b^\perp) = (\alpha, \delta, n) = (\alpha_1, \delta, k)$. Clearly $a \triangle b \in A'$.

Subcase 1.3. $k = 0$. Then by (vi), $\alpha_1 = \delta$. Also, by (iii) $\alpha_1 \leq \alpha_0$; so $\alpha = \alpha_0$. By (ii) and (iv), $m = n$. Let $b = a + d$. Then $\tau(A \restriction b) = (\alpha, \gamma, n) = (\alpha_0, \gamma, m)$. We have $b^\perp \cap A' = \{x \in A' : x \cdot (a + d) = 0\}$ and so $\tau(b^\perp) = (\delta, \delta, 0)$. Clearly $a \triangle b \in A'$.

Case 2. $n = 0$. Then $\alpha = \beta$.

Subcase 2.1. $m = k = 0$. By (v) and (vi), a is as desired.

Subcase 2.2. $m \neq 0$. Then by (ii) and (iv), $\alpha_0 < \alpha_1$. So $\alpha = \alpha_1$. By (iii), $k = 0$. Hence by (vi), $\alpha_1 = \delta$. Since $\alpha_0 \geq \gamma$, we have $\gamma < \alpha$. $\max(\gamma, \delta) = \alpha$, so $\delta = \alpha$. Hence $\tau_s(a^\perp \cap A') = (\delta, \delta, 0)$. Now we apply Corollary 4.32 with A, α, β, m replaced by $a^\perp \cap A', \delta, \alpha_0, m$. This gives $c \in a^\perp \cap A'$ such that $\tau_s(A' \restriction c) = (\alpha_0, 0, m)$ and $\tau_s(\{x \in a^\perp \cap A' : x \cdot c = 0\}) = (\delta, \delta, 0)$.

Subsubcase 2.2.1. $p = 0$. Then $\tau(A \restriction a) = (\gamma, \gamma, 0)$. Let $b = a + c$. Then $(A \restriction b) \cap A' \cong ((A \restriction a) \cap A') \times (A \restriction c)$, and so $\tau(A \restriction b) = (\alpha_0, \gamma, m)$. Now $b^\perp \cap A' = \{x \in A' : x \cdot (a + c) = 0\}$, so $\tau(b^\perp) = (\delta, \delta, 0)$.

Subsubcase 2.2.2. $p \neq 0$.

5.3(2) $\varphi < \alpha$.

In fact, suppose that $\varphi = \alpha$. Since $\gamma < \alpha$, it follows that $\gamma < \mathrm{ar}((A \restriction a) \cap A')$ and so $\mathrm{ar}((A \restriction a) \cap A') = \alpha + 1$. Since $\mathrm{ar}(A') = \alpha$, this is a contradiction.

Let $b = a\backslash f^{-1}(1,0) + c$. Then $A \restriction b \cong (A \restriction (a\backslash f^{-1}(1,0))) \times (A \restriction c)$. Now $\tau(A \restriction (a\backslash f^{-1}(1,0))) = (\gamma,\gamma,0)$ and $\tau(A \restriction c) = (\alpha_0, 0, m)$. Hence $\tau(A \restriction b) = (\alpha_0, \gamma, m)$.

5.3(3) $b^\perp \cap A' = \{x + y : x \cdot (a + c) = 0, y \leq f^{-1}(1,0)\}$.

In fact, $\supseteq$ is clear. Now suppose that $z \in b^\perp \cap A'$ and $z \cdot f^{-1}(1,0) = 0$. Then

$$\begin{aligned} 0 = z \cdot (a\backslash f^{-1}(1,0) + c) &= z \cdot (a\backslash f^{-1}(1,0)) + z \cdot c \\ &= z \cdot (a\backslash f^{-1}(1,0)) + z \cdot a \cdot f^{-1}(1,0) + z \cdot c = z \cdot (a + c). \end{aligned}$$

So 5.3(3) holds.

It follows that $b^\perp \cap A' \cong \{x : x \cdot (a + c) = 0\} \times (A \restriction f^{-1}(1,0))$. Now $\{x : x \cdot (a + c) = 0\} \cong \mathrm{Intalg}_d(\omega^\delta)$ and $(A \restriction f^{-1}(1,0)) \cong \mathrm{Intalg}(\omega^\varphi \cdot p)$. Hence by Proposition 4.27, $\tau(b^\perp) = (\delta, \delta, 0)$.

Subcase 2.3. $k \neq 0$. Then by (iii) and (iv), $\alpha_1 < \alpha_0$ and $m = 0$. Hence $\alpha_0 = \gamma$ by (v). $\delta \leq \alpha_1$ so $\delta < \gamma$. Hence $\alpha = \max(\gamma, \delta) = \gamma$. Hence $\tau(A \restriction a) = (\gamma, \gamma, 0)$. Now we apply Corollary 4.32 with A, α, β, m replaced by $(A \restriction a) \cap A', \gamma, \gamma, 0$. This gives $c \in (A \restriction a) \cap A')$ such that $\tau(A \restriction c) = (\alpha_1, 0, k)$ and $\tau_s(\{x : x \leq a, x \cdot c = 0\}) = (\gamma, \gamma, 0)$.

Subsubcase 2.3.2. $q = 0$. Then $\tau(a^\perp) = (\delta, \delta, 0)$. Let $b = a\backslash c$. Then $\tau(A \restriction b) = (\gamma, \gamma, 0)$.

5.3(4) $b^\perp = \{y + z : y \cdot a = 0, z \leq c\}$.

In fact, $\supseteq$ is clear, and if $x \in b^\perp$ and $x \cdot c = 0$, then $0 = x \cdot (a\backslash c) = x \cdot (a\backslash c) + x \cdot a \cdot c = x \cdot a$, proving $\subseteq$.

It follows that $b^\perp \cap A' \cong (a^\perp \cap A') \times (A \restriction c)$. Hence $\tau(b^\perp) = (\alpha_1, \delta, k)$, as desired.

Subsubcase 2.3.2. $q \neq 0$.

5.3(5) $\xi < \alpha$.

This is proved as for 5.3(2).

Let $b = a\backslash c + g^{-1}(1,0)$. Then $(A \restriction b) \cap A' \cong \mathrm{Intalg}_d(\omega^\gamma) \times \mathrm{Intalg}(\omega^\xi \cdot q) \cong \mathrm{Intalg}_d(\omega^\gamma)$ by Proposition 4.27. So $\tau(A \restriction b) = (\gamma, \gamma, 0)$. Now $b^\perp \cap A' = \{x + y : x \cdot (a + g^{-1}(1,0)) = 0, y \leq c\}$. Hence $b^\perp \cap A' \cong \{x : x \cdot (a + g^{-1}(1,0)) = 0\} \times (A \restriction c)$. Hence $\tau(b^\perp) = (\alpha_1, \delta, k)$. $\square$

Proposition 5.4. *If A is a ABID, $a \in A$, J is an ideal of A, and β is an ordinal, then*

(i) $I_\beta((A \restriction a) \cap J) = (A \restriction a) \cap I_\beta(J)]$.

(ii) There is an isomorphism f of $((A \restriction a) \cap J)/I_\beta((A \restriction a) \cap J)$ onto $(J/I_\beta(J)) \restriction (a \triangle I_\beta(J))$ such that for any $x \in (A \restriction a) \cap J$, $f([x]_{I_\beta((A\restriction a)\cap J)}) = [x]_{I_\beta(J)}$.

Proof. For brevity let $B = (A \restriction a) \cap J$. We prove the proposition by induction on β. It is clear for $\beta = 0$. Now assume it for β. Suppose that $x \in I_{\beta+1}(B)$. Then there exist an $m \in \omega$ and elements $y_i \in B$ for $i < m$ such that each $[y_i]_{I_\beta(B)}$ is an atom in $B/I_\beta(B)$ and $[x]_{I_\beta(B)} = \sum_{i<m}[y_i]_{I_\beta(B)}$. Hence $x \triangle \sum_{i<m} y_i \in I_\beta(B)$, so by the inductive hypothesis $(x \triangle \sum_{i<m} y_i) \in I_\beta(J)$. Also by the inductive hypothesis each $f([y_i]_{I_\beta(B)} = [y_i]_{I_\beta(J)}$ is an atom in $J/I_\beta(J)$. Applying f to the equation $[x]_{I_\beta(B)} = \sum_{i<m}[y_i]_{I_\beta(B)}$ we get $[x]_{I_\beta(J)} = \sum_{i<m}[y_i]_{I_\beta(J}$. Hence $x \in I_{\beta+1}(J)$. Thus we have $\subseteq$ in (i) for $\beta + 1$. Reversing these steps we get $\supseteq$ also. So (i) holds for $\beta + 1$.

For (ii) for $\beta+1$, we have for any $x, x' \in A \upharpoonright a$

$$\begin{aligned}
[x]_{I_{\beta+1}((A\upharpoonright a)\cap J)} = [x']_{I_{\beta+1}((A\upharpoonright a)\cap J)} \quad &\text{iff} \quad (x \triangle x') \in I_{\beta+1}((A \upharpoonright a) \cap J)) \\
&\text{iff} \quad (x \triangle x') \in I_\beta(J) \quad \text{by (i) for } \beta+1 \\
&\text{iff} \quad [x]_{I_\beta(J)} = [x']_{I_\beta(J)}.
\end{aligned}$$

Now (ii) for $\beta+1$ follows.

The limit case is clear. □

Proposition 5.5. *For any ABID A let $\alpha = \sigma'(A)$. Then $\sigma' \upharpoonright A$ is an additive function with range contained in $\alpha+1$.*

Proof. We have $\sigma'(0) = 0$.

5.5(1) $\forall a \in A[\sigma'(a) \leq \sigma'(A)]$.

For, suppose that $a \in A$. $I_{\mathrm{ar}(A)}(A)/I_\alpha(I_{\mathrm{ar}(A)}(A))$ has a largest element $[b]_{I_\alpha(I_{\mathrm{ar}(A)}(A))}$, where $b \in I_{\mathrm{ar}(A)}(A)$. Now we apply Proposition 5.4, with $J = I_{\mathrm{ar}(A)}(A)$. Note that $(a \cdot b) \in (A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A)$. We claim that $[a \cdot b]_{I_\alpha((A\upharpoonright a)\cap I_{\mathrm{ar}(A)}(A))}$ is the greatest element of $((A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A))/I_\alpha((A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A))$. For, suppose that $c \in (A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A)$. Then $[c]_{I_\alpha(I_{\mathrm{ar}(A)}(A))} \in I_{\mathrm{ar}(A)}(A)/I_\alpha(I_{\mathrm{ar}(A)}(A))$, and hence $[c]_{I_\alpha(I_{\mathrm{ar}(A)}(A))} \leq [b]_{I_\alpha(I_{\mathrm{ar}(A)}(A))}$. But also $[c]_{I_\alpha(I_{\mathrm{ar}(A)}(A))} \leq [a]_{I_\alpha(I_{\mathrm{ar}(A)}(A))}$, so $[c]_{I_\alpha(I_{\mathrm{ar}(A)}(A))} \leq [a \cdot b]_{I_\alpha(I_{\mathrm{ar}(A)}(A))}$. Applying f^{-1} from Proposition 5.4 we get $[c]_{I_\alpha((A\upharpoonright a)\cap I_{\mathrm{ar}(A)}(A))} \leq [a \cdot b]_{I_\alpha((A\upharpoonright a)\cap I_{\mathrm{ar}(A)}(A))}$. This proves the claim. Hence (1) holds. It follows that the range of $\sigma' \upharpoonright A$ is contained in $\alpha+1$.

5.5(2) $\forall a, b \in A[a \leq b \to \sigma'(a) \leq \sigma'(b)]$.

In fact, suppose that $a, b \in A$ and $a \leq b$. Let $\beta = \sigma'(b)$. Then $((A \upharpoonright b) \cap I_{\mathrm{ar}(A)}(A))/I_\beta((A \upharpoonright b) \cap I_{\mathrm{ar}(A)}(A))$ has a largest element $[d]_{I_\beta((A\upharpoonright b)\cap I_{\mathrm{ar}(A)}(A))}$. By Proposition 5.4, $[d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$ is the largest element of $(I_{\mathrm{ar}(A)}(A)/I_\beta(I_{\mathrm{ar}(A)}(A))) \upharpoonright [b]_{I_\beta(I_{\mathrm{ar}(A)}(A)))}$. We claim that $(a \cdot d) \triangle I_\beta((A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A))$ is the greatest element of

$$((A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A))/I_\beta((A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A)).$$

For, suppose that $c \in (A \upharpoonright a) \cap I_{\mathrm{ar}(A)}(A)$. Then $[c]_{I_\beta(I_{\mathrm{ar}(A)}(A))} \leq [b]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$, and hence $[c]_{I_\beta(I_{\mathrm{ar}(A)}(A))} \leq [d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$. But also $[c]_{I_\beta(I_{\mathrm{ar}(A)}(A))} \leq [a]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$, so

$$[c]_{I_\beta(I_{\mathrm{ar}(A)}(A))} \leq [a \cdot d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}.$$

Applying f^{-1} from Proposition 5.4 we get $[c]_{I_\beta((A\upharpoonright a)\cap I_{\mathrm{ar}(A)}(A))} \leq [a \cdot d]_{I_\beta((A\upharpoonright a)\cap I_{\mathrm{ar}(A)}(A))}$. This proves the claim. Hence 5.5(2) holds.

Now let $a, b \in A$. By 5.5(2), $\max\{\sigma'(a), \sigma'(b)\} \leq \sigma'(a+b)$. Let $\beta = \max\{\sigma'(a), \sigma'(b)\}$. Then by Proposition 5.4, there exist a largest element $[c]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$ of

$$(I_{\mathrm{ar}}(A))/I_\beta(I_{\mathrm{ar}(A)}(A))) \upharpoonright (a \triangle I_\beta(I_{\mathrm{ar}(A)}(A)),$$

and a largest element $[d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$ of

$$(I_{\mathrm{ar}}(A))/I_\beta(I_{\mathrm{ar}(A)}(A))) \upharpoonright (b \triangle I_\beta(I_{\mathrm{ar}(A)}(A)).$$

Now we claim that $[c+d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}$ is the largest element of

$$(I_{\mathrm{ar}(A)}(A))/I_\beta(I_{\mathrm{ar}(A)}(A))) \restriction ((a+b)\triangle I_\beta(I_{\mathrm{ar}(A)}(A)).$$

For, suppose that $[x]_{I_\beta(I_{\mathrm{ar}(A)}(A))} \in (I_{\mathrm{ar}(A)}(A))/I_\beta(I_{\mathrm{ar}(A)}(A))) \restriction ((a+b)\triangle I_\beta(I_{\mathrm{ar}(A)}(A))$. Then

$$\begin{aligned}[x]_{I_\beta(I_{\mathrm{ar}(A)}(A))} &= [x\cdot a]_{I_\beta(I_{\mathrm{ar}(A)}(A))} + [x\cdot b]_{I_\beta(I_{\mathrm{ar}(A)}(A))}\\ &\leq [c]_{I_\beta(I_{\mathrm{ar}(A)}(A))} + [d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}\\ &= [c+d]_{I_\beta(I_{\mathrm{ar}(A)}(A))}\end{aligned}$$

□

A countable BA A is *normal* iff $\tau(A) = (\mathrm{ar}(A), \mathrm{ar}(A), 0)$ and $A/I_{\mathrm{ar}(A)}(A) \cong \mathrm{Fr}(\omega)$.

Proposition 5.6. *Suppose that A is a countable BA, $d, d' \in A$, and $[d]_{I_{\mathrm{ar}(A)}(A)} = [d']_{I_{\mathrm{ar}(A)}(A)}$. Then $\alpha_*(A \restriction d) = \alpha_*(A \restriction d')$.*

Proof. Note that if $a \in I_{\mathrm{ar}(A)}(A)$ then $\alpha_*(A \restriction a) = 0$. Since $d \cdot -d' \in I_{\mathrm{ar}(A)}(A)$ and $d' \cdot -d \in I_{\mathrm{ar}(A)}(A)$, we have

$$\alpha_*(A \restriction d) = \max(\alpha_*(A \restriction (d\cdot d')), \alpha_*(A \restriction (d \cdot -d'))) = \alpha_*(A \restriction (d\cdot d')),$$

and similarly $\alpha_*(A \restriction d') = \alpha_*(A \restriction (d \cdot d'))$. □

Proposition 5.7. *Suppose that A is normal and g is an isomorphism from $A/I_{ar(A)}(A)$ onto $\mathrm{Fr}(\omega)$. Then there is a function $r_{Ag} : \mathrm{Fr}(\omega) \to (\alpha_*(A)+1)$ such that for all $d \in A$, $r_{Ag}(g([d]_{I_{\mathrm{ar}(A)}(A)}) = \alpha_*(A \restriction d)$. Moreover, if $a, b \in \mathrm{Fr}(\omega)$ and $a \cdot b = 0$, then $r_{Ag}(a+b) = \max(r_{Ag}(a), r_{Ag}(b))$.*

Proof. r_{Ag} is well-defined, since if $g([d]_{I_{\mathrm{ar}(A)}(A)}) = g([d']_{I_{\mathrm{ar}(A)}(A)})$ then $[d]_{I_{\mathrm{ar}(A)}(A)}) = g([d']_{I_{\mathrm{ar}(A)}(A)})$ and so by Proposition 5.6, $\alpha_*(A \restriction d) = \alpha_*(A \restriction d')$. r_{Ag} maps into $\alpha_* + 1$ by Proposition 5.5. If $a, b \in \mathrm{Fr}(\omega)$ and $a \cdot b = 0$, choose $d, d' \in A$ such that $g([d]_{I_{\mathrm{ar}(A)}(A)}) = a$ and $g([d']_{I_{\mathrm{ar}(A)}(A)}) = b$. We may assume that $d \cdot d' = 0$. Then by Proposition 4.14,

$$\begin{aligned}r_{Ag}(a+b) &= g([d+d']_{I_{\mathrm{ar}(A)}(A)}) = \alpha_*(A \restriction (d+d'))\\ &= \max(\alpha_*(A \restriction d), \alpha_*(A \restriction d')) = \max(r_{Ag}(a), r_{Ag}(b)).\end{aligned}$$

□

Theorem 5.8. *Suppose that A_1 and A_2 are normal BAs with the same atomic type, and $g_1 : A_1/I_{\mathrm{ar}(A_1)}(A_1) \to \mathrm{Fr}(\omega)$ and $g_2 : A_2/I_{\mathrm{ar}(A_2)}(A_2) \to \mathrm{Fr}(\omega)$ are isomorphisms. Then the following are equivalent:*

(i) $A_1 \cong A_2$.

(ii) There is an automorphism k of $\mathrm{Fr}(\omega)$ such that $r_{A_1 g_1} = r_{A_2 g_2} \circ k$.

Proof. (i)⇒(ii): Suppose that s is an isomorphism of A_1 onto A_2. Now there is an isomorphism s^+ of $A_1/I_{\mathrm{ar}(A_1)}(A_1)$ onto $A_2/I_{\mathrm{ar}(A_2)}(A_2)$ such that $\forall a \in A_1[s^+([a]_{I_{\mathrm{ar}(A_1)}(A_1)}) =$

$[s(a)]_{I_{\mathrm{ar}(A_2)}(A_2)}$. Let $k = g_2 \circ s^+ \circ g_1^{-1}$. So k is an automorphism of $\mathrm{Fr}(\omega)$. For any $a \in \mathrm{Fr}(\omega)$ let $d \in A_1$ be such that $g_1([d]_{I_{\mathrm{ar}(A_1)}(A_1)}) = a$. Then

$$\begin{aligned} r_{A_2 g_2}(k(a)) &= r_{A_2 g_2}(k(g_1([d]_{I_{\mathrm{ar}(A_1)}(A_1)}))) = r_{A_2 g_2}(g_2(s^+([d]_{I_{\mathrm{ar}(A_1)}(A_1)}))) \\ &= r_{A_2 g_2}(g_2([s(d)]_{I_{\mathrm{ar}(A_2)}(A_2)}) = \alpha_*(A_2 \restriction (s(d)) = \alpha_*(A_1 \restriction d) \\ &= r_{A_1 g_1}(g(([d]_{I_{\mathrm{ar}(A_1)}(A_1)}))) = r_{A_1 g_1}(a). \end{aligned}$$

(ii)⇒(i): Assume that k is an automorphism of $\mathrm{Fr}(\omega)$ such that $r_{A_1 g_1} = r_{A_2 g_2} \circ k$. Define

$$\begin{aligned} R = \{&(a,b) \in A_1 \times A_2 : k(g_1([a]_{I_{\mathrm{ar}(A_1)}(A_1)})) = g_2([b]_{I_{\mathrm{ar}(A_2)}(A_2)}), \\ &\tau(A_1 \restriction a) = \tau(A_2 \restriction b), \tau(A_1 \restriction (-a)) = \tau(A_2 \restriction (-b))\}. \end{aligned}$$

We verify that this is a V-correspondence. If $(0,b) \in R$, then $\tau(A_1 \restriction 0) = \tau(A_2 \restriction b)$, so $b = 0$. Conversely, $b = 0$ clearly implies that $(0,b) \in R$. So (V1) holds. (V2) is similar.

Now by symmetry it suffices to prove (V3). So suppose that $(a,b) \in R$ and $c \in A_1$.

(a) There is a $d_0 \leq b$ such that $g_2([d_0]_{I_{\mathrm{ar}(B)}(B)} = k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)}$, $\tau(A_1 \restriction (a \cdot c)) = \tau(A_2 \restriction d_0)$, $\tau(A_1 \restriction (a \backslash c)) = \tau(A_2 \restriction (b \backslash d_0))$, $\tau(A_1 \restriction (-a + -c)) = \tau(A_2 \restriction (-b + -d_0))$, and $\tau(A_1 \restriction (-a + c)) = \tau(A_2 \restriction (-b + d_0))$.

Assuming that (a) holds, we have

$$\begin{aligned} g_2([b \backslash d_0]_{I_{\mathrm{ar}(A_2)}(A_2)} &= g_2([b]_{I_{\mathrm{ar}(A_2)}(A_2)}) \backslash g_2([d_0]_{I_{\mathrm{ar}(A_2)}(A_2)}) \\ &= k(g_1([a]_{I_{\mathrm{ar}(A_1)}(A_1)} \backslash k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)}) \\ &= k(g_1([a \backslash (a \cdot c)]_{I_{\mathrm{ar}(A_1)}(A_1)}) = k(g_1([a \backslash c]_{I_{\mathrm{ar}(A_1)}(A_1)}). \end{aligned}$$

Hence $(a \cdot c, b \cdot d_0) \in R$ and $(a \backslash c, b \backslash d_0) \in R$.

To prove (a), let $x = k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)}))$. Then $r_{A_1 g_1}(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)})) = r_{A_2 g_2}(k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)})) = r_{A_2 g_2}(x)$ and

$$\begin{aligned} r_{A_1 g_1}(g_1([a \backslash c)]_{I_{\mathrm{ar}(A_1)}(A_1)}) &= r_{A_2 g_2}(k(g_1([a \backslash c]_{I_{\mathrm{ar}(A_1)}(A_1)}))) \\ &= r_{A_2 g_2}(k(g_1([a]_{I_{\mathrm{ar}(A_1)}(A_1)})) \backslash k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)}))) \\ &= r_{A_2 g_2}(g_2([b]_{I_{\mathrm{ar}(A_2)}(A_2)}) \backslash x). \end{aligned}$$

Now since $k(g_1([a]_{I_{\mathrm{ar}(A_1)}(A_1)})) = g_2([b]_{I_{\mathrm{ar}(A_2)}(A_2)}$, there is a $y \leq b$ such that

$$g_2([y]_{I_{\mathrm{ar}(A_2)}(A_2)} = k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)}.$$

Now let $\tau(A_1 \restriction a) = (\alpha, \beta, n)$, $\tau(A_1 \restriction (a \cdot c)) = (\alpha_0, \gamma, m)$, and $\tau(A_1 \restriction (a \backslash c)) = (\alpha_1, \delta, k)$. Now $a_*(A_1 \restriction y) = r_{A_2 g_2}(g_2([y]_{I_{\mathrm{ar}(A_2)}(A_2)} = r_{A_2 g_2}(k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)})) = r_{A_1 g_1}(g_1([a \cdot$

$c]_{I_{\mathrm{ar}(A_1)}(A_1)}))) = \alpha_*(A_1 \restriction (a \cdot c) = \gamma$. Also,

$$\begin{aligned}
\alpha_*(A_2 \restriction (b\backslash y)) &= r_{A_2 g_2}(g_2([b\backslash y]_{I_{\mathrm{ar}(A_2)}(A_2)})) \\
&= r_{A_2 g_2}(g_2([b]_{I_{\mathrm{ar}(A_2)}(A_2)})\backslash g_2([y]_{I_{\mathrm{ar}(A_2)}(A_2)})) \\
&= r_{A_2 g_2}(k(g_1([a]_{I_{\mathrm{ar}(A_1)}(A_1)}))\backslash k(g_1([a \cdot c]_{I_{\mathrm{ar}(A_1)}(A_1)}))) \\
&= r_{A_2 g_2}(k(g_1([a\backslash c]_{I_{\mathrm{ar}(A_1)}(A_1)}))) \\
&= r_{A_1 g_1}(g_1([a\backslash c]_{I_{\mathrm{ar}(A_1)}(A_1)}) = \alpha_*(a\backslash c) = \delta.
\end{aligned}$$

Now we apply Proposition 5.3 with A, a replaced by $A_2 \restriction b, y$. This gives $d_0 \leq b$ such that $[y]_{I_{\mathrm{ar}(A_2 \restriction b)}(A_2 \restriction b)} = [d_0]_{I_{\mathrm{ar}(A_2 \restriction b)}(A_2 \restriction b)}$, $\tau(A_2 \restriction d_0) = (\alpha_0, \gamma, m) = \tau(A_1 \restriction (a \cdot c)$, and $\tau(A_2 \restriction (b\backslash d_0)) = (\alpha_1, \delta, k) = \tau(A_1 \restriction (a\backslash c))$. This gives the first part of (a).

Now let $\tau(A_1 \restriction (-a)) = (\alpha_2, \varphi, q)$. Then by Proposition 4.14, $\tau((A_1 \restriction (-a)) \times (A_1 \restriction (a\backslash c))) = (\alpha_3, \psi, r)$, where $\alpha_3 = \max(\alpha_1, \alpha_2)$, $\psi = \max(\varphi, \delta)$, and

$$r = \begin{cases} q & \text{if } \alpha_1 < \alpha_2, \\ k & \text{if } \alpha_2 < \alpha_1, \\ q + k & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

Since $\tau(A_2 \restriction (-b)) = \tau(A_1 \restriction (-a))$ and $\tau(A_2 \restriction (b\backslash d_0)) = \tau(A_1 \restriction (a\backslash c))$, it follows that

$$\tau((A_1 \restriction (-a)) \times (A_1 \restriction (a\backslash c)) = \tau((A_2 \restriction (-b)) \times (A_2 \restriction (b\backslash d_0)).$$

Now

$$(A_1 \restriction (-a)) \times (A_1 \restriction (a\backslash c)) \cong A_1 \restriction (-a + a\backslash c) = A_1 \restriction (-a + -c)$$

and

$$(A_2 \restriction (-b)) \times (A_2 \restriction (b\backslash d_0) \cong A_2 \restriction (-b + (b\backslash d_0)) = A_2 \restriction (-b + -d_0).$$

Hence $\tau(A_1 \restriction (-a + -c)) = \tau(A_2 \restriction (-b + -d_0))$. Now by Proposition 4.14, $\tau((A_1 \restriction (-a)) \times (A_1 \restriction (a \cdot c))) = (\alpha_4, \rho, u)$ where $\alpha_4 = \max(\alpha_0, \alpha_2)$, $\rho = \max(\gamma, \varphi)$, and

$$u = \begin{cases} q & \text{if } \alpha_0 < \alpha_2, \\ m & \text{if } \alpha_2 < \alpha_0, \\ m + q & \text{if } \alpha_0 = \alpha_2. \end{cases}$$

Since $\tau(A_1 \restriction (-a)) = \tau(A_2 \restriction (-b))$ and $\tau(A_1 \restriction (a \cdot c)) = \tau(A_2 \restriction d_0)$, it follows that

$$\tau((A_1 \restriction (-a)) \times (A_1 \restriction (a \cdot c))) = \tau((A_2 \restriction (-b)) \times (A_2 \restriction d_0)).$$

Now

$$(A_1 \restriction (-a)) \times (A_1 \restriction (a \cdot c)) \cong A_1 \restriction (-a + a \cdot c) = A_1 \restriction (-a + c)$$

and

$$(A_2 \restriction (-b)) \times (A_2 \restriction d_0) \cong A_2 \restriction (-b + d_0).$$

Hence $\tau(A_1 \restriction (-a + c)) = \tau(A_2 \restriction (-b + d_0))$.

This proves (a).

(b) There is a $d_1 \leq -b$ such that $g_2([d_1]_{I_{\mathrm{ar}(A_2)}(A_2)}) = k(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)}))$, $\tau(A \restriction (a+c)) = \tau(B \restriction (b+d_1))$, $\tau(A \restriction ((-a)\cdot(-c)) = \tau(B \restriction ((-b)\cdot(-d_1))$, $\tau(A \restriction (c\backslash a)) = \tau(B \restriction (d_1\backslash b))$, and $\tau(A \restriction (a+-c)) = \tau(B \restriction (b+-d_1))$.

Assuming that (b) holds, we have

$$\begin{aligned} g_2([b+d_1]_{I_{\mathrm{ar}(A_2)}(A_2)}) &= g_2([b]_{I_{\mathrm{ar}(A_2)}(A_2)}) + g_2([d_1]_{I_{\mathrm{ar}(A_2)}(A_2)} \\ &= k(g_1([a]_{I_{\mathrm{ar}(A_2)}(A_2)}) + k(g_1([c\backslash a]_{I_{\mathrm{ar}(A_2)}(A_2)}) \\ &= k(g_1([a+c]_{I_{\mathrm{ar}(A_2)}(A_2)})). \end{aligned}$$

Hence $(a+c, b+d_1) \in R$ and $(c\backslash a, d_1\backslash b) \in R$.

To prove (b), let $x = k(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)}))$. Then

$$r_{A_1 g_1}(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)})) = r_{A_2 g_2}(k(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)}) = r_{A_2 g_2}(x).$$

Now since $k(g_1([-a]_{I_{\mathrm{ar}(A_1)}(A_1)})) = g_2([-b]_{I_{\mathrm{ar}(A_2)}(A_2)})$, there is a $y \leq -b$ such that $g_2([y]_{I_{\mathrm{ar}(A_2)}(A_2)})) = k(g_1(A \restriction ([(-a)\cdot(-c)]_{I_{\mathrm{ar}(A_1)}(A_1)}))$. Say $\tau(A \restriction (-a)) = (\alpha, \beta, n)$, $\tau(A \restriction (-a)\cdot(-c)) = (\alpha_0, \gamma, m)$, and $\tau(A \restriction (c\backslash a)) = (\alpha_1, \delta, k)$. Then

$$\begin{aligned} \sigma'(y) &= r_{A_2 g_2}(g_2([y]_{I_{\mathrm{ar}(A_2)}(A_2)})) = r_{A_2 g_2}(k(g_1(A_1 \restriction ([(-a)\cdot(-c)]_{I_{\mathrm{ar}(A_1)}(A_1)})))) \\ &= r_{A_1 g_1}(g_1(A_1 \restriction ([(-a)\cdot(-c)]_{I_{\mathrm{ar}(A_1)}(A_1)}))) = \sigma'((-a)\cdot(-c)) = \gamma. \end{aligned}$$

Also,

$$\begin{aligned} \sigma'((-b)\cdot(-y)) &= r_{A_2 g_2}(g_2([(-b)\cdot(-y)]_{I_{\mathrm{ar}(A_2)}(A_2)})) \\ &= r_{A_2 g_2}(g_2([-b]_{I_{\mathrm{ar}(A_2)}(A_2)})\cdot g_2([-y]_{I_{\mathrm{ar}(A_2)}(A_2)})) \\ &= r_{A_2 g_2}(k(g_1([-a]_{I_{\mathrm{ar}(A_1)}(A_1)}))\cdot k(g_1([a+c]_{I_{\mathrm{ar}(A_1)}(A_1)})) \\ &= r_{A_2 g_2}(k(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)} \\ &= r_{A_1 g_1}(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)})) \\ &= \sigma'(c\backslash a) = \delta. \end{aligned}$$

Now we apply Proposition 5.3 with A, a replaced by $A_2 \restriction (-b), y$. This gives $d_1 \leq -b$ such that $d_1 \triangle I_{\mathrm{ar}(A_2 \restriction (-b))}(A \restriction (-b)) = y \triangle I_{\mathrm{ar}(A_2 \restriction (-b))}(A \restriction (-b))$, $\tau(A_2 \restriction d_1) = (\alpha_0, \gamma, m)$, and $\tau(A_2 \restriction ((-b)\backslash d_1)) = (\alpha_1, \delta, k)$. As in the proof of (a), $g_2([d_1]_{I_{\mathrm{ar}(A_2)}(A_2)}) = k(g_1([c\backslash a]_{I_{\mathrm{ar}(A_1)}(A_1)}))$.

Then $\tau(A_2 \restriction ((-b)\cdot d_1) = \tau(A_2 \restriction d_1) = (\alpha_0, \gamma, m) = \tau(A_1 \restriction ((-a)\cdot(-c)))$, and $\tau(A_2 \restriction ((-b)\cdot(-d_1)) = (\alpha_1, \delta, k) = \tau(A_1 \restriction (c\backslash a))$. Now let $\tau(A_1 \restriction a) = (\alpha_2, \varphi, p)$. Then by Proposition 4.14,

$$\tau((A_1 \restriction a) \times (A_1 \restriction (c\backslash a)) = (\alpha_3, \psi, q),$$

where $\alpha_3 = \max(\alpha_2, \alpha_1)$, $\psi = \max(\varphi, \delta)$, and

$$q = \begin{cases} p & \text{if } \alpha_1 < \alpha_2, \\ k & \text{if } \alpha_2 < \alpha_1, \\ p + k & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

It follows that

$$\tau((A_1 \restriction a) \times (A_1 \restriction (c \backslash a)) = \tau((A_2 \restriction b) \times (A_2 \restriction (((-d_1) \cdot (-b))).$$

Now $A_1 \restriction (a + c) \cong (A_1 \restriction a) \times (A_1 \restriction (c \backslash a)$ and $A_2 \restriction (b + -d_1)$, so $\tau(A_1 \restriction (a + c)) = \tau(A_2 \restriction (b + -d_1))$.

Next, $\tau((A_1 \restriction a) \times (A_1 \restriction ((-a) \cdot (-c)))) = (\alpha_4, \rho, s)$, where $\alpha_4 = \max(\alpha_2, \alpha_0)$, $\rho = \max(\gamma, \varphi)$, and

$$s = \begin{cases} m & \text{if } \alpha_2 < \alpha_0, \\ p & \text{if } \alpha_0 < \alpha_2, \\ m + p & \text{if } \alpha_0 = \alpha_2. \end{cases}$$

It follows that

$$\tau((A_1 \restriction a) \times (A_1 \restriction ((-a) \cdot (-c)))) = \tau((A_2 \restriction b) \times (A_2 \restriction ((-b) \cdot d_1))).$$

Now $A_1 \restriction (a + -c) \cong (A_1 \restriction a) \times (A_1 \restriction ((-a) \cdot (-c))$ and $A_2 \restriction (b + -d_1) \cong (A_2 \restriction b) \times (A_2 \restriction ((-b) \cdot d_1))$. Hence $\tau(a + -c) = \tau(b + d_1)$. □

If A is a normal BA with atomic type α, and g is an isomorphism of $A/I_{\mathrm{ar}(A)}(A)$ onto $\mathrm{Fr}(\omega)$, then $\mathrm{INV}(A) = (1, \alpha, \{r_{Ag} \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega))\})$.

Theorem B. *If A and B are normal BAs, then $A \cong B$ iff* $\mathrm{INV}(A) = \mathrm{INV}(B)$.

Proof. Let $\mathrm{INV}(A) = (1, \alpha, \{r_{Ag} \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega))\})$ and $\mathrm{INV}(B) = (1, \beta, \{r_{Bk} \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega))\})$. $\Rightarrow$: Suppose that h is an isomorphism of A onto B. Then obviously $\alpha = \beta$. By Theorem 5.8 there is an automorphism l of $\mathrm{Fr}(\omega)$ such that $r_{Ag} = r_{Bk} \circ l$. Then

$(*)$ $\{r_{Bk} \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega))\}) = \{r_{Ag} \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega))\})$.

In fact, if $h \in \mathrm{Aut}(\mathrm{Fr}(\omega))$, then $r_{Bk} \circ h = r_{Bk} \circ l \circ l^{-1} \circ h = r_{Ag} \circ l^{-1} \circ h$. So $\subseteq$ holds in $(*)$. The other inclusion follows similarly.

$\Leftarrow$: Suppose that $\mathrm{INV}(A) = \mathrm{INV}(B)$. In particular, $\alpha = \beta$. We have $r_{Ag} \in \{r_{Bk} \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega))\}$. So there is an $h \in \mathrm{Aut}(\mathrm{Fr}(\omega))$ such that $r_{Ag} = r_{Bk} \circ h$. By Proposition 5.8, $A \cong B$. □

We also define $\mathrm{INV}_1 = \{(1, \alpha, \{r \circ h : h \in \mathrm{Aut}(\mathrm{Fr}(\omega)), r : \mathrm{Fr}(\omega) \to \alpha + 1, r(1) = \alpha\})$. Thus if A is a normal BA, then $\mathrm{INV}(A) \in \mathrm{INV}_1$. The next portion of these notes is aimed at showing that for every $\alpha \in \mathrm{INV}_1$ there is a normal BA A such that $\mathrm{INV}(A) = \alpha$.

If $\langle B_n : n \in \omega \rangle$ is a system of BAs, we define

$$\sum_{n \in \omega} B_n = \left\{ f \in \prod_{n \in \omega} B_n : \{i \in \omega : f_i \neq 0\} \text{ is finite} \right\}$$

Proposition 5.9. *$\sum_{n\in\omega} B_n$ is an ideal in $\prod_{n\in\omega} B_n$.* $\square$

Proposition 5.10. *Let $B \stackrel{\text{def}}{=} \langle B_n : n \in \omega\rangle$ be a system of BAs. For each $x \in \prod_{n\in\omega} B_n$ let $h(x) = \{y \in \sum_{n\in\omega} B_n : y \le x\}$. For $X, Y \in \mathrm{rng}(h)$ let $X \cdot' Y = X \cap Y$ and $X +' Y = \{u+v : u \in X$ and $v \in Y\}$. For $X \in \mathrm{rng}(h)$ let $-'X = \{u \in \sum_{n\in\omega} B_n : \forall x \in X[u\cdot x = 0]\}$. Then*

(i) $B' \stackrel{\text{def}}{=} (\mathrm{rng}(h), +', \cdot', \{0\}, \sum_{n\in\omega} B_n)$ is a BA.
(ii) h is an isomorphism of $\prod_{n\in\omega} B_n$ onto B'.
(iii) $\forall x, y \in \prod_{n\in\omega} B_n[h(x) \cap -'h(y) = \{0\} \cup \{w \in \sum_{n\in\omega} B_n[w \le x$ and $w \cdot y = 0]\}$.

Proof. (iii) is clear. Now it suffices to prove (ii). Suppose that $x, y \in \prod_{n\in\omega} B_n$, and $f \in \sum_{n\in\omega} B_n$. Then

$$f \in h(x) \cdot' h(y) \text{ iff } (f \le x \text{ and } f \le y) \text{ iff } f \le x \cdot y \text{ iff } f \in h(x \cdot y).$$

$$\begin{aligned} f \in h(x) +' h(y) \quad &\text{iff} \quad \exists u \in h(x) \exists v \in h(y)[f = u + v] \\ &\text{iff} \quad \exists u, v[u \le x \text{ and } v \le y \text{ and } f = u + v] \\ &\text{iff} \quad f \le x + y \\ &\text{iff} \quad f \in h(x + y). \end{aligned}$$

$$\begin{aligned} f \in h(-x) \quad &\text{iff} \quad f \le -x \\ &\text{iff} \quad \forall g \in h(x)[f \cdot g = 0] \\ &\text{iff} \quad f \in -'h(x). \end{aligned}$$

Clearly $h(0) = \{0\}$ and $h(1) = \sum_{n\in\omega} B_n$.

Finally, suppose that $x, y \in \prod_{n\in\omega} B_n$ and $x \ne y$; say $x \cdot -y \ne 0$. Choose $f \in h(x \cdot -y)$ with $f \ne 0$. Then $f \in h(x)$ and $f \le -y$, hence $f \in -'h(y)$. So $f \in h(x) \cdot' -'h(y)$. Hence $h(x) \cdot' -'h(y) \ne \{0\}$, and so $h(x) \ne h(y)$. $\square$

Proposition 5.11. *If A is a special ABID and $\sigma'(A) = \alpha + 1$, then there is a system $\langle B_n : n \in \omega\rangle$ of superatomic BAs each of atomic type $(\alpha, 0, 1)$ such that $A \cong \sum_{n\in\omega} B_n$.*

Proof. We know that $A \cong \mathrm{Intalg}_d(\omega^{\alpha+1})$. For each $n \in \omega$ let $B_n = \mathrm{Intalg}([\omega^\alpha \cdot n, \omega^\alpha \cdot (n + 1)))$. Clearly each B_n has atomic type $(\alpha, 0, 1)$. Now for each $\xi < \omega^{\alpha+1}$ and $i \in \omega$ define

$$(f([0, \xi)))_i = \begin{cases} [\omega^\alpha \cdot i, \omega^\alpha \cdot (i + 1)) & \text{if } \omega^\alpha \cdot (i + 1) \le \xi, \\ [\omega^\alpha \cdot i, \xi) & \text{if } \omega^\alpha \cdot (i + 1) \le \xi < \omega^\alpha \cdot (i + 1), \\ 0 & \text{if } \xi < \omega^\alpha \cdot i. \end{cases}$$

Clearly f maps $\{[0, \xi) : \xi < \omega^{\alpha+1}\}$ into $\sum_{n\in\omega} B_n$, and $x \le y$ iff $f(x) \le f(y)$. Since the domain of f generates $\mathrm{Intalg}_d(\omega^{\alpha+1})$ and the range of f generates $\sum_{n\in\omega} B_n$, by Proposition 1.21 f extends to an isomorphism of $\mathrm{Intalg}_d(\omega^{\alpha+1})$ onto $\sum_{n\in\omega} B_n$. $\square$

Proposition 5.12. *If A is a special ABID, $\sigma'(A)$ is a limit ordinal, and $\langle \alpha_m : n \in \omega \rangle$ is a strictly increasing sequence of ordinals with supremum $\sigma'(A)$, then there is a system $\langle B_n : n \in \omega \rangle$ of superatomic BAs such that each B_n has atomic type $(\alpha_{n+1}, 0, 1)$ and $A \cong \sum_{n\in\omega} B_n$.*

Proof. Let $\beta = \sigma'(A)$. Thus $A \cong \mathrm{Intalg}_d(\omega^\beta)$. For each $n \in \omega$ let

$$B_n = \mathrm{Intalg}([\omega^{\alpha_n}, \omega^{\alpha_{n+1}})).$$

Clearly each B_n has atomic type $(\alpha_{n+1}, 0, 1)$. $A \cong \sum_{n\in\omega} B_n$ as in the proof of Proposition 5.11. □

If $B = \langle B_n : n \in \omega \rangle$ is a sequence of BAs and β is an ordinal, we define

$$^*I^B_\beta = \left\{ f \in \prod_{n\in\omega} B_n : \forall n \in \omega [f(n) \in I_\beta(B_n)] \right\}.$$

Proposition 5.13. *If $\langle B_n : n \in \omega \rangle$ is a system of ABIDs and β is an ordinal, then*
(i) $I_\beta(\sum_{n\in\omega} B_n) = \sum_{n\in\omega} I_\beta(B_n)$.
(ii) There is an onto isomorphism

$$f : \left(\sum_{n\in\omega} B_n \right) / I_\beta \left(\sum_{n\in\omega} B_n \right) \to \sum_{n\in\omega} (B_n / I_\beta(B_n))$$

such that for any $x \in \sum_{n\in\omega} B_n$ and $m \in \omega$, $(f([x]_{I_\beta(\sum_{n\in\omega} B_n)})_m = [x_m]_{I_\beta(B_m)}$.

Proof. Induction on β. It is clear for $\beta = 0$. Suppose that $x \in I_{\beta+1}(\sum_{n\in\omega} B_n)$. Then we can write

$$[x]_{I_\beta(\sum_{n\in\omega} B_n)} = \sum_{i<m} \left([y_i]_{I_\beta(\sum_{n\in\omega} B_n)} \right),$$

where each $[y_i]_{I_\beta(\sum_{n\in\omega} B_n)}$ is an atom of $(\sum_{n\in\omega} B_n)/I_\beta(\sum_{n\in\omega} B_n)$. Hence

$$x \triangle \sum_{i<m} y_i \in I_\beta \left(\sum_{n\in\omega} B_n \right) = \sum_{n\in\omega} I_\beta(B_n).$$

Now for each $n \in \omega$, $(x \triangle \sum_{i<m} y_i)_n \in I_\beta(B_n)$ and for each $i < m$, $[y_{in}]_{I_\beta(B_n)}$ is 0 or an atom. Thus $[x_n]_{I_\beta(B_n)} = \left[\sum_{i<m} y_{in}\right]_{I_\beta(B_n)}$. So $x_n \in I_{\beta+1}(B_n)$. This proves $\subseteq$ in (i) for $\beta + 1$.

Now suppose that $x \in \sum_{n\in\omega} I_{\beta+1}(B_n)$. Let $F = \{n \in \omega : x_n \neq 0\}$. For each $n \in F$ we can write

$$[x_n]_{I_\beta(B_n)} = \sum_{j<m_n} ([y_{jn}]_{I_\beta(B_n)}),$$

where each $[y_{jn}]_{I_\beta(B_n)}$ is an atom in $B_n/I_\beta(B_n)$. For each $n \in F$ and $j < m_n$ define $y'_{jn} \in \sum_{m\in\omega} B_m$ by

$$(y'_{jn})_m = \begin{cases} y_{jn} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle([y'_{jn}]_{I_\beta(B_k)} : k \in \omega\rangle$ is an atom in $\sum_{k\in\omega}(B_k/I_\beta(B_k))$. Hence by the inductive hypothesis for (ii), $[y'_{jn}]_{I_\beta\left(\sum_{n\in\omega} B_n\right)}$ is an atom in $(\sum_{n\in\omega} B_n)/I_\beta(\sum_{n\in\omega} B_n)$. Now for any $p \in F$,

$$\begin{aligned}\left[\left(\sum_{q\in F}\sum_{j<m_q} y'_{jq}\right)_p\right]_{I_\beta(B_p)} &= \left[\sum_{j<m_p} y_{jp}\right]_{I_\beta(B_p)} \\ &= \sum_{j<m_p} [y_{jp}]_{I_\beta(B_p)} = [x_p]_{I_\beta(B_p)}\end{aligned}$$

For $p \in \omega\backslash F$ we have $(\sum_{q\in F}\sum_{j<m_q} y'_j)_p = 0 = x_p$. Hence for all $p \in \omega$,

$$\left[\left(\sum_{q\in F}\sum_{j<m_q} y'_{jq}\right)_p\right]_{I_\beta(B_p)} = [x_p]_{I_\beta(B_p)}$$

Hence by (ii) for β, for all $p \in \omega$,

$$\begin{aligned}\left([x]_{I_\beta\left(\sum_{n\in\omega} B_n\right)}\right)_p &= \left(\left[\sum_{q\in F}\sum_{j<m_q} y'_{jq}\right]\right)_p \\ &= \left(\sum_{q\in F}\sum_{j<m_q} [y'_{jq}]_{I_\beta\left(\sum_{n\in\omega} B_n\right)}\right)_p.\end{aligned}$$

Hence

$$[x]_{I_\beta\left(\sum_{n\in\omega} B_n\right)} = \sum_{q\in F}\sum_{j<m_q} [y'_{jq}]_{I_\beta\left(\sum_{n\in\omega} B_n\right)}.$$

This shows that $x \in I_\beta(\sum_{n\in\omega} B_n)$. Hence (i) for $\beta + 1$ holds.

Now for (ii), if $x, x' \in \sum_{n\in\omega} B_n$, then

$$\begin{aligned} x_{I_{\beta+1}\left(\sum_{n\in\omega} B_n\right)} = x'_{I_{\beta+1}\left(\sum_{n\in\omega} B_n\right)} \quad &\text{iff} \quad (x\triangle x') \in I_{\beta+1}\left(\sum_{n\in\omega} B_n\right) \\ &\text{iff} \quad (x\triangle x') \in \sum_{n\in\omega} I_{\beta+1}(B_n) \\ &\text{iff} \quad \forall n \in \omega[(x\triangle x')_n \in I_{\beta+1}(B_n)] \\ &\text{iff} \quad \forall n \in \omega[[x_n]_{I_{\beta+1}(B_n)} = [x'_n]_{I_{\beta+1}(B_n)}]. \end{aligned}$$

Hence (ii) for $\beta+1$ follows.

The limit case is clear. $\square$

Proposition 5.14. *If A is a special ABID and $\langle B_n : n \in \omega\rangle$ is as in Propositions 5.11 or 5.12, then $\prod_{n\in\omega} B_n \subseteq {}^*I^B_{\sigma'(A)}$.*

Proof. Let $\beta = \sigma'(A)$. Then $\beta = \sigma'(\sum_{n\in\omega} B_n)$, and $\sum_{n\in\omega} B_n = I_\beta(\sum_{n\in\omega} B_n) = \sum_{n\in\omega} I_\beta(B_n)$ using Proposition 5.13. Hence $\forall n \in \omega[B_n = I_\beta(B_n)]$. So if $f \in \prod_{n\in\omega} B_n$ then $\forall n \in \omega[f(n) \in I_\beta(B_n)]$, so that $f \in {}^*I^B_\beta$. $\square$

Proposition 5.15. *Let β an ordinal and $B \stackrel{\text{def}}{=} \langle B_n : n \in \omega\rangle$ of BAs. Let h be as in Proposition 5.10, and suppose that $x \in \prod_{n\in\omega} B_n$. Then $h(x) \subseteq I_\beta(\sum_{n\in\omega} B_n)$ iff $x \in {}^*I^B_\beta$.*

Proof. First suppose that $h(x) \subseteq I_\beta(\sum_{n\in\omega} B_n)$. Thus $\forall y \in \sum_{n\in\omega} B_n[y \le x \to y \in I_\beta(\sum_{n\in\omega} B_n)]$, so by Proposition 5.13, $\forall y \in \sum_{n\in\omega} B_n[y \le x \to y \in \sum_{n\in\omega} I_\beta(B_n)]$. Now take any $n \in \omega$. Define $y \in \prod_{m\in\omega} B_m$ by

$$y_m = \begin{cases} x_m & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Then $y \in \sum_{m\in\omega} B_m$ and $y \le x$, so $y \in \sum_{n\in\omega} I_\beta(B_n)]$. Hence $x_n = y_n \in I_\beta(B_n)$. Since n is arbitrary, $x \in {}^*I^B_\beta$.

Second, suppose that $x \in {}^*I^B_\beta$. Take any $y \in h(x)$. Then $y \in \sum_{n\in\omega} B_n$ and $y \le x$. Now $\forall n \in \omega[x_n \in I_\beta(B_n)]$, so $\forall n \in \omega[y_n \in I_\beta(B_n)$. Then by Proposition 5.13, $y \in I_\beta(\sum_{n\in\omega} B_n$. $\square$

Let A be a special ABID. Let $\langle B_n : n \in \omega\rangle$ be as in Propositions 5.11 or 5.12. Then $A^* \stackrel{\text{def}}{=} \prod_{n\in\omega} B_n / \sum_{n\in\omega} B_n$. Now for each $\beta \le \sigma'(A)$ we define $\Psi_\beta(A) = \{[a]_{\sum_{n\in\omega} B_n} : a \in {}^*I^B_\beta\}$. Now if $d \in A^*$, say $d = [a]_{\sum_{n\in\omega} B_n}$, with $a \in \prod_{n\in\omega} B_n$. By Proposition 5.14, $a \in {}^*I^B_{\sigma'(A)}$. Hence $d \in \Psi_{\sigma'(A)}(A)$. For each $d \in A^*$ let $\rho(d) = \min\{\beta : d \in \Psi_\beta(A)\}$.

Proposition 5.16. *Let A be a ABID, J an ideal in A, and β an ordinal. Then*

(i) $J^\beta \stackrel{\text{def}}{=} \{[a]_{I_\beta(A)} : a \in J\}$ is an ideal in $A/I_\beta(A)$.

(ii) There is an isomorphism f of $J/I_\beta(J)$ onto $J^\beta(A)$ such that $f([a]_{I_\beta(J)}) = [a]_{I_\beta(A)}$ for all $a \in J$.

(iii) $I_\beta(J) = J \cap I_\beta(A)$.

Proof. For (i), J^β is clearly closed under $+$. Suppose that $a \in J$, $b \in A$, and $[b]_{I_\beta(A)} \le [a]_{I_\beta(A)}$. Then $b\cdot a \in J$ and $b \triangle (b\cdot a) = b \backslash a \in I_\beta(A)$. So $[b]_{I_\beta(A)} = [a\cdot b]_{I_\beta(A)} \in J^\beta$. So (i) holds.

Now we prove (ii) and (iii) by induction on β. They are clear for $\beta = 0$. For (iii) for $\beta+1$, first suppose that $a \in I_{\beta+1}(J)$. Then we can write $[a]_{I_\beta(J)} = \sum_{i<m}[y_i]_{I_\beta(J)}$, where each $[y_i]_{I_\beta(J)}$ is an atom of $J/I_\beta(J)$. By (ii) for β, $[a]_{I_\beta(A)} = \sum_{i<m}[y_i]_{I_\beta(A)}$. Also, (ii) implies that each $[y_i]_{I_\beta(A)}$ is an atom of $A/I_\beta(A)$. Hence $a \in I_{\beta+1}(A)$. The converse is similar.

For $\beta+1$ for (ii), if $a, a' \in J$ then

$$\begin{aligned} [a]_{I_{\beta+1}(J)} = [a']_{I_{\beta+1}(J)} \quad &\text{iff} \quad (a \triangle a') \in I_{\beta+1}(J) \\ &\text{iff} \quad (a \triangle a') \in I_{\beta+1}(A) \quad \text{by (iii) for } \beta+1, \\ &\text{iff} \quad [a]_{I_{\beta+1}(A)} = [a']_{I_{\beta+1}(A)}. \end{aligned}$$

Now (iii) follows.

The limit step is clear. □

Proposition 5.17. *Let A be a special ABID, let B be as in Proposition 5.11 or 5.12, and let $x \in \prod_{n\in\omega} B_n$. Then $\rho\left([x]_{\sum_{n\in\omega} B_n}\right) = \sigma'(h(x))$.*

Proof. Let $A' = \sum_{n\in\omega} B_n$. By Proposition 4.26 there is an $a \in h(x)$ such that $h(x) \cong (A' \restriction a) \times \{b \in A' : b \cdot a = 0\}$, with $\sigma'(h(x)) = \sigma'(\{b \in A' : b \cdot a = 0\})$. Hence $I_{\sigma'(h(x))}(\{b \in A' : b \cdot a = 0\}) = \{b \in A' : b \cdot a = 0\}$. By Proposition 4.30, $\{b \in A' : b \cdot a = 0\} \subseteq I_{\sigma'(h(x))}(A')$. Now $a \in h(x) \subseteq \sum_{n\in\omega} B_n \subseteq \prod_{n\in\omega} B_n$. So we can form $-a \in \prod_{n\in\omega} B_n$.

(1) $\{b \in A' : b \cdot a = 0\} = h(-a)$.

In fact, ($b \in A'$ and $b \cdot a = 0$) iff ($b \in A'$ and $b \leq -a$) iff $b \in h(-a)$. So (1) holds.

Hence $h(x) \cong (A' \restriction a) \times h(-a)$, so $(\prod_{n\in\omega} B_n \restriction x) \cong (\prod_{n\in\omega} B_n \restriction h^{-1}(a)) \times (\prod_{n\in\omega} B_n \restriction (-a))$. Thus $h(-a) \subseteq I_{\sigma'(h(x))}(A')$. Hence by Proposition 5.15, $-a \in {}^*I^B_{\sigma'(h(x))}$. So $[-a]_{A'} \in \Psi_{\sigma'(h(x))}(A)$. Now $a \in \sum_{n\in\omega} B_n$, so $[-a]_{A'} = [x]_{A'}$. It follows that $\rho([x]_{A'}) \leq \sigma'(h(x))$.

Suppose that $\gamma \stackrel{\text{def}}{=} \rho([x]_{A'}) < \sigma'(h(x))$. Then $[x]_{A'} \in \Psi_\gamma(A)$. Hence there is a $b \in {}^*I^B_\gamma$ such that $[x]_{A'} = [b]_{A'}$. Now $h(b) \subseteq I_\gamma(\sum_{n\in\omega} B_n)$, so $\sigma'(h(b)) \leq \gamma$. Choose $c \in h(b)$ so that $\{x \in h(b) : x \cdot c = 0\}$ is special. Thus $\sigma'(\{x \in h(b) : x \cdot c = 0\}) = \sigma'(h(b))$. Hence $h(b \cdot -c) = \{x \in h(b) : x \cdot c = 0\} \subseteq I_{\sigma'(h(b))}$. Hence by Proposition 5.15, $b \cdot -c \in {}^*I^B_{\sigma'(h(b))}$. Then $x \triangle (b \cdot -c) = x \triangle b \triangle b \triangle (b \cdot -c) \in A'$. It follows that there is a finite $F \subseteq \omega$ such that $\forall n \in \omega \backslash F [x_n = b_n \cdot -c_n]$. Define $e \in \sum_{n\in\omega} B_n$ by

$$e_n = \begin{cases} 1 & \text{if } n \in F, \\ 0 & \text{if } n \notin F. \end{cases}$$

Also, define $e' \in \sum_{n\in\omega} B_n$ by

$$e'_n = \begin{cases} x_n & \text{if } n \in F, \\ 0 & \text{if } n \notin F. \end{cases}$$

Then $h(x) \cong (h(x) \restriction e') \times \{y \in h(x) : y \cdot e' = 0\}$, so $\{y \in h(x) : y \cdot e' = 0\}$ is special with $\sigma'(h(x)) = \sigma'(\{y \in h(x) : y \cdot e' = 0\})$. Also, $\{y \in h(x) : y \cdot e = 0\} = \{y \in h(x) : y \cdot e' = 0\}$.

Now define $e'' \in \sum_{n\in\omega} B_n$ by

$$e''_n = \begin{cases} b_n \cdot -c_n & \text{if } n \in F, \\ 0 & \text{if } n \notin F. \end{cases}$$

Let $M = \{y \in h(b \cdot -c) : y \cdot e'' = 0\}$. Then M is special with $\sigma'(M) = \sigma'(h(b \cdot -c))$.

(2) $\forall y \in A'[y \leq x \cdot -e'$ iff $y \leq b \cdot (-c) \cdot (-e'')]$.

For, suppose that $y \leq x \cdot -e'$. If $n \in F$, then $x_n \cdot -e'_n = 0$, so $y_n = 0$. If $n \in \omega \backslash F$, then $y_{<} x_n = b_n \cdot -c_n$; also $-e''_n = 1$, so $y_n \leq -e''$. This proves $\Rightarrow$.

Suppose that $y \leq b \cdot (-c) \cdot (-e'')]$. If $n \in F$, then $b_n \cdot -c_n \cdot -e''_n = 0$, so $y_n = 0$. If $n \in \omega \backslash F$, then $y_n \leq b_n \cdot -c_n = x_n$. Also, $-e'_n = 1$, so $y_n \leq -e'_n$. This proves $\Leftarrow$.

So (2) holds.

Now $\gamma < \sigma'(h(x)) = \sigma'(\{y \in h(x) : y \cdot e' = 0\}) = \sigma'(h(b) \cdot -c \cdot -e'') = \sigma'(h(b \cdot -c) = \sigma'(h(b)) \leq \gamma$, contradiction. $\square$

Proposition 5.18. *Let A be a special ABID, and let $\langle B_n : n \in \omega \rangle$ be as above. Then ρ maps A^* into $\sigma'(A) + 1$, ρ is additive, and $\rho(d) = 0$ implies that $d = 0$.*

Proof. By Proposition 4.30, for any $x \in \prod_{n \in \omega} B_n$, $I_{\sigma'(A)}(h(x)) = h(x) \cap I_{\sigma'(A)}(A) = h(x) \cap I_{\sigma'(A)+1}(A) = I_{\sigma'(A)+1}(h(x))$ and so by Proposition 5.17, $\rho([x]_{\sum_{n \in \omega} B_n} = \sigma'(h(x)) \leq \sigma'(A)$. Hence ρ maps A^* into $\sigma'(A) + 1$. Clearly $\rho(0) = 0$ and $\rho(d) = 0$ implies that $d = 0$. Now suppose given $d, e \in A^*$; say $d = [d']_{\sum_{n \in \omega} B_n}$ and $e = [e']_{\sum_{n \in \omega} B_n}$. Then

$$\begin{aligned}
\rho(d + e) &= \rho([d']_{\sum_{n \in \omega} B_n} + [e']_{\sum_{n \in \omega} B_n} = \rho([d' + e']_{\sum_{n \in \omega} B_n} \\
&= \sigma'(h(d' + e')) = \sigma'(h(d') + h(e')) \quad \text{by Proposition 5.10} \\
&= \sigma'(h(d')) + \sigma'(h(e')) = \rho([d']_{\sum_{n \in \omega} B_n}) r([e']_{\sum_{n \in \omega} B_n}) = \rho(d) + \rho(e).
\end{aligned}$$ $\square$

Proposition 5.19. *For any special ABID A, any $d \in A^*$, and any $\beta \leq \rho(d)$ there is a $d_0 \in A$ such that $d_0 \leq d$, $\rho(d_0) = \beta$, and $\rho(d \backslash d_0) = \rho(d)$.*

Proof. Suppose that A is a ABID, $d \in A^*$, and $\beta \leq \rho(d)$. Say $\rho(d) = \gamma$. If $\beta = 0$, take $d_0 = 0$. Assume that $\beta \neq 0$. Let $\delta_0 \leq \delta_1 \leq \cdots$ be such that each δ_i is a successor ordinal and $\sup_{n \in \omega} \delta_n = \beta$. Choose $k \in \prod_{n \in \omega} B_n$ so that $d = [k]_{\sum_{n \in \omega} B_n}$.

5.19(1) $\forall \varepsilon < \gamma[\{n \in \omega : k(n) \notin I_\varepsilon(B_n)\}$ is infinite].

Otherwise, let $F = \{n \in \omega : k(n) \notin I_\varepsilon(B_n)\}$. Let $g \restriction (\omega \backslash F) = k \restriction (\omega \backslash F)$ with $\forall n \in \omega[g(n) \in I_\varepsilon(B_n)]$. Then $d = [k]_{\sum_{n \in \omega} B_n} = [g]_{\sum_{n \in \omega} B_n}$, contradicting $\rho(d) = \gamma$.

Case 1. $\forall n \in \omega[\delta_n < \beta]$. Let $i \in \omega$. By 5.19(1), choose $n_i \in \omega$ such that $k(n_i) \notin I_{\delta_i}(B_{n_i})$. Say $\tau_s(B_{n_i} \restriction k(n_i)) = (\xi, 0, m)$. Then $B_{n_i} \restriction k(n_i) \cong \mathrm{Intalg}(\omega^\xi \cdot m)$ and so $\delta_i < \xi$. Say g is an isomorphism from $B_{n_i} \restriction k(n_i)$ onto $\mathrm{Intalg}(\omega^\xi \cdot m)$. Let $a = [0, \omega^{\delta_i})$, an element of $\mathrm{Intalg}(\omega^\xi \cdot m)$. Then $\tau_s(\mathrm{Intalg}(\omega^\xi \cdot m) \restriction a) = (\delta_i, 0, 1)$. Let $d_i \stackrel{\text{def}}{=} g^{-1}(a) \in B_{n_i} \restriction k(n_i)$. Then $\tau_s(B_{n(i)} \restriction d_i) = (\delta_i, 0, 1)$. Now we define

$$h'(n) = \begin{cases} d_{2i} & \text{if } n = n_{2i}, \\ 0 & \text{if } n \notin \{n_0, n_2, \ldots\}. \end{cases}$$

Let $d' = [h']_{\sum_{n \in \omega} B_n}$. Then by Proposition 5.17, $\rho(d') = \sigma'(h(h'))$. Now $h' \in {}^*I_\beta(A)$, so by Proposition 5.15, $h(h') \subseteq I_\beta(A)$. Hence by Proposition 4.30, $h(h') \subseteq I_\beta(h(h'))$, so

$\sigma'(h(h')) \leq \beta$. If $\varepsilon < \beta$, then clearly $h' \notin {}^*I_\varepsilon(A)$. So $\sigma'(h(h')) \not\leq \varepsilon$. Thus $\sigma'(h(h')) = \beta$. So $\rho(d') = \beta$. If $\beta < \gamma$, then $\rho(d \backslash d') = \gamma$ by Proposition 5.18. Suppose that $\beta = \gamma$. Then define

$$h''(n) = \begin{cases} d_{2i+1} & \text{if } n = n_{2i+1}, \\ 0 & \text{if } n \notin \{n_1, n_3, \ldots\}. \end{cases}$$

Let $d'' = [h'']_{\sum_{n\in\omega} B_n}$. Then as above, $\rho(d'') = \beta$. Clearly $d'' \leq d \backslash d'$, so $\rho(d \backslash d') = \beta$.

Case 2. $\beta = \delta + 1$ for some δ. By 5.9(1), choose distinct $n_i \in \omega$ for $i < \omega$ such that $k(n_i) \notin I_\delta(B_{n_i})$. Say $\tau_s(B_{n_i} \restriction k(n_i)) = (\xi, 0, m)$. Then $B_{n(i)} \restriction k(n_i) \cong \mathrm{Intalg}(\omega^\xi \cdot m)$ and so $\delta < \xi$. Say g is an isomorphism from $B_{n_i} \restriction k(n_i)$ onto $\mathrm{Intalg}(\omega^\xi \cdot m)$. Let $a = [0, \omega^\delta)$, an element of $\mathrm{Intalg}(\omega^\xi \cdot m)$. Then $\tau_s(\mathrm{Intalg}(\omega^\xi \cdot m) \restriction a) = (\delta, 0, 1)$. Hence $d_i \stackrel{\text{def}}{=} g^{-1}(\mathrm{Intalg}(\omega^\xi \cdot m) \restriction a) \leq k(n_i)$ and $\tau_s(d_i) = (\delta, 0, 1)$. Now we define

$$h'(n) = \begin{cases} d_{2i} & \text{if } n = n_{2i}, \\ 0 & \text{if } n \notin \{n_0, n_2, \ldots\}. \end{cases}$$

Let $d' = [h']_{\sum_{n\in\omega} B_n}$. By Proposition 5.17, $\rho(d') = \sigma'(h(h'))$. Now $h' \in {}^*I_\beta(A)$, so by Proposition 5.15, $h(h') \subseteq I_\beta(A)$. Hence by Proposition 4.30, $I_\beta(h(h')) = h(h') \cap I_\beta(A) = h(h')$. thus $h(h') \subseteq I_\beta(K)$, so $\sigma'(h(h')) \leq \beta$.

5.19(2) $\sigma'(h(h')) = \beta$.

In fact, suppose that $\sigma'(h(h')) = \varepsilon < \beta$. Since $\rho(d') = \sigma'(h(h'))$, it follows that there is an $e \in {}^*I^B_\varepsilon$ such that $d' = [e]_{\sum_{n\in\omega} B_n}$. Thus $[h']_{\sum_{n\in\omega} B_n} = [e]_{\sum_{n\in\omega} B_n}$, so $h' \triangle e \in \sum_{n\in\omega} B_n$. Let F be finite such that $\forall n \in \omega \backslash F[h'(n) = e(n)]$. Take i with $n_{2i} \notin F$. Then $h'(n_{2i}) = d_{2i} = e(n_{2i}) \in I_\varepsilon(B_n)$, contradiction.

So $\rho(d') = \beta$. If $\beta < \gamma$, then $\rho(d \backslash d') = \gamma$ by Proposition 5.18. Suppose that $\beta = \gamma$. Then define

$$h''(n) = \begin{cases} d_{2i+1} & \text{if } n = n_{2i+1}, \\ 0 & \text{if } n \notin \{n_1, n_3, \ldots\}. \end{cases}$$

Let $d'' = [h'']_{\sum_{n\in\omega} B_n}$. Then as above, $\rho(d'') = \beta$. Clearly $d'' \leq d \backslash d'$, so $\rho(d \backslash d') = \beta$. □

Proposition 5.20. *Let B be a special ABID, $\alpha = \sigma'(B)$, A a countable BA, and $r : A \to \alpha + 1$ an additive function, with $r(1) = \alpha$. Then there is a homomorphism $f : A \to B^*$ such that $\forall d \in A[r(d) = \rho(f(d))]$.*

Proof. Assume the hypotheses. Let $R = \{(a, b) \in A \times B^* : r(a) = \rho(b)$ and $r(-a) = \rho(-b)\}$. We verify the conditions (V5) and (V6) in the definition of a weak V-relation. Condition (V5) is clear. Now suppose that $(a, b) \in R$ and $c \in A$. Let $r(a) = \beta$, $r(-a) = \gamma$, $r(a \cdot c) = \delta$, $r(a \backslash c) = \eta$. By the additivity of r, $\max(\beta, \gamma) = \alpha$ and $\max(\delta, \eta) = \beta$.

Case 1. $\delta \leq \eta$. In Proposition 5.19 replace β and d by δ and b. So $\delta \leq \beta = r(a) = \rho(b)$ gives a $d_0 \leq b$ such that $\rho(d_0) = \delta = r(a \cdot c)$ and $\rho(b \backslash d_0) = \rho(b) = r(a) = \beta = \eta = r(a \backslash c)$. Also, $\rho(b \cdot d_0) = \rho(d_0) = r(a \cdot c)$.

Case 2. $\eta < \delta$. So $\eta \leq \beta = r(a) = \rho(b)$ gives a $d_1 \leq b$ such that $\rho(d_1) = \eta = r(a\backslash c)$ and $\rho(b\backslash d_1) = \rho(b) = r(a) = \beta = \delta = r(a \cdot c)$. Also, $\rho(b \cdot d_1) = \rho(d_1) = r(a\backslash c)$. Setting $d_0 = -d_1$ we get $\rho(b \cdot d_0) = r(a \cdot c)$ and $\rho(b\backslash d_0) = r(a\backslash c)$.

Now

$$\begin{aligned} r(-a + -c) &= r(-a + a\backslash c) = \max(r(-a), r(a\backslash c) \\ &= \max(\rho(-b), \rho(b\backslash d_0)) = \rho(-b + b\backslash d_0) = \rho(-b + -d_0); \\ r(-a + c) &= r(-a + a \cdot c) = \max(r(-a), r(a \cdot c)) \\ &= \max(\rho(-b), \rho(b \cdot d_0)) = \rho(-b + b \cdot d_0) = \rho(-b + d_0). \end{aligned}$$

This shows that $(a \cdot c, d_0), (a\backslash c, b\backslash d_0) \in R$.

Now $-a = c \cdot -a + -c \cdot -a$, so $r(-a) = \max(r(c\backslash a), r(-c \cdot -a)$.

Case 1. $r(-a) = r(-c \cdot -a)$. By Proposition 5.19 there is a $d_1 \leq -b$ such that $\rho(d_1) = r(c\backslash a)$ and $\rho(-b \cdot -d_1) = \rho(-b)$. Thus $\rho(d_1\backslash b) = \rho(d_1) = r(c\backslash a)$. Then $r(a + c) = r(a + (c\backslash a)) = \max(r(a), r(c\backslash a)) = \max(\rho(b), \rho(d_1\backslash b)) = \rho(d_1 + b)$. Also, $r(-a \cdot -c) = r(-a) = \rho(-b) = \rho(-b \cdot -d_1)$. Also, $r(-c + a) = r(-c \cdot -a + a) = \max(r(-c \cdot -a), r(a)) = \max(\rho(-b \cdot -d_1), \rho(b)) = \rho(-b \cdot -d_1 + b) = \rho(-d_1 + b)$. Hence $(a + c, b + d_1), (c\backslash a, d_1\backslash b) \in R$.

Case 2. $r(-a) = r(c\backslash a)$. By Proposition 5.19 there is a $d_1 \leq -b$ such that $\rho(d_1) = r(-c \cdot -a)$ and $\rho(-b \cdot -d_1) = \rho(-b)$. Thus $\rho(d_1\backslash b) = \rho(d_1) = r(-c \cdot -a)$. Next,

$$\begin{aligned} r(a + c) &= r(a + (c\backslash a)) = \max(r(a), r(c\backslash a)) \\ &= \max(r(a), r(-a)) = r(1) = \alpha = \rho(1) = \max(\rho(b), \rho(-b)) \\ &= \max(\rho(b), \rho(-b \cdot -d_1)) = \rho(b + -b \cdot -d_1) = \rho(b + -d_1). \end{aligned}$$

Next, $r(c\backslash a) = r(-a) = \rho(-b) = \rho(-d_1 \cdot -b)$. Finally,

$$\begin{aligned} r(-c + a) &= r(-c \cdot -a + a) = \max(r(-c \cdot -a), r(a)) \\ &= \max(\rho(d_1\backslash b), \rho(b)) = \rho((d_1\backslash b) + b) = \rho(d_1 + b). \end{aligned}$$

This verifies (V6). Hence by Proposition 3.2 we get a homomorphism $f : A \to B^*$ with the additional property described there. Suppose that $a \in A$. Choose $m \in \omega$ and $b \in {}^m A$ so that (i)–(ii) of Proposition 3.2 hold. Then $r(a) = \max\{r(b_i) : i < m\} = \max\{\rho(f(b_i)) : i < m\} = \rho(f(a))$. □

Proposition 5.21. *Suppose that α is an ordinal, and $s : \mathrm{Fr}(\omega) \to \alpha + 1$ is an additive function such that $s(1) = \alpha$. Then there exist a BA A such that $\tau(A) = (\alpha, \alpha, 0)$ and an isomorphism g of $A/I_{\mathrm{ar}(A)}(A)$ onto $\mathrm{Fr}(\omega)$ such that $r_{Ag} = s$.*

Proof. Let $C = \mathrm{Intalg}_d(\omega^\alpha)$. By Proposition 5.20 let f' be a homomorphism from $\mathrm{Fr}(\omega)$ into C^* such that $\forall d \in \mathrm{Fr}(\omega)[s(d) = \rho(f'(d))]$. Let f be an isomorphism from C onto $\sum_{n\in\omega} B_n$. Let h and B' be as in Proposition 5.10. Let

$$\begin{aligned} &D = \{(u, h(x)) : u \in \mathrm{Fr}(\omega), x \in \prod_{n\in\omega} B_n, f'(u) = [x]_{\sum_{n\in\omega} B_n}\}; \\ &I' = \{(0, h(f(c))) : c \in C\}; \\ &\quad \forall (u, h(x)) \in D[w(u, h(x)) = u]; \\ &\quad \forall c \in C[t(c) = (0, h(f(c)))]; \\ &\quad \forall (u, h(x)) \in D[l([(u, h(x))]_I) = u]. \end{aligned}$$

Clearly

5.21(1) D is a subalgebra of the BA $\mathrm{Fr}(\omega) \times \mathrm{rng}(h)$.

5.21(2) I' is an ideal in D.

5.21(3) w is a homomorphism from D into $\mathrm{Fr}(\omega)$

5.21(4) t is an isomorphic embedding of C into D.

5.21(5) l is well-defined, and is an isomorphism of D/I' onto $\mathrm{Fr}(\omega)$.

In fact, clearly l is well-defined, and is a homomorphism of D/I' onto $\mathrm{Fr}(\omega)$. To see that it is one-one, suppose that $l([(u, h(x))]_{I'}) = 0$. Thus $u = 0$. Since $(u, h(x)) \in D$, we have $x \in \sum_{n\in\omega} B_n$. Say $x = f(c)$ with $c \in C$. Then $(u, h(x)) = (0, h(f(c)) \in I'$, as desired.

Now let E, t', l' be such that $C \subseteq E$, t' is an isomorphism of E onto D, $t' \restriction C = t$, and $\forall b \in E[l'([b]_C) = l([t'(b)]_{I'})]$. So l' is an isomorphism from E/C onto $\mathrm{Fr}(\omega)$. Now for any $d \in \mathrm{Fr}\omega)$ write $d = l'([e]_C)$. Note by Proposition 5.1 that $I_{\mathrm{ar}(E)}(E) \subseteq C$. Hence

5.21(6) $$r_{El'}(d) = r_{El'}(l'([e]_C)) = r_{El'}(l'([e]_{I_{\mathrm{ar}(E)}(E)}) = \alpha_*(E \restriction e) = \alpha_*(D \restriction t'(e)).$$

Say $t'(e) = (u, h(x))$. Now

$$f'(d) = f'(l'([e]_C)) = f'(l([t'(e)]_C)) = f'(l([(u, h(x))]_C)) = f'(u) = [x]_{\sum_{n\in\omega} B_n}.$$

Hence by Proposition 5.17 we get

5.21(7) $s(d) = \rho(f'(d)) = \alpha_*(h(x))$.

Now we claim that for any α,

5.21(8) $I_\alpha(D \restriction (u, h(x))) = \{0\} \times I_\alpha(h(x))$.

5.21(9) For any $(p, q) \in D \restriction (u, h(x))$ let $f_\alpha([(p, q)]_{I_\alpha(D\restriction(u,h(x)))}) = (p, [q]_{I_\alpha(h(x))})$. Then f_α is well-defined, and is an isomorphism of $(D \restriction (u, h(x))/I_\alpha(D \restriction (u, h(x))))$ onto $\{(u', [v]_{I_\alpha(h(x))}) : u' \leq u, v \leq h(x), f(u') = v\}$.

Note that 5.12(9) with $\alpha = \alpha_*(D \restriction (u, h(x)))$ gives $\alpha_*(D \restriction t'(e)) = \alpha_*(h(x))$. Together with 5.21(6) and 5.21(7) this finishes the proof.

5.21(8) and 5.21(9) are clear for $\alpha = 0$. Now assume them for α. Suppose that $(s, t) \in I_{\alpha+1}(D \restriction (u, h(x)))$. Then there are atoms $[(z_i, y_i)]_{I_\alpha(D\restriction(u,h(x)))}$ of $(D \restriction (u, h(x)))/I_\alpha(D \restriction (u, h(x)))$ for $i < m$ such that $[(s, t)]_{I_\alpha(B\restriction(u,h(x)))} = \sum_{i<m}[(z_i, y_i)]_{I_\alpha(D\restriction(u,h(x)))}$. Then $(s, t)\triangle\sum_{i<m}(z_i, y_i) \in I_\alpha(D \restriction (u, h(x)))$. So by 5.21(8) for α, $s\triangle\sum_{i<m} z_i = 0$ and $t\triangle\sum_{i<m} y_i \in I_\alpha(v)$. For any $i < m$, by 5.21(9) for α, $[(z_i, y_i)]_{I_\alpha(h(x))}$ is an atom of $\{(u', [v']_{I_\alpha(h(x))} : u' \leq u,\ v' \leq h(x),\ f(u') = v'\}$.

5.21(10) $z_i = 0$.

For, suppose that $z_i \neq 0$. Now $f(z_i) = y_i$. Let w be such that $0 < w < z_i$, and set $u = f(w)$. Then $(0, 0) < [(w, u)]_{I_\alpha(h(x))} < (z_i, [y_i]_{I_\alpha(h(x))}$, contradiction. So 5.2(10) holds.

It follows that $[y_i]_{I_\alpha(h(x))}$ is an atom of $h(x)/I_\alpha(h(x))$. Hence $s = 0$ and $t \in I_{\alpha+1}(h(x))$.

Conversely, suppose that $t \in I_{\alpha+1}(h(x))$. Then there exist atoms $[y_i]_{I_\alpha(h(x))}$ of $h(x)/I_\alpha(h(x))$ for $i < m$ such that $[t]_{I_\alpha(h(x))} = \sum_{i<m}[y_i]_{I_\alpha(h(x))}$. Then $t \triangle \sum_{i<m} y_i \in I_\alpha(h(x))$. Hence by 5.21(8) for α, $(0, t \triangle \sum_{i<m} y_i) \in I_\alpha(D \restriction (u, h(x)))$. Hence $[(0,t)]_{I_\alpha(D\restriction(u,h(x)))} = \sum_{i<m}[(0,y_i)]_{I_\alpha(D\restriction(u,h(x)))}$. Now each $[(0,y_i)]_{I_\alpha(D\restriction(u,h(x)))}$ is an atom of $D \restriction (u, h(x))$ by 5.21(9) for α. Hence $(0,t) \in I_{\alpha+1}(D \restriction (u, h(x)))$.

This proves 5.21(8) for $\alpha + 1$.

For 5.21(9) for $\alpha + 1$, suppose that $(p,q), (p',q') \in D \restriction (u, h(x))$. Then

$$\begin{aligned}
&[(p,q)]_{I_{\alpha+1}(D\restriction(u,h(x)))} = [(p',q')]_{I_{\alpha+1}(D\restriction(u,h(x)))}\\
&\text{iff } ((p,q)\triangle(p',q')) \in I_{\alpha+1}(D \restriction (u,h(x)))\\
&\text{iff } p = p' \text{ and } q \triangle q' \in I_{\alpha+1}(h(x)) \quad \text{by } 5.21(8)\\
&\text{iff } p = p' \text{ and } [q]_{I_\alpha(h(x))} = [q']_{I_\alpha(h(x))}.
\end{aligned}$$

5.21(9) for $\alpha + 1$ now easily follows.

The limit step is clear.

So 5.21(8) and 5.21(9) hold. □

For any additive function $s : \mathrm{Fr}(\omega) \to \omega_1$ the BA of Proposition 5.21 is denoted by B_s; an isomorphism $g : B_s^* \to \mathrm{Fr}(\omega)$ is denoted by g_s. Thus $r_{B_s g_s} = s$.

Proposition 5.22. *If A is a countable non-superatomic BA, then there exist BAs B, C such that $A \cong B \times C$ and:*

(i) B is normalized.

(ii) Either $|C| = 1$ or C is superatomic with atomic type of the form $(\alpha, 0, n)$ with $\alpha \geq \sigma'(B)$.

Proof. Let $\tau(A) = (\alpha, \beta, n)$.

Case 1. $n = 0$. Then by Proposition 5.10, $\alpha = \beta$. Hence $I_{\mathrm{ar}(A)}(A)$ is special. Since $A/I_{\mathrm{ar}(A)}(A)$ is atomless, A is normalized. We can then take $B = A$ and $|C| = 1$.

Case 2. $n > 0$. Now by Proposition 4.22 there is an element $d \in I_{\mathrm{ar}(A)}(A)$ with $\tau_s(I_{\mathrm{ar}(A)}(A) \restriction d) = (\alpha, 0, n)$ and $\tau_s(d^\perp) = (\beta, \beta, 0)$. Now

$$\begin{aligned}
d^\perp &= \{a \in I_{\mathrm{ar}(A)}(A) : a \cdot d = 0\} = I_{\mathrm{ar}(A)}(A) \cap (A \restriction (-d))\\
&= I_{\mathrm{ar}(A)}(A \restriction (-d)) = I_{\mathrm{ar}(A\restriction(-d))}(A \restriction (-d)).
\end{aligned}$$

Now

$$\tau(A \restriction d) = \tau_s(I_{\mathrm{ar}(A\restriction d)}(A \restriction d)) = \tau_s(I_{\mathrm{ar}(A)}(A) \restriction d) = (\alpha, 0, n)$$

and

$$\tau(A \restriction (-d)) = \tau_s(I_{\mathrm{ar}(A\restriction(-d))}(A \restriction (-d))) = \tau_s(d^\perp) = (\beta, \beta, 0).$$ □

Proposition 5.23. *Suppose that A is a countable nonsuperatomic BA, and $A \cong B \times C \cong B' \times C'$, and*

(i) B and B' are normalized.

(ii) $|C| = |C'| = 1$, or there exist ordinals α, β and positive integers n, p such that $\alpha \geq \sigma'(B)$, $\beta \geq \sigma'(B')$, $\tau(C) = (\alpha, 0, n)$, and $\tau(C') = (\beta, 0, p)$.

Then $B \cong B'$ and $C \cong C'$.

Proof. Assume the hypotheses. If $|C| = |C'| = 1$, then the desired conclusion is clear. Assume now that α, β are ordinals and n, p are positive integers such that $\alpha \geq \sigma'(B)$, $\beta \geq \sigma'(B')$, $\tau(C) = (\alpha, 0, n)$, and $\tau(C') = (\beta, 0, p)$. Say $\beta \leq \alpha$. Let $f : A \to B \times C$ be an isomorphism. Then by Proposition 4.11, $f[I_{\alpha+1}(A)] = I_{\alpha+1}(B \times C) = I_{\alpha+1}(B) \times I_{\alpha+1}(C)$. Similarly, let $g : A \to B' \times C'$ be an isomorphism. Then by Proposition 4.11, $g[I_{\alpha+1}(A)] = I_{\alpha+1}(B' \times C') = I_{\alpha+1}(B') \times I_{\alpha+1}(C')$. Now $\tau_s(I_{\alpha+1}(B)) = (\sigma'(B), \sigma'(B), 0)$. By the above, $\tau_s(C) = (\alpha, 0, n)$. Also, $\tau_s(I_{\alpha+1}(B')) = (\sigma'(B'), \sigma'(B'), 0)$. By the above, $\tau_s(C') = (\beta, 0, p)$. Then by Proposition 4.14, $\tau_s(I_{\alpha+1}(B) \times I_{\alpha+1}(C)) = (\alpha, \sigma'(B), n)$. Similarly, $\tau_s(I_{\alpha+1}(B') \times I_{\alpha+1}(C')) = (\beta, \sigma'(B'), p)$. Now $I_{\alpha+1}(B) \times I_{\alpha+1}(C) \cong I_{\alpha+1}(B') \times I_{\alpha+1}(C')$, so $\alpha = \beta$, $\sigma'(B) = \sigma'(B')$, and $n = p$. Then $\tau_s(I_{\alpha+1}(B)) = \tau_s(I_{\alpha+1}(B'))$, so $I_{\alpha+1}(B) \cong I_{\alpha+1}(B')$ by Proposition 4.28. Similarly, $I_{\alpha+1}(C) \cong I_{\alpha+1}(C')$. Now $C = I_{\alpha+1}(C)$ and $C' = I_{\alpha+1}(C')$. Hence $C \cong C'$. Now $A/I_{\alpha+1}(A) \cong B/I_{\alpha+1}(B) \times C/I_{\alpha+1}(C) \cong B/I_{\alpha+1}(B)$, since $|C/I_{\alpha+1}(C)| = 1$. Similarly, $A/I_{\alpha+1}(A) \cong B'/I_{\alpha+1}(B)$. Let h be an isomorphism from $B/I_{\alpha+1}(B)$ onto $\mathrm{Fr}(\omega)$. Now since $I_{\alpha+1}(B) \cong I_{\alpha+1}(B')$, let B'' and j be such that j is an isomorphism from B' onto B'' and $j[I_{\alpha+1}(B')] = I_{\alpha+1}(B)$. Let k be an isomorphism from $B''/I_{\alpha+1}(B)$ onto $\mathrm{Fr}(\omega)$. For each $e \in A$ let $t([e_{I_{\alpha+1}(A)}) = 1^{st}([f(e)]_{I_{\alpha+1}(B)}$. Then t is an isomorphism from $A/I_{\alpha+1}(A)$ onto $B/I_{\alpha+1}(B)$.

Let l be an isomorphism of $A/I_{\alpha+1}(A)$ onto $\mathrm{Fr}(\omega)$. Now if $x \in \mathrm{Fr}(\omega)$ is given, choose $e \in A$ so that $l([e]_{I_{\alpha+1}(A)}) = x$. Then $l \circ t^{-1}$ is an isomorphism of $B/I_{\alpha+1}(B)$ onto $\mathrm{Fr}(\omega)$. Hence

$$l(t^{-1}(t([e]_{I_{\alpha+1}(A)})) = l([e]_{I_{\alpha+1}(A)}) = x.$$

Hence

$$\begin{aligned} r_{B,l\circ t^{-1}}(x) &= r_{B,l\circ t^{-1}}(l(t^{-1}(t(e \triangle I_{\alpha+1}(A)))) \\ &= r_{B,l\circ t^{-1}}(l(t^{-1}(1^{st}(f(e)) \triangle I_{\alpha+1}(B) = \sigma'(1^{st}(f(e)) = \sigma'(e). \end{aligned}$$

Let g' be an isomorphism of A onto $B'' \times C$. For each $e \in A$ let $s([e]_{I_{\alpha+1}(A)}) = 1^{st}([g'(e)]_{I_{\alpha+1}(B)}$. Then s is an isomorphism from $A/I_{\alpha+1}(A)$ onto $B''/I_{\alpha+1}(B)$. As above, $r_{B''}(x) = \sigma'(e)$. Hence $B \cong B'' \cong B'$. □

Now if A is a BA which is nonsuperatomic and is not normalized, then $\mathrm{INV}(A) = (2, \beta, \alpha, n, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))$, where there is a superatomic BA C such that $A \cong B \times C$, with B normalized, $\tau(B) = (\beta, \beta, 0)$, $\tau(C) = (\alpha, 0, n)$, and $\alpha \geq \beta$. Note by Proposition 5.23 that the entries in $\mathrm{INV}(A)$ do not depend on the particular B, C chosen.

Theorem C. *For any BAs A, A' the following are equivalent:*

(i) $A \cong A'$.

(ii) $\mathrm{INV}(A) = \mathrm{INV}(A')$.

Proof. For A, A' superatomic, or A, A' normal, see Theorems A and B. Now suppose that A, A' are not superatomic, and not normal.

(i)$\Rightarrow$(ii): Suppose that $A \cong A'$. By Proposition 5.22 there are BAs B, C such that $A \cong B \times C$, B is normalized, and C is superatomic with some type $(\alpha, 0, n)$, with $\alpha \geq \sigma'(B)$. Then clearly $\mathrm{INV}(A) = \mathrm{INV}(A')$.

(ii)$\Rightarrow$(i). Suppose that $\mathrm{INV}(A) = (2, \beta, \alpha, n, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))$, where there is a superatomic BA C such that $A \cong B \times C$, with B normalized, $\tau(B) = (\beta, \beta, 0)$, $\tau(C) = (\alpha, 0, n)$, and $\alpha \geq \beta$ and $\mathrm{INV}(A') = (2, \beta', \alpha', n', \{r'_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))$, where there is a superatomic BA C' such that $A' \cong B' \times C'$, with B' normalized, $\tau(B') = (\beta, \beta, 0)$, $\tau(C') = (\alpha', 0, n')$, and $\alpha' \geq \beta'$. Then $\alpha = \alpha'$, $\beta = \beta'$, $n = n'$, and there is an $f \in \mathrm{Aut}(\mathrm{Fr}(\omega))$ such that $r_B = r_{B'} \circ f$. Now $\tau_s(I_\beta(B)) = \tau_s(I_\beta(B'))$, so by Proposition 4.23, $I_\beta(B) \cong I_\beta(B')$; say that h is an isomorphism from $I_\beta(B)$ onto $I_\beta(B')$. Then there exists an isomorphism h' of B onto a BA B'' such that $I_\beta(B') \subseteq B''$ and $h' \restriction I_\beta(B) = h$:

$$\begin{tikzcd}
B \arrow[r, "h'"] \arrow[d, dash, "\subseteq"'] & B'' \arrow[d, dash, "\subseteq"] \\
I_\beta(B) \arrow[r, "h"'] & I_\beta(B')
\end{tikzcd}$$

Now since B is normalized, there is an isomorphism l from $B/I_\beta(B)$ onto $\mathrm{Fr}(\omega)$. Similarly, there is an isomorphism k from $B'/I_\beta(B')$ onto $\mathrm{Fr}(\omega)$. For any $a \in B''$ define $g([a]_{I_\beta(B')}) = f(l([h'^{-1}(a)]_{I_\beta(B)}))$. Then g is well-defined, since if $a \in I_\beta(B')$ then $h'^{-1}(a) = h^{-1}(a) \in I_\beta(B)$. Clearly then g is an isomorphism from $B''/I_\beta(B')$ onto $\mathrm{Fr}(\omega)$. Hence if $b \in B''$ then

$$\begin{aligned}
r_{B''g}(g([b]_{I_\beta(B')})) &= \sigma'(b) = \sigma'(h'^{-1}(b)) \\
&= r_{B, l \circ h'^{-1}}(l([h'^{-1}(b)]_{I_\beta(B)})) = r_{B', f \circ l \circ h'^{-1}}([f(l(h'^{-1}(b)]_{I_\beta(B)})) \\
&= r_{B'g}(g([b]_{I_\beta(B')}))
\end{aligned}$$

Thus $r_{B''g} = r_{B'g}$, and so by Proposition 5.8, $B'' \cong B'$. Hence $B \cong B'$.

$C \cong C'$ by Proposition 4.28. It follows that $A \cong A'$. $\square$

Also, we define

$$\begin{aligned}
\mathrm{INV} = \{&(0, \alpha, n) : \alpha \text{ an ordinal}, n \in \omega \backslash \{0\}\} \\
&\cup \{(1, \alpha, \{r \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\}) : \alpha \text{ a nonzero ordinal}, \\
&\qquad r : \mathrm{Fr}(\omega) \to \alpha + 1 \text{ is additive}, r(1) = \alpha\} \\
&\cup \{(2, \beta, \alpha, n, \{r \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\}) : \alpha, \beta \text{ are ordinals}, n \in \omega \backslash \{0\}, \\
&\qquad \alpha \geq \beta \neq 0,\ r : \mathrm{Fr}(\omega) \to \beta + 1 \text{ is additive}, r(1) = \beta\}
\end{aligned}$$

Theorem 5.24. *If* $\Gamma \in \mathrm{INV}$*, then there is a BA* A *such that* $\mathrm{INV}(A) = \Gamma$.

Proof. Suppose that $\Gamma \in \mathrm{INV}$. For $1^{st}(\Gamma) = 0$ or 1, apply Propositions 4.19 and 5.21. Now suppose that $1^{st}(\Gamma) = 2$. Say $\Gamma = (2, \beta, \alpha, n, \{r \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$, where

α, β are ordinals, $n \in \omega\backslash\{0\}$, $\alpha \geq \beta \neq 0$, $r : \mathrm{Fr}(\omega) \to \beta + 1$ is additive, and $r(1) = \beta$. Let $A_0 = \mathrm{Intalg}_d(\omega^\beta)$. Thus A_0 is a special ABID such that $\tau(A_0) = (\beta, \beta, 0)$. By Proposition 5.21 let B be a BA such that $\tau(B) = (\beta, \beta, 0)$ and let g be an isomorphism of B^* onto $\mathrm{Fr}(\omega)$ such that $r_{Bg} = r$. Let $A = B \times C$, where $C = \mathrm{Intalg}(\omega^\alpha \cdot n)$. Then by Proposition 4.11, $(B \times C)/I_\beta(B \times C) \cong (B/I_\beta(B)) \times (C/I_\beta(C))$. Thus B is normalized and $\tau(C) = (\alpha, 0, n)$. So $\mathrm{Inv}(A) = (2, \beta, \alpha, n, \{r \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$. $\square$

Proposition 5.25. *If A is a superatomic BA with $\tau(A) = (\alpha, 0, n)$ and B is a normalized BA with $\tau(B) = (\beta, \beta, 0)$, and $\alpha < \beta$, then $A \times B \cong B$.*

Proof. We have $(A \times B)/I_\beta(A \times B) \cong (A/I_\beta(A)) \times (B/I_\beta(B)) \cong (B/I_\beta(B)) \cong \mathrm{Fr}(\omega)$. Also, $I_\beta(A \times B) = I_\beta(A) \times I_\beta(B)$. By Proposition 4.14, $\tau_s(I_\beta(A \times B)) = (\beta, \beta, 0)$. Thus $A \times B$ is normalized. By Proposition 4.27, $I_\beta(A \times B) \cong I_\beta(B)$. Let f be an isomorphism of $(A \times B)/I_\beta(A \times B)$ onto $\mathrm{Fr}(\omega)$. Let g be an isomorphism of $A \times B$ onto a BA C such that $g[I_\beta(A \times B)] = I_\beta(B)$. Let $g'([(a, b)]_{I_\beta(A\times B)}) = [g((a, b))]_{I_\beta(C)})$ for all $(a, b) \in A \times B$. Thus g' is an isomorphism of $(A \times B)/I_\beta(A \times B)$ onto $C/I_\beta(C)$. Let h be an isomorphism of $B/I_\beta(B)$ onto $\mathrm{Fr}(\omega)$. Now $f \circ g'^{-1}$ is an isomorphism of $C/I_\beta(C)$ onto $\mathrm{Fr}(\omega)$. Let $x \in \mathrm{Fr}(\omega)$. Choose $c \in C$ such that $f(g'^{-1}([c]_{I_\beta(C)})) = x$. Then

$$r_{C,f\circ g'^{-1}}(x) = r_{C,f\circ g'^{-1}}(f(g'^{-1}([c]_{I_\beta(C)}))) = \sigma'(C \restriction c) = \sigma'((A \times B) \restriction g^{-1}(c)).$$

Say $g^{-1}(c) = (a, b)$. Now there is an automorphism l of $\mathrm{Fr}(\omega)$ such that for all $(u, v) \in A \times B$, $l(f(](u, v)]_{I_\beta(A\times B)})) = h([v]_{I_\beta(B)})$. Hence

$$\begin{aligned} r_{Bh}(l(x)) &= r_{Bh}(l(f(g'^{-1}([c]_{I_\beta(C)}))))) = r_{Bh}(l(f(g'^{-1}([g(g^{-1}(c))]_{I_\beta(C)}))))) \\ &= r_{Bh}(l(f(g'^{-1}([g((a, b))]_{I_\beta(C)}))))) = r_{Bh}(l(f([(a, b)]_{I_\beta(A\times B)}))) \\ &= r_{Bh}(h([b]_{I_\beta(B)}) = \sigma'(B \restriction b) = \sigma'((A \times B) \restriction g^{-1}(c)). \end{aligned}$$

It follows that $B \cong C \cong A \times B$. $\square$

Proposition 5.26. *Suppose that B and C are normalized, with $\tau(B) = (\beta, \beta, 0)$ and $\tau(C) = (\gamma, \gamma, 0)$. Then*

(i) if $\beta < \gamma$, then $B \times C \cong C$;

(ii) if $\gamma < \beta$, then $B \times C \cong B$;

(iii) if $\beta = \gamma$, then $\tau(B \times C) = (\beta, \beta, 0)$, $B \times C$ is normalized, and $\forall x \in \mathrm{Fr}(\omega)\exists y, z \in \mathrm{Fr}(\omega)[x = y + z$ and $r_{A\times B,u}(x) = \max(r_{A,v}(y), r_{B,w}(z))]$ for some u, v, w.

Proof. (i): Assume that $\beta < \gamma$. By Proposition 4.14, $\tau(B \times C) = (\gamma, \gamma, 0)$, and by Proposition 4.11, $I_\gamma(B \times C) \cong I_\gamma(C)$. Then

$$(B \times C)/I_\gamma(B \times C) \cong (B/I_\gamma(B)) \times (C/I_\gamma(C)) \cong C/I_\gamma(C) \cong \mathrm{Fr}(\omega).$$

Hence $B \times C$ is normalized. Let f be an isomorphism of $B \times C$ onto a BA D such that $f[I_\gamma(B \times C)] = I_\gamma(D)$. Let g be an isomorphism from $C/I_\gamma(C)$ onto $\mathrm{Fr}(\omega)$, and let h be an

isomorphism from $D/I_\gamma(D)$ onto $\mathrm{Fr}(\omega)$. Define $f'([(u,v)]_{I_\gamma(B\times C)}) = [f(u,v)]_{I_D(D)}$. Then f' is an isomorphism of $(B\times C)/I_\gamma(B\times C)$ onto $D/I_\gamma(D)$. Define

$$k(h(f'([(u,v)]_{I_\gamma(B\times C)}))) = g([v]_{I_\gamma(C)}).$$

Then k is an automorphism of $\mathrm{Fr}(\omega)$. Now given $x \in \mathrm{Fr}(\omega))$, choose $(b,c) \in B\times C$ such that $h(f'([(b,c)]_{I_\gamma(B\times C)})) = x$. Let $g([c]_{I_\gamma(C)}) = y$. Then $h\circ f'$ is an isomorphism from $(B\times C)/I_\gamma(B\times C)$ onto $\mathrm{Fr}(\omega)$. Further, $k(x) = g([c]_{I_\gamma(C)}$, and

$$r_{D,h\circ f'}(x) = r_{D,h\circ f'}(h(f'((b,c)\triangle I_\gamma(B\times C))) = \sigma'(f(b,c));$$
$$r_{Cg}(k(x)) = r_{Cg}(g([c]_{I_\gamma(C)}) = r_{Cg}(y) = \sigma'(c) = \sigma'(f(b,c)).$$

Hence $B\times C \cong C$.

(ii): this is symmetric to (i).

(iii): By Proposition 4.14, $\tau(B\times C) = (\beta,\beta,0)$. Also,

$$(B\times C)/I_\beta(B\times C) \cong (B/I_\beta(B)) \times (C/I_\beta(C)) \cong \mathrm{Fr}(\omega)\times\mathrm{Fr}(\omega) \cong \mathrm{Fr}(\omega).$$

So $B\times C$ is normalized.

Let f be an isomorphism of $(B\times C)/I_\beta(B\times C)$ onto $\mathrm{Fr}(\omega)$. Given $x\in\mathrm{Fr}(\omega)$ choose $(b,c)\in B\times C$ such that $f([(b,c)]_{I_\beta(B\times C)}) = x$. Then

$$r_{B\times C,f}(x) = r_{B\times C,f}(f([(b,c)]_{I_\beta(B\times C)})) = \sigma'(b,c) = \max(\sigma'((b,0)),\sigma'((0,c)).$$

Now $[(b,0)]_{I_\beta(B\times C)} + [(0,c)]_{I_\beta(B\times C)} = [(b,c)]_{I_\beta(B\times C)}$. Let $f([(b,0)]_{I_\beta(B\times C)} = y$ and $f([(0,c)]_{I_\beta(B\times C)} = z$. Then $x = y+z$. Let g be an isomorphism of $B/I_\beta(B)$ onto $((B\times C)/I_\beta(B\times C)) \restriction ([(1,0)]_{I_\beta(B\times C)})$ and h an isomorphism of $C/I_\beta(C)$ onto $((B\times C)/I_\beta(B\times C)) \restriction ([(0,1)]_{I_\beta(B\times C)})$. Then $r_{B,f\circ g}(y) = r_{B,f\circ g}(f(g([b]_{I_\beta(B)})) = \sigma'((b,0))$ and similarly $r_{C,f\circ h}(z) = \sigma'((0,c))$. So $r_{B\times C,f}(x) = \max(r_{B,f\circ g}(y), r_{C,f\circ h}(z))$. □

Proposition 5.27. *Let A, B be nontrivial countable BAs.*

(i) $\mathrm{INV}(A) = (0,\alpha,n)$, $\mathrm{INV}(B) = (0,\beta,m) \Rightarrow \mathrm{INV}(A\times B) = (0,\max(\alpha,\beta),p)$, *where*

$$p = \begin{cases} m & \text{if } \alpha<\beta, \\ n & \text{if } \beta<\alpha, \\ m+n & \text{if } \alpha=\beta. \end{cases}$$

(ii) If $\mathrm{INV}(A) = (0,\alpha,n)$ *and* $\mathrm{INV}(B) = (1,\beta,\{r_B\circ f : f\in\mathrm{aut}(\mathrm{Fr}(\omega))\})$, *where B is normalized and* $\tau(B) = (\beta,\beta,0)$, *then*

(a) if $\alpha<\beta$, then $A\times B\cong B$ and so $\mathrm{INV}(A\times B) = \mathrm{INV}(B)$,

(b) if $\beta\le\alpha$, then $\mathrm{INV}(A\times B) = (2,\beta,\alpha,n,\{r_B\circ f : f\in\mathrm{aut}(\mathrm{Fr}(\omega)\})$.

(iii) If $\mathrm{INV}(A) = (0,\alpha,n)$ *and* $\mathrm{INV}(B) = (2,\beta,\gamma,m,\{r_B\circ f : f\in\mathrm{aut}(\mathrm{Fr}(\omega)\})$, *then*

(a) if $\alpha<\beta$, then $A\times B\cong B$ and so $\mathrm{INV}(A\times B) = \mathrm{INV}(B)$,

(b) if $\beta\le\alpha$, then $\mathrm{INV}(A\times B) = (2,\beta,\alpha,n,\{r_B\circ f : f\in\mathrm{aut}(\mathrm{Fr}(\omega))\})$,

(iv) If $\mathrm{INV}(A) = (0,\alpha,m)$ *and* $\mathrm{INV}(B) = (2,\beta,\gamma,n,\{r_B\circ f : f\in\mathrm{aut}(\mathrm{Fr}(\omega))\})$, *then*

(a) if $\alpha<\beta$, then $A\times B\cong B$, and so $\mathrm{INV}(A\times B) = \mathrm{INV}(B)$.

(b) if $\beta \leq \alpha$, then $\mathrm{INV}(A \times B) = (2, \beta, \delta, p, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where* $\delta = \max(\alpha, \gamma)$ *and*

$$p = \begin{cases} n & \text{if } \alpha < \gamma, \\ m & \text{if } \gamma < \alpha, \\ m+n & \text{if } \alpha = \gamma. \end{cases}$$

(v) If $\mathrm{INV}(A) = (1, \beta, \{r_A \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where A is normalized and* $\tau(A) = (\beta, \beta, 0)$ *and* $\mathrm{INV}(B) = (1, \gamma, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where B is normalized and* $\tau(B) = (\gamma, \gamma, 0)$*, then* $\mathrm{INV}(A \times B)$ *is given by Proposition 5.29.*

(vi) If $\mathrm{INV}(A) = (1, \alpha, \{r_A \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where A is normalized and* $\tau(A) = (\alpha, \alpha, 0)$ *and* $\mathrm{INV}(B) = (2, \beta, \gamma, n, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, then*

(a) if $\alpha < \beta$, then $A \times B \cong B$ and $\mathrm{INV}(A \times B) = \mathrm{INV}(B)$*;*

(b) if $\gamma < \alpha$, then $A \times B \cong A$ and $\mathrm{INV}(A \times B) = \mathrm{INV}(A)$*;*

(c) if $\beta < \alpha \leq \gamma$, then $\mathrm{INV}(A \times B) = (2, \alpha, \gamma, n, \{r_A \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*;*

(d) If $\alpha = \beta$, then $\mathrm{INV}(A \times B) = (2, \alpha, \gamma, n, \{r_{A \times B} \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*;*

(vii) If $\mathrm{INV}(A) = (2, \alpha, \gamma, n, \{r_A \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$ *and* $\mathrm{INV}(B) = (2, \beta, \delta, m, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, then*

(a) if $\gamma < \beta$ then $A \times B \cong B$ and $\mathrm{INV}(A \times B) = \mathrm{INV}(B)$*;*

(b) if $\alpha < \beta \leq \gamma$, then $\mathrm{INV}(A \times B) = (2, \beta, \varepsilon, p, \{r_B \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where ε and p are determined by Proposition 5.14;*

(c) If $\beta < \alpha$, then $\mathrm{INV}(A \times B) = (2, \alpha, \varepsilon, p, \{r_A \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where ε and p are determined by Proposition 5.14;*

(d) If $\beta = \alpha$, then $\mathrm{INV}(A \times B) = (2, \varphi, \varepsilon, p, \{r_{A \times B)} \circ f : f \in \mathrm{aut}(\mathrm{Fr}(\omega))\})$*, where φ, ε and p are determined by Proposition 5.14.*

Proof. (i): by Proposition 4.14.

(ii)(a): by Proposition 5.25.

(ii)(b): clear.

(iii): Let C be normalized and D be superatomic such that $B \cong C \times D$, $\tau(C) = (\beta, \beta, 0)$, and $\tau(D) = (\gamma, 0, m)$.

(a): Then $A \times C \cong C$ by Proposition 5.25, and hence $A \times B \cong B$.

(b): obvious.

(iii) Let C be normalized and D be superatomic such that $B \cong C \times D$, $\tau(C) = (\beta, \beta, 0)$, and $\tau(D) = (\gamma, 0, m)$.

(a): clear.

(b): clear.

(iv)(a): by Proposition 5.25

(iv)(b): by Proposition 4.14.

(v): by Proposition 5.26.

(vi)(a): by Proposition 5.25.

(vi)(b): by Propositions 5.24, 5.25.

(vi)(c): by Proposition 5.25.

(vi)(d): by Proposition 5.25.

(vii)(a): by Propositions 5.24, 5.25.

(vii)(b): by Proposition 5.25.

(vii)(c),(d): by Propositions 5.24, 5.25. □

6. More examples

Example 6.1. *If $|A| = 1$, then $\tau(A) = \tau_s(A) = (0, 0, 0)$.* □

Example 6.2. *If A is finite with $n > 0$ atoms, then $\tau(A) = \tau_s(A) = (0, 0, n)$.* □

Example 6.3. *If $A = \mathrm{Intalg}(\omega^\alpha \cdot n)$, with α a nonzero ordinal and n a nonzero natural number, then $\tau(A) = \tau_s(A) = (\alpha, 0, n)$.*

Proof. We claim that for all $\beta < \alpha$,

6.3(1) $\quad \mathrm{atoms}(A/I_\beta(A)) = \{[[\omega^\alpha \cdot i + \omega^\beta \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot (\xi + 1))]_{I_\beta(A)} : i < n, \xi < \omega^\alpha\}$.

6.3(2) $A/I_\beta(A)$ is generated by the increasing sequence

$$\begin{aligned}
&\{[[0, \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}^\frown \\
&[[0, \omega^\alpha + \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}^\frown \\
&\dots \\
&[[0, \omega^\alpha \cdot (n-1) + \omega^\beta \cdot \xi)]_{I_\beta(A)} : \xi < \omega^\alpha\}.
\end{aligned}$$

We prove these statements by induction on β. They are clear for $\beta = 0$. Assume them for β. Then they are clear for $\beta + 1$. Now assume that γ is limit $< \omega^\alpha$ and (1) and (2) hold for all $\beta < \gamma$. Then for any $i < n$ and $\xi < \omega^\alpha$, $[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi + 1))]_{I_\gamma(A)}$ is an atom of $A/I_\gamma(A)$. In fact, first note that each member of $I_\gamma(A)$ is a finite union of intervals $[\varphi, \psi)$ with $\psi - \varphi < \gamma$; hence $[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot (\xi + 1))]_{I_\gamma(A)} \neq 0$. Now suppose that $i < n$, $\xi < \omega^\alpha$, and $\delta < \omega^\gamma$. Choose $\beta < \gamma$ so that $\delta < \omega^\beta$. Say $\beta + \varepsilon = \gamma$. Then

$$\begin{aligned}
&[[\omega^\alpha \cdot i + \omega^\gamma \cdot \xi, \omega^\alpha \cdot i + \omega^\gamma \cdot \xi + \delta))]_{I_\gamma(A)} \\
&= [[\omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi + \delta)]_{I_\gamma(A)} \\
&\leq [[\omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi + \omega^\beta)]_{I_\gamma(A)} \\
&\leq [[\omega^\alpha \cdot i + \omega^\beta \cdot \omega^\varepsilon \cdot \xi, \omega^\alpha \cdot i + \omega^\beta \cdot (\omega^\varepsilon \cdot \xi + 1))]_{I_\gamma(A)} = 0
\end{aligned}$$

Now 6.3(1) and 6.3(2) follow for γ.

So 6.3(1) and 6.3(2) hold for all $\beta < \alpha$. It follows that $A/I_\alpha(A)$ is generated by

$$\{[[0, \omega^\alpha)]_{I_\alpha(A)}, \dots, [[\omega^\alpha \cdot (n-1), \infty)]_{I_\alpha(A)}\}.$$

So $A/I_\alpha(A)$ is finite with n atoms. Hence $\tau(A) = \tau_s(A) = (\alpha, 0, n)$. □

Example 6.4. *Let $L = \omega \times \mathbb{Q}$ ordered by second differences, and with a new zero $0'$ adjoined. Let $A = \mathrm{Intalg}_d(L)$. Then $\tau(A) = (1, 1, 0)$.*

Proof.

atoms$(A) = \{(\{n\}, q) : n \in \omega, q \in \mathbb{Q}\}$;
$I_1(A)$ is generated by $\{(\{n\}, q) : n \in \omega, q \in \mathbb{Q}\}$;
$A/I_1(A)$ is atomless;
$I_2(A) = I_1(A)$;
$I_1(A)/I_0(A)$ is not principal.
$\tau(A) = (1, 1, 0)$.
ar$(A) = 1$.
A is normal.
$g : A/I_{\mathrm{ar}(A)}(A) \to \mathrm{Intalg}(\mathbb{Q})$ is defined by $g([[0', (m, q)]]_{I_1(A)} = [0, q)$; with h an isomorphism of Intalg($\mathbb{Q}$ onto Fr(ω) we have INV$(A) = (1, 1, \{r_{A,h\circ g} \circ k : k \in \mathrm{Aut}(\mathrm{Fr}(\omega))\}$. □

Example 6.5. *Let $L = \omega^2 \times \mathbb{Q}$ ordered by second differences, and with a new zero $0'$ adjoined. Let $A = \mathrm{Intalg}(L)$. Then $\tau(A) = (2, 2, 0)$.*

Proof.

atoms$(A) = \{(\{\alpha\}, q) : \alpha < \omega_2, q \in \mathbb{Q}\}$;
$I_1(A)$ is generated by $\{(\{\alpha\}, q) : \alpha < \omega_2, q \in \mathbb{Q}\}$;
atoms$(A/I_1(A)) = \{[([\omega \cdot n, \sigma \cdot (n + 1)), q)]_{I_1(A)} : n \in \omega, q \in \mathbb{Q}\}$;
$I_2(A)$ is generated by $I_1(A) \cup \{([\omega \cdot n, \sigma \cdot (n + 1)), q) : n \in \omega, q \in \mathbb{Q}\}$;
$A/I_2(A)$ is atomless;
$I_3(A) = I_2(A)$;
$I_2(A)/I_1(A)$ is not principal;
$\tau(A) = (2, 2, 0)$.
ar$(A) = 2$.
A is normal.
$g : A/I_{\mathrm{ar}(A)}(A) \to \mathrm{Intalg}(\mathbb{Q})$ is defined by $g([[0', (\alpha, q)]]_{I_1(A)} = [0, q)$; with h an isomorphism of Intalg($\mathbb{Q}$ onto Fr(ω) we have INV$(A) = (2, 2, \{r_{A,h\circ g} \circ k : k \in \mathrm{Aut}(\mathrm{Fr}(\omega))\}$. □

Example 6.6. *Let $A = \mathrm{Intalg}(\omega^2) \times \mathrm{Intalg}(\mathbb{Q})$. Then $\tau(A) = (2, 0, 1)$.*

Proof.

atoms$(A) = \{(\{\alpha\}, 0) : \alpha < \omega^2\}$;
$I_1(A)$ is generated by $\{(\{\alpha\}, 0) : \alpha < \omega^2\}$;
atoms$(A/I_1(A)) = \{[([\omega \cdot n, \omega \cdot (n + 1)), 0)]_{I_1(A)} : n \in \omega\}$;
$I_2(A)$ is generated by $I_1(A) \cup \{([\omega \cdot n, \omega \cdot (n + 1)), 0) : n \in \omega\}$;
atoms$(A/I_2(A)) = \{[(1, 0)]_{I_2(A)}\}$;
$I_3(A)$ is generated by $I_2(A) \cup \{(1, 0)\}$;
$A/I_3(A)$ is atomless;
$I_4(A) = I_3(A)$
$I_3(A)/I_0(A)$ is generated by $[(1, 0)]_{I_0(A)}$;
$A/I_2(A)$ has the one atom $[(1.0)]_{I_2(A)}$.
$\tau(A) = (2, 0, 1)$. □

Example 6.7. *Let $A = \mathrm{Intalg}(\omega) \times \mathrm{Intalg}(\omega^2 \times \mathbb{Q})$. Then $\tau(A) = (2, 2, 0)$.*

Proof.

atoms$(A) = \{(\{m\}, 0) : m \in \omega\} \cup \{(0, (\{\alpha\}, q)) : \alpha < \omega^2, q \in \mathbb{Q}\}$;
$I_1(A)$ is generated by $\{(\{m\}, 0) : m \in \omega\} \cup \{(0, (\{\alpha\}, q)) : \alpha < \omega^2, q \in \mathbb{Q}\}$;
atoms$(A/I_1(A)) = \{[(1, 0)]_{I_1(A)}\} \cup \{[(0, ([\omega \cdot n, \omega \cdot (n + 1)), q)]_{I_1(A)} : n \in \omega, q \in \mathbb{Q}\}$;
$I_2(A)$ is generated by $I_1(A) \cup \{(1, 0)\} \cup \{(0, ([\omega \cdot n, \omega \cdot (n + 1)), q) : n \in \omega, q \in \mathbb{Q}\}$;
$A/I_2(A)$ is atomless;
$I_3(A) = I_2(A)$;
$I_2(A)/I_0(A)$ and $I_2/I_1(A)$ are not principal;
$\tau(A) = (2, 2, 0)$. □

Example 6.8. *Let* $A = \mathrm{Intalg}(\omega^2) \times \mathrm{Intalg}(\omega^2 \times \mathbb{Q})$. *Then* $\tau(A) = (2, 2, 1)$.

Proof.

atoms$(A) = \{(\{\alpha\}, 0) : \alpha < \omega^2\} \cup \{(0, (\{\alpha\}, q) : \alpha < \omega^2, q \in \mathbb{Q}\}$;
$I_1(A)$ is generated by $\{(\{\alpha\}, 0) : \alpha < \omega^2\} \cup \{(0, \{(\alpha, q)\}) : \alpha < \omega^2, q \in \mathbb{Q}\}$;
atoms$(A/I_1(A)) = \{[[\omega \cdot n, \omega \cdot (n+1)), 0)]_{I_1(A)} : n \in \omega\} \cup \{[(0, (0, [\omega \cdot n, \omega \cdot (n+1)), q))]_{I_1(A)} : n \in \omega, q \in \mathbb{Q}\}$;
$I_2(A)$ is generated by $I_1(A) \cup \{[\omega \cdot n, \omega \cdot (n+1)), 0) : n \in \omega\} \cup \{(0, (0, [\omega \cdot n, \omega \cdot (n+1)), q)) : n \in \omega, q \in \mathbb{Q}\}$;
atoms$(A/I_2(A)) = \{[(1, 0)]_{I_2(A)}\}$;
$I_3(A)$ is generated by $I_2(A) \cup \{(1, 0)\}$;
$A/I_3(A)$ is atomless;
$I_4(A) = I_3(A)$;
$I_3(A)/I_0(A)$ is not principal; $I_3(A)/I_1(A)$ is not principal; $I_3(A)/I_2(A)$ is the principal ideal determined by $[(1, 0)]_{I_2(A)}$;
$\tau(A) = (2, 2, 1)$. □

Example 6.9. *Let* $L = (\omega \times \mathbb{Q}) + (\omega^2 \times \mathbb{Q})$ *with a new zero* $0'$ *adjoined,* $A = \mathrm{Intalg}(L)$. $\tau(A) = (2, 2, 0)$.

atoms$(A) = \{(\{m\}, q) : q \in \mathbb{Q}\} \cup \{(\{\alpha\}, q) : \alpha < \omega^2, q \in \mathbb{Q}\}$.
$I_1(A)$ is generated by $\{(\{m\}, q) : q \in \mathbb{Q}\} \cup \{(\{\alpha\}, q) : \alpha < \omega^2, q \in \mathbb{Q}\}$.
atoms$(A/I_1(A)) = \{[([\omega \cdot n, \omega \cdot (n + 1), q))]_{I_1(A)} : n \in \omega, q \in \mathbb{Q}\}$.
$I_2(A)$ is generated by $I_1(A) \cup \{[\omega \cdot n, \omega \cdot (n + 1), q) : n \in \omega, q \in \mathbb{Q}\}$.
$A/I_2(A)$ is atomless.
$I_3(A) = I_2(A)$.
$\varphi(A) = (2, 2, 0)$. □

II. Ketonen's theorem

7. Monoids and measures

An *m-monoid* is an algebra $(M, +, 0)$ such that $+$ is a commutative and associative binary operation on M, $0 \in M$, $\forall a \in M[a + 0 = a]$ and $\forall a, b \in M[a + b = 0 \to a = b = 0]$.

For any m-monoid $(M, +, 0)$ we set $M^* = M \backslash \{0\}$. Note that M^* is closed under $+$, so that $(M^*, +)$ is a commutative semigroup.

A *morphism* of an m-monoid $(M,+,0)$ to an m-monoid $(M',+',0')$ is a function $f: M \to M'$ such that $\forall x,y \in M[f(x+y) = f(x)+f(y)]$, $f(0)=0'$, and $f[M^*] \subseteq M'^*$.

Proposition 7.1. *If L is a linear order with smallest element 0, then $(L, \max, 0)$ is an m-monoid, where $\max(a,b)$ is the maximum of a and b.* □

$\mathscr{W}$ is ω_1 with a new smallest element o adjoined. So $(\mathscr{W}, \max, o)$ is an m-monoid.

If A is a BA and M is an m-monoid, then an *M-measure* on A is a mapping $\sigma: A \to M$ such that

7.1(1) $\sigma(x+y) = \sigma(x)+\sigma(y)$ for each pair (x,y) of disjoint elements of A.

7.1(2) $\forall x \in A[\sigma(x)=0 \text{ iff } x=0]$.

The set of all M-measures on $\mathrm{Fr}(\omega)$ will be denoted by $\mathscr{M}(M)$. $\mathscr{M}$ is $\mathscr{M}(\mathscr{W})$.

Proposition 7.2. *If M, N are m-monoids, $\Phi: M \to N$ is a morphism, and $\sigma \in \mathscr{M}(M)$, then $\Phi \circ \sigma \in \mathscr{M}(N)$.*

Proof. Clearly $\Phi \circ \sigma : \mathrm{Fr}(\omega) \to N$. If x,y are disjoint elements of $\mathrm{Fr}(\omega)$, then

$$\Phi(\sigma(x+y)) = \Phi(\sigma(x)+\sigma(y)) = \Phi(\sigma(x)) + \Phi(\sigma(y)).$$

Also, if $x \in \mathrm{Fr}(\omega)$, then

$$\Phi(\sigma(x)) = 0 \quad \text{iff} \quad \sigma(x) = 0 \quad \text{iff} \quad x = 0.$$ □

Proposition 7.3. *If k is an automorphism of $\mathrm{Fr}(\omega)$ and $\sigma \in \mathscr{M}(M)$, then $\sigma \circ k \in \mathscr{M}(M)$.*

Proof. Assume that k is an automorphism of $\mathrm{Fr}(\omega)$ and $\sigma \in \mathscr{M}(M)$. If x,y are disjoint elements of $\mathrm{Fr}(\omega)$, then $k(x)$ and $k(y)$ are disjoint, and

$$\sigma(k(x+y)) = \sigma(k(x)+k(y)) = \sigma(k(x)) + \sigma(k(y)).$$

For any $x \in \mathrm{Fr}(\omega)$,

$$\sigma(k(x)) = 0 \quad \text{iff} \quad k(x) = 0 \quad \text{iff} \quad x = 0.$$ □

8. Derived monoids

For any set M we denote by ${}^{<<\omega}M$ the set of all finite nonempty sequences of members of M. If $(M,+,0)$ is an m-monoid, then for any $m \in \omega\backslash\{0\}$ the operation $+$ on ${}^mM$ is defined coordinatewise. We define the *trace map* $T: {}^{<<\omega}M \to M$ by

$$T(a) = \sum_{i<m} a_i \quad \text{where } a \in {}^mM \text{ with } m \in \omega\backslash\{0\}.$$

If M is an m-monoid, $m,n \in \omega\backslash\{0\}$, $a \in {}^mM$, and $b \in {}^nM$, then a is a *refinement* of b, in symbols $a \prec b$, iff there is a mapping $\lambda: m \to n$ such that $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. Note that if $a \prec b$ with m,n,λ as indicated, then $\sum_{j<n} b_j = \sum_{i<m} a_i$.

Proposition 8.1. *$\prec$ is reflexive.*

Proof. Suppose that $a \in {}^mM$. Let $\lambda(i) = i$ for all $i < m$. Then for all $j < m$, $a_j = \sum\{a_i : \lambda(i) = j\}$. $\square$

Proposition 8.2. *$\prec$ is transitive.*

Proof. Suppose that $a \prec b \prec c$. Say $a \in {}^mM$, $b \in {}^nM$, and $c \in {}^pM$. Then there exist $\lambda : m \to n$ and $\mu : n \to p$ such that $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$ and $\forall k < p[c_k = \sum\{b_j : j < n, \mu(j) = k\}]$. Then for all $k < p$,

$$\begin{aligned} c_k &= \sum\{b_j : j < n, \mu(j) = k\} = \sum\left\{\sum\{a_i : i < m, \lambda(i) = j\} : j < n, \mu(j) = k\right\} \\ &= \sum\{a_i : i < m : \mu(\lambda(i)) = k\}. \end{aligned}$$ $\square$

Proposition 8.3. *If M is a m-monoid, $m \in \omega\backslash\{0\}$, $a \in {}^mM$, and λ is a permutation of m, let $b = a \circ \lambda = \langle a_{\lambda(0)}, a_{\lambda(1)}, \ldots, a_{\lambda(m-1)}\rangle$. Then $a \prec b \prec a$.*

Proof. $\forall j < n[b_j = a_{\lambda(j)} = \sum\{a_i : i < n, \lambda^{-1}(i) = j\}]$ and $\forall j < n[a_j = b_{\lambda^{-1}(j)} = \sum\{b_i : i < n, \lambda(i) = j\}]$. $\square$

Proposition 8.4. *If M is a m-monoid, $m \in \omega\backslash\{0\}$, and $a \in {}^mM$, then $a^\frown\langle 0\rangle = \langle a_0, \ldots, a_{m-1}, 0\rangle$ and $a \prec a^\frown\langle 0\rangle \prec a$.*

Proof. Clearly $a^\frown\langle 0\rangle = \langle a_0, \ldots, a_{m-1}, 0\rangle$. Now let $b = a^\frown\langle 0\rangle$. Let $\lambda : m \to m+1$ be the inclusion map. Then $\forall j < m+1[b_j = \sum\{a_i : i < n, \lambda(i) = j\}]$. So $a \prec b$. Now let $\mu : m+1 \to m$ be such that $\mu \restriction m$ is the identity and $\mu(m) = m-1$. Then $\forall j < m[a_j = \sum\{b_i : i < m+1, \mu(i) = j\}]$, so $b \prec a$. $\square$

Proposition 8.5. *If M is a m-monoid, $m, n \in \omega\backslash\{0\}$, $a \in {}^mM$, and $b \in {}^nM$, and $a \prec b$, then $T(a) = T(b)$.* $\square$

Let $f : \omega \times \omega \to \omega$ be a bijection. Let $(M, +, 0)$ be an m-monoid. The *derived monoid* $\triangle M$ of M is the set of all $\alpha \in [{}^{<<\omega}M]^\omega$ satisfying the following conditions:

(i) (*Collection property, (C.P.)*): For all $a, b \in {}^{<<\omega}M$, if $a \in \alpha$ and $a \prec b$, then $b \in \alpha$.

(ii) (*Refinement property, (R.P.)*): For all $m, n \in \omega\backslash\{0\}$ and all $a \in {}^mM$ and $b \in {}^nM$, if $a, b \in \alpha$ then there is a $c \in {}^{m\cdot n}M$ such that $c \in \alpha$, $\forall i < m[a_i = \sum\{c_{f(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum\{c_{f(i,j)} : i < m\}]$.

(iii) (*Splitting property, (S.P.)*) $\forall m \in \omega\backslash\{0\}$ and all $a \in {}^mM$, if $a \in \alpha$ and $a_0 \neq o$, then there exist $b, c \in M\backslash\{0\}$ such that $a_0 = b + c$ and $\langle b, c, a_1, \ldots, a_{m-1}\rangle \in \alpha$.

For any $\alpha, \beta \in \triangle M$ let

$$\alpha + \beta = \{a + b : a \in \alpha, b \in \beta, \mathrm{dmn}(a) = \mathrm{dmn}(b)\}.$$

Also, let $O = \{\langle 0\rangle, \langle 0, 0\rangle, \ldots\}$.

Proposition 8.6. *If $\alpha \in \triangle M$, then for each $m \in \omega\backslash\{0\}$, α has a member of length m.*

Proof. α is infinite, so fix $a \in \alpha$; say a has length m. Let $\lambda : m \to 1$ be the obvious function. Let $b = \{(0, \sum\{a_i : i < m\})\}$. Then $b_0 = \sum\{a_i : i < m, \lambda(i) = 0\}$, so $a \prec b$. Hence by the C.P., $b \in \alpha$. Thus α has an element of length 1. By Proposition 8.4 and the C.P. it has elements of each positive length. $\square$

Proposition 8.7. *If $\alpha, \beta \in \triangle M$, then $\alpha + \beta \in \triangle M$. Also $O \in \triangle M$. Moreover, $(\triangle M, +, O)$ is an m-monoid.*

Proof. By Proposition 8.6, $\alpha + \beta \in [^{<<\omega}M]^{\omega}$. Now to check C.P. for $\alpha + \beta$, suppose that $a, b \in {}^{<<\omega}M$, $a \in \alpha + \beta$, and $a \prec b$. Say $a = c + d$ and $m \in \omega\backslash\{0\}$ with $a, c, d \in {}^{m}M$, Say $n \in \omega\backslash\{0\}$, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. Define c' and d' with domain n by setting, for $j < n$, $c'_j = \sum\{c_i : i < m, \lambda(i) = j\}$ and $d'_j = \sum\{d_i : i < m, \lambda(i) = j\}$. Then $c \prec c'$ and $d \prec d'$, so by C.P. for α and β, $c' \in \alpha$ and $d' \in \beta$. Clearly $b = c' + d'$, so $b \in \alpha + \beta$. This proves C.P. for $\alpha + \beta$.

For R.P., suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^{m}M$, $b \in {}^{n}M$, and $a, b \in \alpha + \beta$. Say $c, d \in {}^{m}M$, $e, f \in {}^{n}M$, $c, e \in \alpha$, $d, f \in \beta$, $a = c + d$, and $b = e + f$. Choose $g, h \in {}^{m \cdot n}M$ such that $g \in \alpha$, $h \in \beta$, $\forall i < m[c_i = \sum\{g_{f(i,j)} : j < n\}]$, $\forall j < n[e_j = \sum\{g_{f(i,j)} : i < m\}]$, $\forall i < m[d_i = \sum\{h_{f(i,j)} : j < n\}]$, and $\forall j < n[f_j = \sum\{h_{f(i,j)} : i < m\}]$. Let $k = g + h$. Then $k \in {}^{m \cdot n}M$, $k \in \alpha + \beta$,

$$\forall i < m\big[a_i = c_i + d_i = \sum\{k_{f(i,j)} : j < n\} \text{ and}$$
$$\forall j < n[b_j = e_j + f_j = \sum\{k_{f(i,j)} : i < m\}\big].$$

This proves R.P. for $\alpha + \beta$.

For S.P., suppose that $m \in \omega\backslash\{0\}$, $a \in {}^{m}M$, $a \in \alpha + \beta$, and $a_0 \neq 0$. Say $a = b + c$ with $b \in \alpha$ and $c \in \beta$.

Case 1. $b_0 = 0 \neq c_0$. Choose $d, e \in M\backslash\{0\}$ such that $c_0 = d + e$ and

$$\langle d, e, c_1, c_2, \ldots c_{m-1}\rangle \in \beta.$$

Also, let $f = \langle 0, 0, b_1, b_2, \ldots b_{m-1}\rangle$. Define $\lambda : m \to m + 1$ by $\lambda(i) = i + 1$ for all $i < m$. Then $f_0 = 0 = \sum\{b_i : \lambda(i) = 0\}$, $f_1 = 0 = \sum\{b_i : i < n, \lambda(i) = 1\} = b_0$, and for $j \geq 2$, $f_j = b_{j-1} = \sum\{b_i : i < n, \lambda(i) = j\}$. So $b \prec f$, hence $f \in \alpha$. Clearly $a_0 = d + e$ and

$$\langle d, e, a_1, a_2, \ldots, a_{m-1}\rangle = \langle d, e, c_1, c_2, \ldots c_{m-1}\rangle + f;$$

hence $\langle d, e, a_1, a_2, \ldots, a_{m-1}\rangle \in \alpha + \beta$.

Case 2. $b_0 \neq 0 = c_0$. This is symmetric to Case 1.

Case 3. $b_0 \neq 0 \neq c_0$. Choose $d, e \in M\backslash\{0\}$ such that $c_0 = d + e$ and

$$\langle d, e, c_1, c_2, \ldots c_{m-1}\rangle \in \beta$$

and choose $d', e' \in M\backslash\{0\}$ such that $b_0 = d' + e'$ and $\langle d', e', b_1, b_2, \ldots b_{m-1}\rangle \in \alpha$. Then $a_0 = b_0 + c_0 = d + e + d' + e'$ and

$$\langle d + d', e + e', a_1, a_2, \ldots, a_{m-1}\rangle \in \alpha + \beta.$$

This proves S.P. for $\alpha + \beta$.

Hence $\alpha + \beta \in \triangle M$. Clearly $O \in \triangle M$. Clearly then $(\triangle M, +, O)$ is an m-monoid. $\square$

Proposition 8.8. *If $a \in \alpha \in \triangle M$, with $a \in {}^m M$, and if λ is a permutation of m, then $a \circ \lambda \in \alpha$.*

Proof. By Proposition 8.3 and C.P. $\square$

Proposition 8.9. *$\forall m \in \omega \backslash \{0\}$, all $a \in {}^m M$, and all $i < m$, if $a \in \alpha$ and $a_i \neq o$, then there exist $b, c \in M \backslash \{0\}$ such that $a_i = b + c$ and $\langle a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{m-1} \rangle \in \alpha$.*

Proof. let $\lambda : m \to m$ be the permutation $(i, 0)$. Then

$$\langle a_i, a_1, \ldots, a_{i-1}, a_0, a_{i+1}, \ldots, a_{m-1} \rangle \in \alpha$$

by Proposition 8.8. By S.P. let $b, c \in M \backslash \{0\}$ be such that $a_i = b + c$ and

$$\langle b, c, a_1, \ldots, a_{i-1}, a_0, a_{i+1}, \ldots, a_{m-1} \rangle \in \alpha.$$

Let $d = \langle b, c, a_1, \ldots, a_{i-1}, a_0, a_{i+1}, \ldots, a_{m-1} \rangle$, and let λ be the permutation of $m + 1$ such that

$$\begin{aligned} &\lambda(0) = i + 1, \lambda(1) = 2, \ldots, \lambda(i - 1) = i, \\ &\lambda(i) = 0, \lambda(i + 1) = 1, \lambda(i + 2) = i + 2, \ldots, \lambda(m) = m. \end{aligned}$$

Then

$$d \circ \lambda = \langle d_{i+1}, d_2, \ldots, d_{i-1}, b, c, d_{i+1}, \ldots, d_m \rangle = \langle a_0, a_1, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{m-1} \rangle.$$

Since $d \in \alpha$, also $d \circ \lambda \in \alpha$ by Proposition 8.8. $\square$

Proposition 8.10. *If α is a countable subset of ${}^{<<\omega} M$ satistying R.P. and $a, b \in \alpha$, then $T(a) = T(b)$.*

Proof. Say $a \in {}^m M$ and $b \in {}^n M$. Let $c \in \alpha$ be obtained by R.P.; thus $\forall i < m[a_i = \sum\{c_{f(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum\{c_{f(i,j)} : i < m\}$. We claim that $a \prec c$. Let $\lambda(f(i, j)) = i$ for all $i < m, j < n$. Then

$$\forall i < m[a_i = \sum\{c_{f(i,j)} : j < n\} = \sum\{c_k : k < m \cdot n, \lambda(k) = i\}.$$

So $a \prec c$. By symmetry, $b \prec c$. Hence by Proposition 8.5, $T(a) = T(c) = T(b)$. $\square$

Now for any $\alpha \in \triangle M$ we let $T(\alpha) = T(a)$ for any $a \in \alpha$; this is justified by Proposition 8.10.

Proposition 8.11. *$T : \triangle M \to M$ is a morphism of m-monoids.*

Proof. For preservation of $+$, let $\alpha, \beta \in \triangle M$. Choose $a \in \alpha$ and $b \in \beta$ with $\mathrm{dmn}(a) = \mathrm{dmn}(b)$. Then $(a+b) \in (\alpha+\beta)$, so $T(\alpha+\beta) = T(a+b) = T(a)+T(b) = T(\alpha)+T(\beta)$.

Clearly $T(O) = 0$. If $\alpha \neq O$, choose $a \in \alpha$ with $\mathrm{dmn}(a) = 1$. Then $a_0 \neq 0$, so $T(\alpha) = T(a) \neq 0$. □

Proposition 8.12. *If α is a countable subset of ${}^{<<\omega}M$ satisfying C.P. and R.P., then $\langle T(\alpha)\rangle \in \alpha$.*

Proof. Take any $a \in \alpha$; say a has domain m. Let $\lambda : m \to 1$ be the obvious function. Let $b = \{(0, \sum\{a_i : i < m\})\}$. Then $b_0 = \sum\{a_i : i < m, \lambda(i) = 0\}$, so $a \prec b$. Hence by the C.P., $b \in \alpha$. □

9. The derived monoid of $\mathscr{W}$

Recall that $\mathscr{W}$ is ω_1 with a new smallest element o adjoined; it is an m-monoid.

For each $\alpha \in (\triangle\mathscr{W})^*$ define $\theta_\alpha : \omega_1 \to \omega+1$ by setting, for each $\zeta < \omega_1$,

$$\theta_\alpha(\zeta) = lub\{|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| : a \in \alpha\}.$$

Proposition 9.1. *If $\alpha \in (\triangle\mathscr{W})^*$, and $a \in \alpha$ with $a \in {}^m\mathscr{W}$, then $T(\alpha) = \max\{a_i : i < m, a_i \neq o, a \in \alpha\}$, a countable ordinal.* □

Proposition 9.2. *Suppose that $\alpha \in (\triangle\mathscr{W})^*$ and $T(\alpha) = \eta$. Then*

(i) $\theta_\alpha(\zeta) = 0$ for all $\zeta > \eta$.

(ii) $0 < \theta_\alpha(\eta) \leq \omega$.

(iii) If $\xi = \min\{\zeta < \omega_1 : \theta_\alpha(\zeta) \neq 0\}$, then

(a) $\xi \leq \eta$.

(b) For each $n \in \omega$ there is an $a \in \alpha$ such that a has at least n entries equal to ξ.

(c) $\theta_\alpha(\xi) = \omega$.

(d) $\forall\zeta < \xi[\theta_\alpha(\zeta) = 0]$.

Proof. (i), (ii), and (iii)(a) are clear. We prove (iii)(b) by induction on n. It is clear for $n = 0, 1$. Now assume it for n. Say $a \in \alpha$ and a has at least n entries equal to ξ. Say $a \in {}^m\mathscr{W}$, $i < m$, and $a_i = \xi$. By Proposition 8.10 there are $b, c \in \mathscr{W}^*$ such that $\xi = a_i = b+c$ and $\langle a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{m-1}\rangle \in \alpha$. Since $\max(b,c) = \xi$ and by the minimality of ξ both $\xi \leq b$ and $\xi \leq c$, we have $b = c = \xi$. Thus $\langle a_0, \ldots, a_{i-1}, b, c, a_{i+1}, \ldots, a_{m-1}\rangle$ has one more ξ than a, and hence it has at least $n+1$ entries equal to ξ. This proves (iii)(b). (iii)(c) follows from (iii)(b). (iii)(d) is obvious. □

Let $\mathcal{N}^*$ be the set of all mappings $\theta : \omega_1 \to \omega+1$ such that there exist $\xi \leq \eta$ ordinals in ω_1 such that $\theta(\zeta) = 0$ for $\zeta < \xi$ and $\eta < \zeta < \omega_1$, with $\theta(\xi) = \omega$ and $1 \leq \theta(\eta) \leq \omega$.
Let $\mathcal{N} = \mathcal{N}^*$ together with a new element o. We define $+$ on $\mathcal{N}$ by setting, for $\theta, \psi \in \mathcal{N}^*$, $o + o = o$; $\theta + o = \theta = \theta + o$; $(\theta+\psi)(\zeta) = \theta(\zeta) + \psi(\zeta)$ for each $\zeta < \omega_1$.

Proposition 9.3. *Suppose that $\alpha \in \triangle M$, $m, n \in \omega\backslash\{0\}$, $a \in {}^mM$, $b \in {}^nM$, $a, b \in \alpha$, λ and μ are permutations of m, n respectively, $c \in {}^{m\cdot n}M$, $c \in \alpha$, $\forall i < m[a_{\lambda(i)} = \sum\{c_{f(i,j)} :$*

$j < n\}]$, $\forall j < n[b_{\mu(j)} = \sum\{c_{f(i,j)} : i < m\}]$, $d \in {}^{m\cdot n}M$, *and* $\forall i < m \forall j < n[d_{f(i,j)} = c_{f(\lambda^{-1}(i),\mu^{-1}(j))}]$.

Then $d \in \alpha$, $\forall i < m[a_i = \sum\{d_{f(i,j)} : j < n\}]$, *and* $\forall j < n[b_j = \sum\{d_{f(i,j)} : i < m\}]$.

Proof. Let ρ be the permutation of $m \times n$ such that $\rho(f(i,j)) = f(\lambda^{-1}(i), \mu^{-1}(j))$ for all $i < m$ and $j < n$. Then $d = c \circ \rho$. Hence $d \in \alpha$ by Proposition 8.8.

Now take any $i < m$. Then

$$\begin{aligned} a_i = a_{\lambda(\lambda^{-1}(i))} &= \sum\{c_{f(\lambda^{-1}(i),j)} : j < n\} \\ &= \sum\{c_{f(\lambda^{-1}(i),\mu^{-1}(j))} : j < n\} = \sum\{d_{f(i,j)} : j < n\}. \end{aligned}$$

By symmetry, $b_j = \sum\{d_{f(i,j)} : i < m\}$ for all $j < n$. □

Proposition 9.4. *Suppose that* $\alpha \in \triangle M$, $m, n \in \omega\backslash\{0\}$, $a \in {}^mM$, $b \in {}^nM$, $m < n$, $a' \in {}^nM$, $a \subseteq a'$, $a, a', b \in \alpha$, $\forall i \in n\backslash m[a'_i = o]$, $c \in {}^{n\cdot n}M$, $c \in \alpha$, $\forall i < n[a'_i = \sum\{c_{f_{nn}(i,j)} : j < n\}]$, $\forall j < n[b_j = \sum\{c_{f_{nn}(i,j)} : i < n\}$, $d \in {}^{m\cdot n}M$, *and* $\forall i < m \forall j < n[d_{f_{mn}(i,j)} = c_{f_{nn}(i,j)}]$.

Then $d \in \alpha$, $\forall i < m[a_i = \sum\{d_{f_{mn}(i,j)} : j < n\}]$ *and* $\forall j < n[b_j = \sum\{d_{f_{mn}(i,j)} : i < m\}]$.

Proof. Define $\lambda : n \cdot n \to m \cdot n$ by setting, for any $i < n$ and $j < n$,

$$\lambda(f_{nn}(i,j)) = \begin{cases} f_{mn}(i,j) & \text{if } i < m, \\ f_{mn}(m-1,j) & \text{if } m \le i < n. \end{cases}$$

Now $\forall i \in n\backslash m[o = a'_i = \sum\{c_{f_{nn}(i,j)} : j < n\}$ and hence

$(*)$ $\forall i \in n\backslash m \forall j < n[c_{f_{nn}(i,j)} = o]$.

Now $\forall i < m \forall j < n \forall k < n \forall l < n[\lambda(f_{nn}(k,l)) = f_{mn}(i,j)$ iff $(k = i$ and $l = j)$ or $(m \le k < n$ and $i = m-1$ and $l = j)]$. So if $i < m-1$ and $j < n$ then

$$\sum\{c_{f_{nn}(k,l)} : \lambda(f_{nn}(k,l)) = f_{mn}(i,j)\} = c_{f_{nn}(i,j)} = d_{f_{mn}(i,j)},$$

while if $i = m-1$ and $j < n$ then

$$\begin{aligned} &\sum\{c_{f_{nn}(k,l)} : \lambda(f_{nn}(k,l)) = f_{mn}(i,j)\} \\ &= c_{f_{nn}(i,j)} + \sum\{c_{f_{nn}(k,l)} : k \in n\backslash m, j < n\} = c_{f_{nn}(i,j)} = d_{f_{mn}(i,j)}. \end{aligned}$$

Hence $c \prec d$, so $d \in \alpha$ by C.P. Next, for any $i < m$, $a_i = a'_i = \sum\{c_{f_{nn}(i,j)} : j < n\} = \sum\{d_{f_{mn}(i,j)} : j < n\}$. By $(*)$, for all $j < n$, $b_j = \sum\{c_{f_{nn}(i,j)} : i < n\} = \sum\{c_{f_{nn}(i,j)} : i < m\} = \sum\{d_{f_{mn}(i,j)} : i < m\}$. □

Note by symmetry that the modification of Proposition 9.4 by replacing $m < n$ by $n < m$ also holds.

Proposition 9.5. *The following statement implies R.P.:*

For all $\alpha \in \triangle\mathscr{W}$, all $m \in \omega\backslash\{0\}$ and for all $a, b \in {}^{m}\mathscr{W}$, if $a, b \in \alpha$, $a_0 \geq a_1 \geq \cdots \geq a_{m-1}$ and $b_0 \geq b_1 \geq \cdots \geq b_{m-1}$, then there is a $c \in {}^{m\cdot m}\mathscr{W}$ such that $c \in \alpha$ and $\forall i < m[a_i = \sum\{c_{f_{mm}(i,j)} : j < m\}]$ and $\forall j < m[b_j = \sum\{c_{f_{mm}(i,j)} : i < m\}]$.

Proof. Assume the indicated statement, and suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^{m}\mathscr{W}$, $b \in {}^{n}\mathscr{W}$, and $a, b \in \alpha$. Let λ be a permutation of m and μ a permutation of n, such that $a_{\lambda(0)} \geq a_{\lambda(1)} \geq \cdots \geq a_{\lambda(m-1)}$ and $b_{\mu(0)} \geq b_{\mu(1)} \geq \cdots \geq b_{\mu(n-1)}$. Let $a' = a \circ \lambda$ and $b' = b \circ \mu$. Wlog $m < n$. Let $a'' \in {}^{n}\mathscr{W}$ be such that $a' \subseteq a''$ and $\forall i \in n\backslash m[a''_i = o]$. Then by the assumed statement, there is a $c \in {}^{n\cdot n}\mathscr{W}$ such that $c \in \alpha$ and $\forall i < n[a''_i = \sum\{c_{f_{nn}(i,j)} : j < n\}]$ and $\forall j < n[b'_j = \sum\{c_{f_{nn}(i,j)} : i < n\}]$. Define $d \in {}^{m\cdot n}\mathscr{W}$ by setting $d_{f_{mn}(i,j)} = c_{f_{nn}(i,j)}$ for all $i < m$ and $j < n$, Then by Proposition 9.4, $d \in \alpha$, $\forall i < m[a'_i = \sum\{d_{f_{mn}(i,j)} : j < n\}]$ and $\forall j < n[b'_j = \sum\{d_{f_{mn}(i,j)} : i < m\}]$. Now define $e \in {}^{m\times n}\mathscr{W}$ by setting, for $i < m$ and $j < n$, $e_{f_{mn}(i,j)} = d_{f_{mn}(\lambda^{-1}(i),\mu^{-1}(j))}$. Now $\forall i < m[a_{\lambda(i)} = a'_i = \sum\{d_{f_{mn}(i,j)} : j < n\}]$ and $\forall j < n[b_{\mu(j)} = b'_j = \sum\{d_{f_{mn}(i,j)} : i < m\}]$. Hence by Proposition 9.3, $e \in \alpha$, $\forall i < m[a_i = \sum\{e_{f_{mn}(i,j)} : j < n\}]$, and $\forall j < n[b_j = \sum\{e_{f_{mn}(i,j)} : i < m\}]$. $\square$

Proposition 9.6. *Let $f(O) = o$, and for any $\alpha \in (\triangle\mathscr{W})^*$ let $f(\alpha) = \theta_\alpha$. Then f is an isomorphism of $(\triangle\mathscr{W}, +, O)$ onto $(\mathcal{N}, +, o)$.*

Proof. By Proposition 9.2, f maps $\triangle\mathscr{W}$ into $\mathcal{N}$. Now we check that f preserves $+$. Suppose that $x, y \in \triangle\mathscr{W}$.

Case 1. $x = y = O$. Then $f(x + y) = f(O + O) = f(O) = o = o + o = f(x) + f(y)$.

Case 2. $x = \alpha \in (\triangle\mathscr{W})^*$ and $y = o$. Then $f(x+y) = f(\alpha) = \theta_\alpha = \theta_\alpha + o = f(x)+f(y)$.

Case 3. $x = o$ and $y = \beta \in (\triangle\mathscr{W})^*$. Symmetric to Case 2.

Case 4. $x = \alpha \in (\triangle\mathscr{W})^*$ and $y = \beta \in (\triangle\mathscr{W})^*$. Then $f(x + y) = f(\alpha + \beta) = \theta_{\alpha+\beta}$, $f(x) = f(\alpha) = \theta_\alpha$, and $f(y) = f(\beta) = \theta_\beta$. So we want to show that $\theta_{\alpha+\beta} = \theta_\alpha + \theta_\beta$. Take any $\zeta < \omega_1$. Then if $a \in \alpha$, $b \in \beta$, and $\mathrm{dmn}(a) = \mathrm{dmn}(b)$, then

$$\{i \in \mathrm{dmn}(a) : a_i + b_i = \zeta\} \subseteq \{i \in \mathrm{dmn}(a) : a_i = \zeta\} \cup \{i \in \mathrm{dmn}(b) : b_i = \zeta\},$$

and so $\theta_{\alpha+\beta}(\zeta) \leq \theta_\alpha(\zeta) + \theta_\beta(\zeta)$. On the other hand, if $a \in \alpha$ and $b \in \beta$, then $a^\frown c \in \alpha$ and $d^\frown b \in \beta$, where c is the sequence of o's with domain $\mathrm{dmn}(b)$ and d is the sequence of o's with domain $\mathrm{dmn}(a)$. Then $(a^\frown c) + (d^\frown b) \in \alpha + \beta$, and

$$\begin{aligned}|\{i \in \mathrm{dmn}(a) + \mathrm{dmn}(b) : ((a^\frown c) + (d^\frown b))_i = \zeta\}| \\ = |\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| + |\{i \in \mathrm{dmn}(b) : b_i = \zeta\}|.\end{aligned}$$

It follows that $\theta_\alpha(\zeta) + \theta_\beta(\zeta) \leq \theta_{\alpha+\beta}(\zeta)$.

So f preserves $+$.

Now we define a function g with domain $\mathcal{N}$ which will turn out to be the inverse of f. Let $g(o) = O$. Now suppose that $\theta \in \mathcal{N}^*$. Say $\xi \leq \eta$ are ordinals in ω_1 such that $\theta(\zeta) = 0$ for $\zeta < \xi$ and $\eta < \zeta < \omega_1$, with $\theta(\xi) = \omega$ and $1 \leq \theta(\eta) \leq \omega$. Let

$$g(\theta) = \alpha = \{a \in {}^{<<\omega}\mathscr{W} : T(a) = \eta \text{ and } \forall\zeta < \omega_1[|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| \leq \theta(\zeta)]\}.$$

Now $\langle\eta\rangle$, $\langle\eta, o\rangle$, $\langle\eta, o, o\rangle$, $\ldots \in \alpha$, so α is infinite. Each member of α is a finite sequence of members of $\{o\} \cup (\eta + 1)$, and so α is countable. Now we check that $\alpha \in \triangle\mathscr{W}$.

C.P.: Suppose that $a \in \alpha$, $b \in {}^{<<\omega}\mathscr{W}$, and $a \prec b$. Say $\mathrm{dmn}(a) = m$, $\mathrm{dmn}(b) = n$, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. Then $T(b) = T(a) = \eta$. For any $\zeta < \eta$ we have $|\{j < n : b_j = \zeta\}| \leq |\{i < m : a_i = \zeta\}| \leq \theta(\zeta)$. So C.P. holds.

S.P.: Suppose that $m \in \omega\backslash\{0\}$, $a \in {}^{m}\mathscr{W}$, $a \in \alpha$ and $a_0 \neq o$. Now $\xi \leq a_0$ and $a_0 = a_0 + \xi$. Clearly

$$\langle a_0, \xi, a_1, a_2, \ldots, a_{m-1}\rangle \in \alpha.$$

So S.P. holds.

R.P.: By Proposition 9.5 it suffices to prove the following statement:

($*$) For all $m \in \omega\backslash\{0\}$ and for all $a, b \in {}^{m}\mathscr{W}$, if $a, b \in \alpha$, $a_0 \geq a_1 \geq \cdots \geq a_{m-1}$ and $b_0 \geq b_1 \geq \cdots \geq b_{m-1}$, then there is a $c \in {}^{m\cdot m}\mathscr{W}$ such that $c \in \alpha$, $\forall i < m[a_i = \sum\{c_{f_{mm}(i,j)} : j < m\}]$, and $\forall j < m[b_j = \sum\{c_{f_{mm}(i,j)} : i < m\}]$.

We prove the statement by induction on m. First assume that $m = 1$ and $a, b \in {}^{m}\mathscr{W}$ with $a, b \in \alpha$. Since $\mathrm{dmn}(a) = \mathrm{dmn}(b) = 1$ and $T(a) = T(b)$, it follows that $a_0 = \eta = b_0$, hence $a = b$. Then we can take $c = a$.

Now assume the result for m, and suppose that $a, b \in {}^{m+1}\mathscr{W}$, $a, b \in \alpha$, $a_0 \geq a_1 \geq \cdots \geq a_m$ and $b_0 \geq b_1 \geq \cdots \geq b_m$. Let $a' = a \restriction m$ and $b' = b \restriction m$. Let $\lambda : m + 1 \to m$ be such that $\lambda \restriction m$ is the identity, and $\lambda(m) = m - 1$. Then $\forall j < m[a'_j = a_j = \sum\{a_i : i < m + 1, \lambda(i) = j\}]$, So $a \prec a'$, hence $a' \in \alpha$ by C.P. Similarly, $b' \in \alpha$. By the inductive assumption there is a $c \in {}^{m\cdot m}\mathscr{W}$ with $c \in \alpha$ such that $\forall i < m[a'_i = \sum\{c_{f(i,j)} : j < m\}]$, and $\forall j < m[b'_j = \sum\{c_{f(i,j)} : i < m\}]$. Note that $a_m = \xi$ or $a_m = o$; similarly, $b_m = \xi$ or $b_m = o$. Let $d \in {}^{(m+1)\cdot(m+1)}\mathscr{W}$ extend c, with

$$\begin{aligned} d_{f(m,0)} &= a_m; \\ d_{f(m,i)} &= o \text{ if } 0 < i \leq m; \\ d_{f(0,m)} &= b_m; \\ d_{f(i,m)} &= o \text{ if } 0 < i < m. \end{aligned}$$

We have $a_m = \sum\{d_{f(m,i)} : i \leq m\}$ and $b_m = \sum\{d_{f(i,m)} : i \leq m\}$. Also, if $i < m$, then $\sum_{j\leq m} d_{ij} = \sum_{j<m} c_{ij} + d_{im} = a_i + o = a_i$, and similarly $\sum_{i\leq m} d_{ij} = b_j$. Also, $d \in \alpha$ by the definition of $g(\theta)$, proving ($*$) for $m + 1$.

So R.P. holds.

Hence g maps into $\triangle\mathscr{W}$.

Now we claim that $f(g(\theta)) = \theta$ for all $\theta \in \mathcal{N}$. For, $f(g(o)) = f(O) = o$. Now let $\theta \in \mathcal{N}^*$. Let $\alpha = g(\theta)$. Choose $\xi \leq \eta$ ordinals in ω_1 such that $\forall\zeta < \xi[\theta(\zeta) = 0]$, $\forall\zeta > \eta[\theta(\zeta) = 0]$, $\theta(\xi) = \omega$ and $1 \leq \theta(\eta) \leq \omega$. Now suppose that $\zeta < \omega_1$.

Case 1. $\zeta < \xi$. Then by the definition of $g(\theta)$, for any $a \in \alpha$, $|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| \leq \theta(\zeta) = 0$, so $(f(g(\theta))(\zeta) = \theta_\alpha(\zeta) = 0 = \theta(\zeta)$.

Case 2. $\zeta = \xi$. Clearly there are members of α with an arbitrarily large number of ζ's, so $\theta_\alpha(\xi) = \omega = \theta(\xi)$.

Case 3. $\xi < \zeta \leq \eta$. Then by the definition of $g(\theta)$, $\forall a \in \alpha[|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| \leq \theta(\zeta)]$. Hence $\theta_\alpha(\zeta) \leq \theta(\zeta)$. Now we consider subcases.

Subcase 3.1. $\zeta < \eta$ and $\theta(\zeta) < \omega$. Let $\mathrm{dmn}(a) = \theta(\zeta) + 1$ with $a_0 = \eta$ and $a_i = \zeta$ for $0 < i \leq \theta(\zeta)$. Then $a \in \alpha$, so $\theta_\alpha(\zeta) = \theta(\zeta)$.

Subcase 3.2. $\zeta = \eta$ and $\theta(\zeta) < \omega$. Let $\mathrm{dmn}(a) = \theta(\zeta)$ with $a_i = \zeta$ for $i < \theta(\zeta)$. Then $a \in \alpha$, so $\theta_\alpha(\zeta) = \theta(\zeta)$.

Subcase 3.3. $\theta(\zeta) = \omega$ and $\zeta = \eta$. For each $m \in \omega\backslash\{0\}$ let a have domain m with constant value η. Then $\theta_\alpha(\eta) = \omega = \theta(\eta)$.

Subcase 3.4. $\theta(\zeta) = \omega$ and $\zeta \neq \eta$. For each $m \in \omega\backslash\{0\}$ let a have domain $m+1$ with $a_0 = \eta$ and $a_i = \zeta$ for all $i = 1, \ldots, m$. This shows that $\theta_\alpha(\zeta) = \omega = \theta(\zeta)$.

Case 4. $\eta < \zeta$. Since $T(\alpha) = \eta$, for any $a \in \alpha$ we have $\{i \in \mathrm{dmn}(a) : a_i = \zeta\} = \emptyset$. Hence $\theta_\alpha(\zeta) = 0 = \theta(\zeta)$.

This proves the claim that $f(g(\theta)) = \theta$ for all $\theta \in \mathcal{N}$.

Now to prove that $g(f(\alpha)) = \alpha$ for all $\alpha \in \triangle\mathscr{W}$, first note that $g(f(O)) = g(o) = O$. Now suppose that $\alpha \in (\triangle\mathscr{W})^*$; we want to show that $\alpha_{\theta_\alpha} = \alpha$. Let $\eta = T(\alpha)$. If $a \in \alpha$, then $T(a) = \eta$ and $\forall \zeta < \omega_1[|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| \leq \theta_\alpha(\zeta)]$, hence $a \in \alpha_{\theta_\alpha}$. Thus $\alpha \subseteq \alpha_{\theta_\alpha}$.

Now since $\alpha \in (\triangle\mathscr{W})^*$, θ_α is given by

$$\theta_\alpha(\zeta) = lub\{|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| : a \in \alpha\}.$$

Then we have ξ and η given by Proposition 9.2, and

$$\alpha_{\theta_\alpha} = \{a \in {}^{<<\omega}\mathscr{W} : T(a) = \eta \text{ and } \forall \zeta < \eta[|\{i \in \mathrm{dmn}(a) : a_i = \zeta\}| \leq \theta_\alpha(\zeta)]\}.$$

Now suppose that $a \in \alpha_{\theta_\alpha}$. Say $\zeta_0 > \cdots > \zeta_n$ are the distinct values of a, with multiplicities $k_0, \ldots, k_n \geq 1$. Note that when R.P. is applied to a' and b', obtaining c, if $\zeta < \omega_1$ occurs l times in a', then it occurs at least l times in c. Now for each $i \leq n$ we have $k_i \leq \theta_\alpha(\zeta_i)$, so there is a $b_i \in \alpha$ such that $|\{j \in \mathrm{dmn}(b_i) : b_{ij} = \zeta_i\}| \geq k_i$. Applying R.P. to all the b_i in succession, we end up with $c \in \alpha$ such that $\forall i \leq n[|\{j \in \mathrm{dmn}(c) : c_j = \zeta_i\}| \geq k_i]$. Note that c may have values in addition to $\zeta_0, \ldots, \zeta_n$. Say $m = \mathrm{dmn}(c)$. By Proposition 8.8 we may assume that the entries of c are in decreasing order; say

$$\begin{aligned}
&c_i = \zeta_0 \text{ for } i = 0, \ldots, k_0 - 1;\ k_0 \leq l_1;\\
&c_i = \zeta_1 \text{ for } i = l_1, \ldots, l_1 + k_1 - 1;\ l_1 + k_1 \leq l_2;\\
&\ldots\ldots\ldots\ldots\ldots\\
&c_i = \zeta_n \text{ for } i = l_n, \ldots, l_n + k_n - 1;\ l_n + k_n \leq m.
\end{aligned}$$

Let $p = \mathrm{dmn}(a)$, and define $\lambda : m \to p$ as follows:

$$\begin{aligned}
&\lambda(j) = j \text{ for } 0 \leq j < k_0;\\
&\lambda(j) = k_0 - 1 \text{ for } k_0 \leq j < l_1;\\
&\lambda(j) = j \text{ for } l_1 \leq j < l_1 + k_1;\\
&\lambda(j) = l_1 + k_1 - 1 \text{ for } l_1 + k_1 \leq j < l_2;\\
&\ldots\ldots\ldots\ldots\ldots\\
&\lambda(j) = j \text{ for } l_n \leq j < l_n + k_n;\\
&\lambda(j) = l_n + k_n - 1 \text{ for } l_n + k_n \leq j < m.
\end{aligned}$$

Now let $d_j = \sum\{c_i : i < m, \lambda(i) = j\}$ for all $j < n$. Thus $c \prec d$, so $d \in \alpha$ by C.P. Now d has entries, in order, $\zeta_0, \ldots \zeta_n$ with multiplicities $k_0, \ldots, k_n$. a is a permutation of d, so $a \in \alpha$ by Proposition 8.8. $\square$

10. Derived measures

For the notion of an m-monoid, M-measure, and $\mathscr{M}(M)$ see section 7. We apply these notions with $A = \mathrm{Fr}(\omega)$. For any $\sigma \in \mathscr{M}(M)$ we define $\triangle\sigma$ with domain $\mathrm{Fr}(\omega)$ by

$$(\triangle\sigma)(x) = \{\langle \sigma(y_0), \ldots, \sigma(y_{n-1})\rangle : n \in \omega, x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{n-1}\}.$$

Proposition 10.1. *If $\sigma \in \mathscr{M}(M)$, then $\triangle\sigma \in \mathscr{M}(\triangle M)$.*

Proof. Recall from Proposition 8.7 that $\triangle M$ is an m-monoid. Clearly $(\triangle\sigma)(0) = O$, the zero of $\triangle M$. Conversely, if $(\triangle\sigma)(x) = O$, then $x = 0$. Now suppose that $x \neq 0$. Then clearly $(\triangle\sigma)(x)$ is a countable subset of ${}^{<<\omega}M$. To show that $(\triangle\sigma)(x) \in \triangle M$ we need to check C.P., R.P. and S.P.

C.P.: Suppose that $a, b \in {}^{<<\omega}M$, $a \in (\triangle\sigma)(x)$, and $a \prec b$. Say $a = \langle \sigma(y_0), \ldots, \sigma(y_{m-1})\rangle$ with $m \in \omega$ and $x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$, b has length n, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. For each $j < n$ let $z_j = \sum\{y_i : i < m, \lambda(i) = j\}$. Then $x = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{n-1}$ and for all $j < n$,

$$b_j = \sum\{\sigma(y_i) : i < n, \lambda(i) = j\} = \sigma\left(\sum\{y_i : i < n, \lambda(i) = j\}\right) = \sigma(z_j).$$

Thus $b \in (\triangle\sigma)(x)$. This proves C.P.

R.P.: Suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^mM$, $b \in {}^nM$, and $a, b \in (\triangle\sigma)(x)$. Say $x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$, $\forall i < m[a_i = \sigma(y_i)]$, $x = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{n-1}$, and $\forall i < n[b_i = \sigma(z_i)]$. For $i < m$ and $j < n$ let $w_{f(i,j)} = y_i \cdot z_j$. Then $x = \sum_{k<m\cdot n} w_k$ (disjoint sum). Let $c_{f(i,j)} = \sigma(y_i \cdot z_j)$ for all $i < m$ and $j < n$. Then $\forall i < m[a_i = \sum\{c_{f(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum\{c_{f(i,j)} : i < m\}$. So R.P. holds.

S.P.: Suppose that $m \in \omega\backslash\{0\}$, $a \in {}^mM$, $a_0 \neq o$, and $a \in (\triangle\sigma)(x)$. Say $x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$ and $a = \langle \sigma(y_0), \sigma(y_1), \ldots, \sigma(y_{m-1}\rangle$. Thus $\sigma(y_0) \neq o$. So $y_0 \neq 0$. Write $y_0 = u \dot{+} v$ with $u, v \neq 0$. Then $x = u \dot{+} v \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$. Hence

$$\langle \sigma(u), \sigma(v), \sigma(y_1), \ldots, \sigma(y_{m-1}\rangle \in (\triangle\sigma)(x).$$

This proves S.P.

Thus $(\triangle\sigma)(x) \in \triangle M$.

To prove that $(\triangle\sigma)$ preserves $+$, suppose that $x, y \in \mathrm{Fr}(\omega)$ are disjoint. Suppose that $u \in (\triangle\sigma)(x) + (\triangle\sigma)(y)$. Say $a \in (\triangle\sigma)(x)$, $b \in (\triangle\sigma)(y)$, with $\mathrm{dmn}(a) = \mathrm{dmn}(b)$, $x = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{m-1}$, $a = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1})\rangle$, $y = w_0 \dot{+} w_1 \dot{+} \cdots \dot{+} w_{m-1}$,

$$b = \langle \sigma(w_0), \sigma(w_1), \ldots, \sigma(w_{n-1})\rangle,$$

and $a + b = u$. Now

$$x + y = (z_0 \dot{+} w_0) \dot{+} (z_1 \dot{+} w_1) \dot{+} \cdots \dot{+} (z_{m-1} \dot{+} w_{m-1})$$

and

$$a + b = \langle \sigma(z_0 \dot{+} w_0), \sigma(z_1 \dot{+} w_1), \ldots, \sigma(z_{m-1} \dot{+} w_{m-1}) \rangle,$$

hence $u = a + b \in (\triangle\sigma)(x + y)$. If follows that $(\triangle\sigma)(x) + (\triangle\sigma)(y) \subseteq (\triangle\sigma)(x + y)$.

Conversely, suppose that $v \in (\triangle\sigma)(x + y)$. Say $x + y = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{m-1}$ and $v = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1}) \rangle$. Then $x = (x \cdot z_0) \dot{+} (x \cdot z_1) \dot{+} \cdots \dot{+} (x \cdot z_{m-1})$, and hence $a \stackrel{\text{def}}{=} \langle \sigma(x \cdot z_0), \sigma(x \cdot z_1), \ldots, \sigma(x \cdot z_{m-1}) \rangle \in (\triangle\sigma)(x)$. Similarly, $b \stackrel{\text{def}}{=} \langle \sigma(y \cdot z_0), \sigma(y \cdot z_1), \ldots, \sigma(y \cdot z_{m-1}) \rangle \in (\triangle\sigma)(y)$. So $a + b \in (\triangle\sigma)(x) + (\triangle\sigma)(y)$. Now

$$\begin{aligned} a + b &= \langle \sigma(x \cdot z_0) + \sigma(y \cdot z_0), \sigma(x \cdot z_1) + \sigma(y \cdot z_1), \ldots, \sigma(x \cdot z_{m-1}) + \sigma(y \cdot z_{m-1}) \rangle \\ &= \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1}) \rangle = v. \end{aligned}$$

Thus $v \in (\triangle\sigma)(x) + (\triangle\sigma)(y)$. □

Proposition 10.2. $\forall x \in \mathrm{Fr}(\omega)[T((\triangle\sigma)(x)) = \sigma(x)]$.

Proof. We have $(\triangle\sigma)(x) \in \triangle M$ by Proposition 10.1. Hence by the definition before Proposition 8.11, $T((\triangle\sigma)(x)) = T(a)$ for any element a of $(\triangle\sigma)(x)$. So suppose that $x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$ and $a = \langle \sigma(y_0), \sigma(y_1), \ldots, \sigma(y_{m-1}) \rangle$. Hence $T((\triangle\sigma)(x)) = T(a) = \sum_{i<m} \sigma(y_i) = \sigma(\sum_{i<m} y_i) = \sigma(x)$. □

The definition of $\triangle\sigma$ can also be made with respect to any BA isomorphic to $\mathrm{Fr}(\omega)$, and 10.1,10.2 hold.

Proposition 10.3. *If $z \in \mathrm{Fr}(\omega)$, $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \restriction z$ is an isomorphism, and $\sigma : \mathrm{Fr}(\omega) \restriction z \to M$ is an M-measure on $\mathrm{Fr}(\omega) \restriction z$, then $\triangle(\sigma \circ k) = (\triangle'\sigma) \circ k$. Here $\triangle'$ refers to the above construction with $A = \mathrm{Fr}(\omega) \restriction z$.*

Proof. Let $x \in \mathrm{Fr}(\omega)$. First suppose that $a \in (\triangle(\sigma \circ k))(x)$. Say $x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$ and $a = \langle \sigma(k(y_0)), \sigma(k(y_1)), \ldots, \sigma(k(y_{m-1}) \rangle$. Then

$$k(x) = k(y_0) \dot{+} k(y_1) \dot{+} \cdots \dot{+} k(y_{m-1}),$$

and so $a \in (\triangle'\sigma)(k(x))$.

Conversely, suppose that $a \in (\triangle'\sigma)(k(x))$. Say $k(x) = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{m-1}$ and $a = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{m-1}) \rangle$. Let $y_i = k^{-1}(z_i)$ for all $i < m$. Then $x = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1}$ and $a = \langle \sigma(k(y_0)), \sigma(k(y_1)), \ldots, \sigma(l(y_{m-1})) \rangle$. Hence $a \in (\triangle(\sigma \circ k))(x)$. □

Proposition 10.4. *Suppose that σ, τ are M measures with respect to $\mathrm{Fr}(\omega)$ and k is an automorphism of $\mathrm{Fr}(\omega)$. Then (i) implies (ii):*

(i) $\sigma = \tau \circ k$.

(ii) $(\triangle\sigma) = (\triangle\tau) \circ k$.

Proof. Assume that k is an automorphism k of $\mathrm{Fr}(\omega)$ such that $\sigma = \tau \circ k$. Then by Proposition 10.3, $(\triangle\tau) \circ k = \triangle(\tau \circ k) = \triangle\sigma$. □

The measure $\triangle\sigma$ is the *first derivative* of σ.

11. Existence of measures

Proposition 11.1. *If $\sigma, \tau \in \mathscr{M}(M)$, k is an automorphism of* $\mathrm{Fr}(\omega)$*, and $\sigma = \tau \circ k$, then* $(\triangle\sigma)(1) = (\triangle\tau)(1)$.

Proof. Let $a \in (\triangle\sigma)(1)$. Say

$$1 = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{m-1} \text{ and } a = \langle \sigma(y_0), \sigma(y_1), \ldots, \sigma(y_{m-1}) \rangle.$$

Thus

$$a = \langle \tau(k(y_0)), \tau(k(y_1)), \ldots, \tau(k(y_{m-1})) \rangle \text{ and } 1 = k(y_0)) \dot{+} k(y_1) \dot{+} \cdots \dot{+} k(y_{m-1}),$$

so $a \in (\triangle\tau)(1)$. The converse is symmetric. □

Proposition 11.2. *Suppose that M is an m-monoid, $\alpha \subseteq {}^{<<\omega}M$ satisfies C.P. and R.P., $m, n \in \omega \backslash \{0\}$, $a \in {}^{m}M$, $b \in {}^{n}M$, $a, b \in \alpha$, and $p \geq n$.*

Then there is a $c \in {}^{m \cdot p}M$ such that $c \in \alpha$, $\forall i < m[\sum_{j<p} c_{f_{mp}(i,j)} = a_i]$, and $c \prec b$.

Proof. By R.P. let $d \in {}^{m \cdot n}M$ be such that $d \in \alpha$, $\forall i < m[a_i = \sum\{d_{f_{mn}(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum\{d_{f_{mn}(i,j)} : i < m\}]$. Define $c \in {}^{m \cdot p}M$ so that $c_{f_{mp}(i,j)} = d_{f_{mn}(i,j)}$ for $i < m$ and $j < n$, and $c_{f_{mp}(i,j)} = 0$ for $i < m$ and $n \leq j < p$. Define $\lambda : m \cdot n \to m \cdot p$ by $\lambda(f_{mn}(i,j)) = f_{mp}(i,j)$ for all $i < m$ and $j < n$. Now if $i < m$, $j < p$, $k < m$, $l < n$, then $\lambda(f_{mn}(k,l)) = f_{mp}(k,l)$, and so $\lambda(f_{mn}(k,l)) = f_{mp}(i,j)$ iff $j < n$, $k = i$, and $l = j$. so $\forall i < m \forall j < p[c_{f_{mp}(i,j)} = \sum\{d_{f_{mp}(k,l)} : k < m, l < n, \lambda(f_{mn}(k,l)) = f_{mp}(i,j)\}$. Hence $d \prec c$ and so $c \in \alpha$. We have

$$\forall i < m \left[\sum_{j<p} c_{f_{mp}(i,j)} = \sum_{j<n} c_{f_{mp}(i,j)} = \sum_{j<n} d_{f_{mn}(i,j)} = a_i \right].$$

Now define $\mu : m \cdot p \to n$ by setting, for $i < m$ and $j < p$,

$$\mu(i,j) = \begin{cases} j & \text{if } j < n, \\ n-1 & \text{if } n \leq j < p. \end{cases}$$

Now take any $j < n$, and any $k < m$, $l < p$.

Case 1. $j < n-1$. Then $\mu(k,l) = j$ iff $l = j$, so

$$\sum\{c_{f_{mp}(k,l)} : \mu(k,l) = j\} = \sum\{c_{f_{mp}(i,j)} : i < m\} = \sum\{d_{f_{mn}(i,j)} : i < m\} = b_j.$$

Case 2. $j = n-1$. Then $\mu(k,l) = j$ iff $n-1 \leq j < p$ Hence

$$\begin{aligned} \sum\{c_{f_{mp}(k,l)} : \mu(k,l) = j\} &= \sum\{c_{f_{mp}(i,n-1)} : i < m\} + \sum\{c_{f_{mp}(k,l)} : k < m, n \leq l < p\} \\ &= \sum\{c_{f_{mp}(i,n-1)} : i < m\} = b_{n-1}. \end{aligned}$$

□

Proposition 11.3. *Suppose that M is an m-monoid, $\alpha \subseteq {}^{<<\omega}M$ satisfies C.P., R.P., and S.P., $m, n \in \omega\backslash\{0\}$, $a \in {}^{m}M$,$b \in {}^{n}M$, $a, b \in \alpha$, and $p \geq n$. Also suppose that $\forall i < m[a_i \neq 0]$*

Then there is a $c \in {}^{m\cdot p}M$ such that $c \in \alpha$, $\forall i < m[\sum_{j<p} c_{f_{mp}(i,j)} = a_i]$, $c \prec b$, and $\forall i < m \forall j < p[c_{f_{mp}(i,j)} \neq 0]$.

Proof. Assume all the hypotheses. By R.P. let $d \in {}^{m\cdot n}M$ be such that $d \in \alpha$, $\forall i < m[a_i = \sum\{d_{f_{mn}(i,j)} : j < n\}]$ and $\forall j < n[b_j = \sum\{d_{f_{mn}(i,j)} : i < m\}]$. Then for every $i < m$ there is a $j(i) < n$ such that $d_{f_{mn}(i,j(i))} \neq 0$. For each $i < m$ let $r(i) = |\{j < n : d_{f_{mn}(i,j)} \neq 0\}|$. Thus $1 \leq r(i) \leq n$. Let g be a bijection from $\prod_{i<m}(n + p - r(i))$ onto some integer q. Applying Proposition 8.9 many times, for each $i < m$ we get elements $e^i_0, \ldots, e^i_{p-r(i)}$ such that each $e^i_s \neq 0$ and $d_{f_{mn}(i,j(i))} = e^i_0 + \ldots + e^i_{p-r(i)}$ and $d' \in \alpha$, where for each $i < m$ and $k < n + p - r(i)$ we have

$$d'_{g(i,k)} = \begin{cases} d_{f_{mn}(i,k)} & \text{if } k < j(i), \\ e^i_l & \text{if } k = j(i) + l \text{ with } l \leq p - r(i), \\ d_{f_{mn}(i,k-p+r(i))} & \text{if } j(i) + p - r(i) < k < n + p - r(i). \end{cases}$$

Note that for each $i < m$ there are $n - r(i)$ 0's among $\{d_{f_{mn}(i,j)} : j < n\}$. So if we delete all 0's in d' we obtain a sequence $c \in \alpha$ of length $\sum_{i<m}(r(i) + p - r(i)) = \sum_{i<m} p = m \cdot p$. We claim that c is as desired.

For each $i < m$,

$$a_i = \sum\{d_{f_{mn}(i,j)} : j < n\} = \sum\{d'_{g(i,j)} : j < n + p - r(i)\} = \sum_{j<n} c_{f_{mp}(i,j)}.$$

Now define $\lambda : q \to n$ as follows, with $i < m$:

$$\lambda(g(i,v)) = \begin{cases} v & \text{if } v < j(i); \\ j(i) & \text{if } v = j(i) + l \text{ with } l \leq p - r(i); \\ v - p + r(i) & \text{if } j(i) + p - r(i) < v < n + p - r(i). \end{cases}$$

Then for all $k < n$,

$$\begin{aligned} &\sum\{d'_{g(i,v)} : i < m, j < n + p - r(i), \lambda(g(i,v)) = k\} \\ &\quad = \sum\{d'_{g(i,k)} : i < m, v = k < j(i)\} \\ &\qquad + \sum\{d'_{g(i,v)} : i < m, k = j(i), v = j(i) + l, l \leq p - r(i)\} \\ &\qquad + \sum\{d'_{g(i,v)} : k = v - p + r(i), j(i) + p - r(i) < v < n + p - r(i)\} \\ &\quad = \sum\{d'_{g(i,k)} : i < m, v = k < j(i)\} \\ &\qquad + \sum\{d'_{g(i,v)} : i < m, k = j(i), v = j(i) + l, l \leq p - r(i)\} \\ &\qquad + \sum\{d'_{g(i,v)} : i < m, j(i) < k < n, j(i) + p - r(i) < v < n + p - r(i)\} \end{aligned}$$

$$
\begin{aligned}
&= \sum\{d_{f_{mn}(i,k)} : i < m, k < j(i)\} \\
&\quad + \sum\{e^i_l : i < m, k = j(i), l \leq p - r(i)\} \\
&\quad + \sum\{d_{f_{mn}(i,k)} : i < m, j(i) < k\} \\
&= \sum\{d_{f_{mn}(i,k)} : i < m\} = b_k.
\end{aligned}
$$

Thus $d' \prec b$. By Propositions 8.3 and 8.4, $c \prec d'$. So $c \prec b$. Clearly $\forall i < m \forall j < p[c_{f_{mp}(i,j)} \neq 0]$. □

We let $\mathscr{D} = {}^{<\omega}2$, and for each $n \in \omega$, $\mathscr{D}_n = {}^n2$. If $a : \mathscr{D}_n \to M$, then we let $a(\mathscr{D}_n) = \langle a(lex_{\mathscr{D}_n}(i)) : i < |\mathscr{D}_n|\rangle$, where $lex_{\mathscr{D}_n}$ enumerates $\mathscr{D}_n$ in lexicographic order. Thus $a(\mathscr{D}_n) \in {}^{2^n}M$. Let $a_n = a(\mathscr{D}_n)$. So $a_n \in {}^{2^n}M$; $a_n = \langle a(lex_{\mathscr{D}_n}(i)) : i < 2^n\rangle$.

Let $\alpha \in [{}^{<<\omega}M]^\omega$. An *$\alpha$-tree* is a mapping $a : \mathscr{D} \to M$ such that:

(t1) $\forall n \in \omega[a_n \in \alpha]$.

(t2) If $f \in \mathscr{D}_n$ and $n \leq m$, then $a(f) = \sum\{a(g) : f \subseteq g \in {}^m2\}$.

An α-tree a is *dense* iff :

(t3) $\forall b \in \alpha \exists n \in \omega[a_n \prec b]$.

An α-tree a is *homogeneous* iff $\forall f \in \mathscr{D}[a(f) \neq 0]$.

Proposition 11.4. *If $\alpha \in [{}^{<<\omega}M]^\omega$ and $\forall a, b \in \alpha[T(a) = T(b) \overset{\text{def}}{=} T(\alpha)]$, and if a is an α-tree, then $a_0 = \langle T(\alpha)\rangle$.*

Proof. By (t1) we have $a_0 \in \alpha$, so $T(a_0) = T(\alpha)$. Now $a_0 = a(\mathscr{D}_0) = a(\{\emptyset\}) = \langle a(\emptyset)\rangle$, so $T(\alpha) = T(a_0) = a(\emptyset)$, hence $a_0 = \langle T(\alpha)\rangle$. □

Proposition 11.5. *If $\alpha \in [{}^{<<\omega}M]^\omega$, a is an α-tree, and $n \leq m$, then $a_m \prec a_n$.*

Proof. Define $\lambda : 2^m \to 2^n$ by setting, for any $i < 2^m$, $\lambda(i) = lex^{-1}_{\mathscr{D}_n}(lex_{\mathscr{D}_m}(i) \restriction n)$. Then for any $j < 2^n$,

$$
\begin{aligned}
\sum\{a_m(i) : i < 2^m \text{ and } \lambda(i) = j\} &= \sum\{a_m(i) : i < 2^m \text{ and } lex^{-1}_{\mathscr{D}_n}(lex_{\mathscr{D}_m}(i) \restriction n) = j\} \\
&= \sum\{a_m(i) : i < 2^m \text{ and } lex_{\mathscr{D}_m}(i) \restriction n = lex_{\mathscr{D}_n}(j)\} \\
&= \sum\{a(g) : g \in {}^m2 \text{ and } g \restriction n = lex_{\mathscr{D}_n}(j)\} \\
&= a(lex_{\mathscr{D}_n}(j)) = a_n(j).
\end{aligned}
$$
□

Proposition 11.6. *Let M be an m-monoid, and α a countable subset of ${}^{<<\omega}M$ which satisfies C.P. and R.P. Then there is a dense α-tree a.*

Proof. Note that $\langle T(\alpha)\rangle \in \alpha$ by the proof of Proposition 8.6. Let $\alpha = \{b_0, b_1, \ldots\}$ with $b_0 = \langle T(\alpha)\rangle$. We define m_n and $a : \bigcup_{k \leq m_n} \mathscr{D}_k \to M$ by induction on n so that the following conditions hold:

11.6(1) $\forall k \leq m_n [a_k \in \alpha]$.

11.6(2) $\forall k \leq m_n \forall f \in \mathscr{D}_k [a(f) = \sum\{a(g) : g \in \mathscr{D}_{m_n}, f \subseteq g\}]$.

11.6(3) a_{m_n} refines each b_j with $j \leq n$.

11.6(4) $n \leq m_n$.

Note that $\mathscr{D}_0 = {}^0 2 = \{\emptyset\}$. We let $m_0 = 0$ and $a(\emptyset) = \langle T(\alpha) \rangle$. Thus $a_0 = a(\mathscr{D}_0) = a(\emptyset) = \langle T(\alpha) \rangle$. So 11.6(1)–11.6(4) hold for $n = 0$. Now suppose that n and $a : \bigcup_{k \leq m_n} \mathscr{D}_k \to M$ have been defined so that 11.6(1)–11.6(4) hold. Note that a_{m_n} and b_{n+1} are in α. Say $b_{n+1} \in {}^p M$. Take any $2^s \geq p + 1$. Thus $s \geq 1$ since $p \geq 1$. Now we apply Proposition 11.2 with m, n, a, b, α, p replaced by $2^{m_n}, p, a_{m_n}, b_{n+1}, \alpha, 2^s$. This gives a $c \in {}^{2^{m_n} \cdot 2^s} M$ such that

11.6(5) $c \in \alpha$,

11.6(6) $\forall i < 2^{m_n} [\sum_{j < 2^s} c_{f_{2^{m_n} 2^s}(i,j)} = a_{m_n}(i) = a(lex_{\mathscr{D}_{m_n}}(i))]$,

and

11.6(7) $c \prec b_{n+1}$.

Let $m_{n+1} = m_n + s$. Since $s > 0$, we have $n + 1 \leq m_{n+1}$. So 11.6(4) holds for $n + 1$. For each $g \in \mathscr{D}_{m_{n+1}}$ let $a(g) = c_{f_{2^{m_n} 2^s}(i,j)}$ with $i = lex^{-1}_{\mathscr{D}_{m_n}}(g \restriction m_n)$, and $j = lex^{-1}_{\mathscr{D}_s}(\langle g_{m_n+k} : k < s \rangle)$. For $m_n < k < m_{n+1}$ and $g \in \mathscr{D}_k$ let $a(g) = \sum\{a(h) : h \in \mathscr{D}_{m_{n+1}}, g \subseteq h\}$. Now suppose that $g \in \mathscr{D}_{m_n}$. Let $lex_{\mathscr{D}_{m_n}}(i) = g$. Then

$$\text{11.6(8)} \qquad a(g) = a(lex_{\mathscr{D}_{m_n}}(i)) = \sum_{j < 2^s} c_{f_{2^{m_n} 2^s}(i,j)} = \sum\{a(h) : h \in \mathscr{D}_{m_{n+1}}, g \subseteq h\}.$$

Now for $k \leq m_n$ and $g \in \mathscr{D}_k$ we have

$$\begin{aligned} a(g) &= \sum\{a(h) : h \in \mathscr{D}_{m_n}, g \subseteq h\} \quad \text{by 11.6(2) for } n \\ &= \sum\{a(h) : h \in \mathscr{D}_{m_{n+1}}, g \subseteq h\} \quad \text{by 11.6(8)} \end{aligned}$$

For $m_n < k < m_{n+1}$ and $g \in \mathscr{D}_k$ we have

$$a(g) = \sum\{a(h) : h \in \mathscr{D}_{m_{n+1}}, g \subseteq h\}.$$

Hence 11.6(2) holds for $n + 1$.

Now suppose that $k < 2^{m_{n+1}}$. Let $g = lex_{\mathscr{D}_{m_{n+1}}}(k)$. Then by the above, $a(g) = c_{f_{2^{m_n} 2^s}(i,j)}$ with $i = lex^{-1}_{\mathscr{D}_{m_n}}(g \restriction m_n)$, and $j = lex^{-1}_{\mathscr{D}_s}(\langle g_{m_n+k} : k < s \rangle)$. Now for $i < 2^{m_n}$ and $j < 2^s$ let $\mu(f_{2^{m_n} 2^s}(i,j)) = lex_{\mathscr{D}_{2^{m_n}}}(i)^\frown lex_{\mathscr{D}_{2^s}}(j)$. Then

$$\forall k < 2^{m_{n+1}} [a(lex_{\mathscr{D}_{m_{n+1}}}(k)) = \sum\{c_{f_{2^{m_n} 2^s}(i,j)} : i < 2^{m_n}, j < 2^s, \mu(f_{2^{m_n} 2^s}(i,j)) = k\}].$$

It follows that $c \prec a_{m_{n+1}}$, and so $a_{m_{n+1}} \in \alpha$. Next, suppose that $m_n \leq k < m_{n+1}$. For any $h \in \mathscr{D}_{m_{n+1}}$ let $\lambda(lex^{-1}_{\mathscr{D}_{m_{n+1}}}(h)) = lex^{-1}_{\mathscr{D}_k}(h \restriction k)$. Thus $\lambda : 2^{m_{n+1}} \to 2^k$. For any $i < 2^k$, with $g = lex_{\mathscr{D}_k}(i)$ we have

$$\begin{aligned}
a_k(i) &= a(lex_{\mathscr{D}_k}(i)) = a(g) \\
&= \sum\{a(h) : h \in \mathscr{D}_{m_{n+1}}, g \subseteq h\} \\
&= \sum\{a(lex_{\mathscr{D}_{m_{n+1}}}(j)) : g \subseteq lex_{\mathscr{D}_{m_{n+1}}}(j))\} \\
&= \sum\{a(lex_{\mathscr{D}_{m_{n+1}}}(j)) : (lex_{\mathscr{D}_{m_{n+1}}}(j)) \restriction k = g\} \\
&= \sum\{a(lex_{\mathscr{D}_{m_{n+1}}}(j)) : lex_{\mathscr{D}_{m_{n+1}}}(j)) \restriction k = lex_{\mathscr{D}_k}(i)\} \\
&= \sum\{a(lex_{\mathscr{D}_{m_{n+1}}}(j)) : \lambda(j) = i\}
\end{aligned}$$

The last equation holds since

$$\lambda(j) = \lambda(lex^{-1}_{\mathscr{D}_{m_{n+1}}}(lex_{\mathscr{D}_{m_{n+1}}}(j)) = lex^{-1}_{\mathscr{D}_k}(lex_{\mathscr{D}_{m_{n+1}}}(j) \restriction k),$$

hence $\lambda(j) = i$ iff $lex_{\mathscr{D}_{m_{n+1}}}(j)) \restriction k = lex_{\mathscr{D}_k}(i)$.

Thus $\langle a(lex_{\mathscr{D}_{m_{n+1}}}(j)) : j < 2^{m_{n+1}}\rangle \prec \langle a(lex_{\mathscr{D}_k}(j)) : j < 2^k\rangle$, so $\langle a(lex_{\mathscr{D}_k}(j)) : j < 2^k\rangle \in \alpha$. Hence 11.6(1) holds for $n+1$.

Now if $s \leq n$, then $\langle a(lex_{\mathscr{D}_n}(i)) : i < 2^{m_n}\rangle \prec b_s$ by 11.6(3) for n. By the above with $k = 2^{m_n}$ we have

$$\langle a(lex(j)) : j < 2^{m_{n+1}}\rangle \prec \langle a(lex(j)) : j < 2^{m_n}\rangle \prec b_s.$$

Thus 11.6(3) holds for $n+1$. □

Proposition 11.7. *Let M be an m-monoid, and $\alpha \neq O$ a countable subset of ${}^{<<\omega}M$ which satisfies C.P., R.P., and S.P. Then there is a dense homogeneous α-tree a.*

Proof. We add to the proof of Proposition 11.6. We apply Proposition 11.3 instead of Proposition 11.2, and obtain in addition to the properties of c given in the proof of Proposition 11.6 the condition $\forall i < 2^{m_n} \forall j < 2^s [c_{f_{2^{m_n}2^s}(i,j)} \neq 0]$. Clearly this gives the homogeneous property. □

Proposition 11.8. (1.16.4) *If M is an m-monoid and $O \neq \alpha \in \triangle M$, then there is a $\sigma \in \mathscr{M}(M)$ such that $(\triangle\sigma)(1) = \alpha$.*

Proof. For each $n \in \omega$ and each $f \in \mathscr{D}_n$ let

11.8(1) $x(f) = \{g \in {}^\omega 2 : f \subseteq g\}$.

Then $\{x(f) : f \in \mathscr{D}\}$ is a base for clop(${}^\omega 2$), which is a denumerable atomless BA.

11.8(2) If $k \leq i \in \omega$ and $f \in \mathscr{D}_k$, then $x(f) = \bigcup\{x(g) : g \in \mathscr{D}_i, f \subseteq g\}$, and this union is pairwise disjoint.

11.8(3) If $1 = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{n-1}$ with each y_i clopen in ${}^{\omega}2$, then there is a $k \in \omega$ and a partition $\mathscr{D}_k = G_0 \dot{\cup} G_1 \dot{\cup} \ldots \dot{\cup} G_{n-1}$ such that $\forall j < n[y_j = \sum\{x(f) : f \in G_j\}]$.

In fact, since $\{x(f) : f \in \mathscr{D}\}$ is a base for $\mathrm{clop}({}^{\omega}2)$ and each y_i is clopen and hence compact, there exist finite subsets $\mathscr{D}^i$ of $\mathscr{D}$ for $i < n$ such that $\forall i < n[y_i = \bigcup\{x(f) : f \in \mathscr{D}^i\}]$. Let k be greater than the domain of f, for all $f \in \bigcup_{i<n} \mathscr{D}^i$. By 11.8(2) we may assume that each $\mathscr{D}^i$ is a subset of $\mathscr{D}_k$. Clearly the sets $\mathscr{D}^i$ are pairwise disjoint. For any $f \in {}^{k}2$ the set $x(f)$ is a nonempty set, and hence $x(f) \cap y_i \neq \emptyset$ for some $i < n$. So we must actually have $f \in \mathscr{D}^i$. This proves 11.8(3).

By Proposition 11.7 let a be a dense homogeneous α-tree. Now let $z \in \mathrm{clop}({}^{\omega}2)$. Then by compactness there is a finite $X \subseteq \mathscr{D}$ such that $z = \bigcup\{x(f) : f \in X\}$. Hence there is a $k \in \omega$ and a $G \subseteq \mathscr{D}_k$ such that $z = \sum\{x(f) : f \in G\}$. (disjoint sum) We then define

$$\sigma(z) = \sum\{a(f) : f \in G\}.$$

This definition does not depend on the particular choice of k and G. For, suppose that also $l \in \omega$, $H \subseteq \mathscr{D}_l$, and $z = \sum\{x(f) : f \in H\}$; we claim that $\sum\{a(f) : f \in G\} = \sum\{a(f) : f \in H\}$. For, suppose that $k \leq l$, and let $K = \{f \in \mathscr{D}_l : \exists g \in G[g \subseteq f\}$. Then $z = \sum\{x(f) : f \in K\}$ by 11.8(2). Since $x(f) \cap x(g) = \emptyset$ for distinct $f, g \in \mathscr{D}_l$, it follows that $K = H$. Then by 11.8(2) we get

$$\sum\{a(f) : f \in G\} = \sum\{a(g) : g \in K\} = \sum\{a(f) : f \in H\}.$$

Now to show that σ is additive, suppose that $z, z' \in \mathrm{clop}({}^{\omega}2)$ and $z \cdot z' = 0$. Choose k, k', G, G' such that $G \subseteq \mathscr{D}_k$, $G' \subseteq \mathscr{D}_{k'}$, $z = \sum\{x(f) : f \in G\}$, and $z' = \sum\{x(f) : f \in G'\}$; thus $\sigma(z) = \sum\{a(f) : f \in G\}$ and $\sigma(z') = \sum\{a(f) : f \in G'\}$. Wlog $k \leq k'$. Let $H = \{f \in \mathscr{D}_{k'} : \exists g \in G[g \subseteq f]\}$. Then by 11.8(2), $z = \sum\{x(f) : f \in H\}$ and $\sigma(z) = \sum\{a(f) : f \in H\}$. Since $z \cdot z' = 0$ we have $H \cap G' = \emptyset$. Now $z + z' = \sum\{x(f) : f \in H \cup G'\}$, so

$$\sigma(z+z') = \sum\{a(f) : f \in H \cup G'\} = \sum\{a(f) : f \in H\} + \sum\{a(f) : f \in G'\} = \sigma(z) + \sigma(z').$$

Since a is homogeneous, clearly $\sigma(z) = 0$ iff $z = 0$. So $\sigma \in \mathscr{M}(M)$.

It remains only to show that $(\triangle\sigma)(1) = \alpha$. Suppose that $b \in (\triangle\sigma)(1)$. Say $n \in \omega$, $1 = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{n-1}$ and $b = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{n-1}) \rangle$. Now by 11.8(3) let $k \in \omega$ and $G_0, \ldots, G_{n-1}$ be such that $\mathscr{D}_k = G_0 \dot{\cup} G_1 \dot{\cup} \ldots \dot{\cup} G_{n-1}$ and $\forall j < n[z_j = \sum\{x(f) : f \in G_j\}$. So $\forall j < n[\sigma(z_j) = \sum\{a(f) : f \in G_j\}$. Define $\lambda : 2^k \to n$ by $\lambda(m) = j$ iff $lex(m) \in G_j$. Then $\forall j < n[\sigma(z_j) = \sum\{a(lex(m)) : m < 2^{\kappa}, \lambda(m) = j\}$. Hence $a_k \prec b$, hence $b \in \alpha$ by (t1) and C.P. This shows that $(\triangle\sigma)(1) \subseteq \alpha$.

For the other inclusion, first note that

11.8(4) $\forall f \in \mathscr{D}[\sigma(x(f)) = a(f)]$

In fact, let $f \in \mathscr{D}$. Let $z = x(f)$ and $G = \{f\}$. Then by definition $\sigma(x(f)) = a(f)$.

11.8(5) $\forall k \in \omega[a_k \in (\triangle\sigma)(1)]$.

For, let $k \in \omega$. Then

$$a_k = a(\mathscr{D}_k) = \langle a(lex(i)) : i < 2^k \rangle = \langle \sigma(x(lex(i))) : i < 2^k \rangle \in (\triangle\sigma)(1)$$

since $1 = \sum\{x(lex(i)) : i < 2^k\}$, a disjoint sum. So 11.8(5) holds.

Now take any $b \in \alpha$. Since a is dense there is a $k \in \omega$ such that $a_k \prec b$. Now $(\triangle\sigma)(1) \in \triangle M$ by Proposition 10.1, so $b \in (\triangle\sigma)(1)$ by (t1) and C.P.

So $(\triangle\sigma)(1) = \alpha$. □

12. Stable measures

Proposition 12.1. *If $x, y \in \mathrm{Fr}(\omega)$, σ is a M-measure, and $(\triangle\sigma)(x) = (\triangle\sigma)(y)$, then $\sigma(x) = \sigma(y)$.*

Proof. By Proposition 10.2. □

An M-measure σ is *stable* iff $\forall x, y \in \mathrm{Fr}(\omega)$, $\sigma(x) = \sigma(y)$ implies that $(\triangle\sigma)(x) = (\triangle\sigma)(y)$.

Proposition 12.2. *Let M be an m-monoid, and σ, τ stable M-measures. Then the following are equivalent:*

(i) There is an automorphism k of $\mathrm{Fr}(\omega)$ such that $\sigma = \tau \circ k$.

(ii) $(\triangle\sigma)(1) = (\triangle\tau)(1)$.

Proof. (i)⇒(ii): by Proposition 10.4.

(ii)⇒(i): Assume (ii). Define xRy iff $\sigma(x) = \tau(y)$.

12.2(1) R is a V-relation.

Recall from Section 3 the notion of a V-relation.

$1R1$: By Proposition 10.2, $\sigma(1) = T((\triangle\sigma)(1)) = T(\triangle\tau(1)) = \tau(1)$.

$0R0$: By definition, $\sigma(0) = 0 = \tau(0)$.

For (V9), we want to show that $\forall x \in \mathrm{Fr}(\omega)[xR0 \to x = 0]$. Suppose that $\sigma(x) = \tau(0)$. Since $\tau(0) = 0$, we have $\sigma(x) = 0$, and hence $x = 0$.

(V10) is symmetric to the above.

To complete the proof of 12.2(1) by symmetry it suffices to prove (V11). So suppose that $\sigma(x) = \tau(y \dot{+} z)$. Now $1 = y \dot{+} z \dot{+} (-y \cdot -z)$, so by definition $\langle \tau(y), \tau(z), \tau(-y \cdot -z) \rangle \in \triangle\tau(1) = (\triangle\sigma)(1)$. Hence there exist u_0, u_1, u_2 such that $1 = u_0 \dot{+} u_1 \dot{+} u_2$, $\tau(y) = \sigma(u_0)$, $\tau(z) = \sigma(u_1)$, and $\tau(-y \cdot -z) = \sigma(u_2)$. Let $v = u_0 + u_1$. Then $\sigma(v) = \sigma(u_0) + \sigma(u_1) = \tau(y) + \tau(z) = \tau(y + z) = \sigma(x)$. Then since σ is stable, $(\triangle\sigma)(v) = (\triangle\sigma)(x)$. Now $v = u_0 + u_1$, so $\langle \sigma(u_0, \sigma(u_1) \rangle \in (\triangle\sigma)(v) = (\triangle\sigma)(x)$. Hence there exist w_0, w_1 such that $x = w_0 \dot{+} w_1$, $\tau(y) = \sigma(u_0) = \sigma(w_0)$, and $\tau(z) = \sigma(u_1) = \sigma(w_1)$. Hence $w_0 Ry$ and $w_1 Rz$. This proves (V11). So we have proved 12.2(1).

Now by 12.2(1) and Theorem 3.3, there is an automorphism k of $\mathrm{Fr}(\omega)$ such that for all $x \in \mathrm{Fr}(\omega)$ there exist $n \in \omega$ and a disjoint $y \in {}^n\mathrm{Fr}(\omega)$ such that $x = \sum_{i \in n} y_i$ and $\forall i < n[y_i R k(y_i)]$. So $\forall i < n[\sigma(y_i) = \tau(k(y_i))$. Hence

$$\sigma(x) = \sigma\left(\sum_{i<n} y_i\right) = \sum_{i<n} \sigma(y_i) = \sum_{i<n} \tau(k(y_i)) = \tau\left(k\left(\sum_{i<n} y_i\right)\right) = \tau(k(x)).$$

Thus $\sigma = \tau \circ k$. □

13. Fragments

Let M be an m-monoid. For $\alpha \subseteq {}^{<<\omega}M$ we define

$$\Phi(\alpha) = \{b \in {}^{<<\omega}M : \exists c \in {}^{<\omega}M[b^\frown c \in \alpha]\}.$$

If also $a \in M$, then

$$\Phi_a(\alpha) = \{b \in \Phi(\alpha) : T(b) = a\}.$$

Elements of $\Phi(\alpha)$ are called *fragments* of α; elements of $\Phi_a(\alpha)$ are called *a-fragments* of α.

Proposition 13.1. *If α is a countable subset of M, then $\Phi(\alpha)$ and $\Phi_a(\alpha)$ are countable.* □

Proposition 13.2. *(i) $\alpha \subseteq \Phi(\alpha)$.*
(ii) $\Phi(\alpha \cup \beta) = \Phi(\alpha) \cup \Phi(\beta)$.
(iii) $\Phi(\emptyset) = \emptyset$.
(iv) $\Phi(\Phi(\alpha)) = \Phi(\alpha)$.
(v) If $\alpha \subseteq \beta$, then $\Phi(\alpha) \subseteq \Phi(\beta)$. □

Proposition 13.3. *If σ is an M-measure, then $\Phi((\triangle\sigma)(1)) = \bigcup_{x \in \mathrm{Fr}(\omega)} (\triangle\sigma)(x)$.*

Proof. Suppose that $b \in \Phi((\triangle\sigma)(1))$. Choose $c \in {}^{<\omega}M$ such that $b^\frown c \in (\triangle\sigma)(1)$. Say b has length m and c has length n. Then we can write

$$1 = x_0 \dot{+} \cdots \dot{+} x_{m-1} \dot{+} x_m \dot{+} \cdots \dot{+} x_{m+n-1}, \text{ and}$$
$$\langle \sigma(x_0), \ldots, \sigma(x_{m-1}), \sigma(x_m), \ldots \sigma_{m+n-1} \rangle = b^\frown c.$$

Let $y = x_0 \dot{+} \cdots \dot{+} x_{m-1}$. Then $b = \langle \sigma(x_0), \ldots, \sigma(x_{m-1}) \rangle \in (\triangle\sigma)(y)$. This proves $\subseteq$ in the proposition.

Conversely, suppose that $x \in \mathrm{Fr}(\omega)$ and $b \in (\triangle\sigma)(x)$. Say $x = y_0 \dot{+} \cdots \dot{+} y_{m-1}$ and $b = \langle \sigma(y_0), \ldots, \sigma(y_{m-1}) \rangle$. Then $1 = y_0 \dot{+} \cdots \dot{+} y_{m-1} \dot{+} (-x)$, and so

$$\langle \sigma(y_0), \ldots, \sigma(y_{m-1}), \sigma(-x) \rangle \in (\triangle\sigma)(1),$$

and hence $b \in \Phi((\triangle\sigma)(1))$. □

Proposition 13.4. *If σ is an M-measure and $a \in M$, then*

$$\Phi_a((\triangle\sigma)(1)) = \bigcup_{\substack{x \in \mathrm{Fr}(\omega) \\ \sigma(x)=a}} (\triangle\sigma)(x).$$

Proof. Suppose that $b \in \Phi_a((\triangle\sigma)(1))$. Then $T(b) = a$. Choose $c \in {}^{<<\omega}M$ such that $b^\frown c \in (\triangle\sigma)(1)$. Say b has length m and c has length n. Then we can write

$$1 = x_0 \dot{+} \cdots \dot{+} x_{m-1} \dot{+} x_m \dot{+} \cdots \dot{+} x_{m+n-1}, \text{ and}$$
$$\langle \sigma(x_0), \ldots, \sigma(x_{m-1}), \sigma(x_m), \ldots \sigma_{m+n-1} \rangle = b^\frown c.$$

Let $y = x_0 \dot{+} \cdots \dot{+} x_{m-1}$. Then $b = \langle \sigma(x_0), \ldots, \sigma(x_{m-1}) \rangle \in (\triangle\sigma)(y)$. Also, $\sigma(y) = \sigma(x_0) + \cdots + \sigma(x_{m-1}) = T(b) = a$. This proves $\subseteq$ in the proposition.

Conversely, suppose that $x \in \mathrm{Fr}(\omega)$, $\sigma(x) = a$, and $b \in (\triangle\sigma)(x)$. Say $x = y_0 \dot{+} \cdots \dot{+} y_{m-1}$ and $b = \langle \sigma(y_0), \ldots, \sigma(y_{m-1}) \rangle$. Then $1 = y_0 \dot{+} \cdots \dot{+} y_{m-1} \dot{+} (-x)$, and so

$$\langle \sigma(y_0), \ldots, \sigma(y_{m-1}), \sigma(-x) \rangle \in (\triangle\sigma)(1),$$

and hence $b \in \Phi((\triangle\sigma)(1))$. Also,

$$T(b) = \sigma(y_0) + \cdots + \sigma(y_{m-1}) = \sigma(y_0 + \cdots + y_{m-1}) = \sigma(x) = a. \qquad \square$$

Proposition 13.5. *If $\alpha \subseteq {}^{<<\omega}M$ has C.P., then $\Phi(\alpha)$ and $\Phi_a(\alpha)$ have C.P.*

Proof. Assume that $\alpha \subseteq {}^{<<\omega}M$ has C.P. Suppose that $x \in \Phi(\alpha)$ and $x \prec y$. Choose $z \in {}^{<<\omega}M$ so that $x^\frown z \in \alpha$. Say $x \in {}^mM$, $y \in {}^nM$, and $z \in {}^pM$. Since $x \prec y$, there is a $\lambda : m \to n$ such that $\forall j < n[y_j = \sum\{x_i : i < m, \lambda(i) = j\}]$. Let $\mu : m + p \to n + p$ extend λ by setting $\mu(m + k) = n + k$ for all $k < p$. Let $w = y^\frown z$ and $v = x^\frown z$. Then $\forall j < n + p[w_j = \sum\{v_i : i < m + p, \lambda(i) = j\}$. Hence $(x^\frown z) \prec (y^\frown z)$ so, since α has C.P., $y^\frown z \in \alpha$. It follows that $y \in \Phi(\alpha)$.

For $\Phi_a(\alpha)$, we assume in the above argument that in addition $T(x) = a$. By Proposition 8.5, $T(y) = a$. $\square$

Proposition 13.6. *If $\alpha \subseteq {}^{<<\omega}M$ has S.P., then $\Phi(\alpha)$ and $\Phi_a(\alpha)$ have S.P.*

Proof. Assume that $\alpha \subseteq {}^{<<\omega}M$ has S.P. Suppose that $m \in \omega\backslash\{0\}$, $a \in {}^mM$, $a \in \Phi(\alpha)$, and $a_0 \neq o$. Choose $c \in {}^{<<\omega}M$ so that $a^\frown c \in \alpha$. Choose $u, v \in M\backslash\{0\}$ such that $a_0 = u + v$ and $\langle u, v, a_1, \ldots, a_{m-1}, c_0, c_1 \ldots \rangle \in \alpha$. Then $\langle u, v, a_1, \ldots, a_{m-1} \rangle \in \Phi(\alpha)$.

For $\Phi_d(\alpha)$, assuming that $T(a) = d$, we have

$$T(\langle u, v, a_1, \ldots, a_{m-1} \rangle) = u + v + a_1 + a_2 + \cdots + a_{m-1} = a_0 + a_1 + \cdots + a_{m-1} = T(a) = d. \quad \square$$

Lemma 13.7. *Let $\sigma \in \mathscr{M}(M)$ be stable, and let $\alpha = (\triangle\sigma)(1)$. Suppose that $a \in M$, $c \in {}^{<<\omega}M$, and $\langle a \rangle^\frown c \in \alpha$. Then*

(i) $\Phi_a(\alpha) \in \triangle M$.
(ii) $\forall b \in \Phi_a(\alpha)[b^\frown c \in \alpha]$.

Proof. Assume the hypotheses. To prove (i), by Proposition 10.1 it suffices to find $x \in \mathrm{Fr}(\omega)$ such that $\Phi_a(\alpha) = (\triangle\sigma)(x)$. Now by Proposition 13.4,

$$\Phi_a(\alpha) = \Phi_a((\triangle\sigma)(1)) = \bigcup_{\substack{x \in \mathrm{Fr}(\omega) \\ \sigma(x)=a}} (\triangle\sigma)(x)$$

Note that if $x, y \in \mathrm{Fr}(\omega)$ and $\sigma(x) = a = \sigma(y)$, then $(\triangle\sigma)(x) = (\triangle\sigma)(y)$ since σ is stable. Since $\langle a\rangle^\frown c \in \alpha = (\triangle\sigma)(1)$, there is an $x \in \mathrm{Fr}(\omega)$ such that $\sigma(x) = a$. So (i) follows.

Now for (ii), suppose that $b \in \Phi_a(\alpha)$ with $\mathrm{dmn}(b) = m$. Let $\mathrm{dmn}(c) = n$. Since $\langle a\rangle^\frown c \in \alpha = (\triangle\sigma)(1)$, there are pairwise disjoint $x, y_0, \ldots, y_{n-1}$ such that

$$1 = x \dot{+} y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{n-1}$$

and $a = \sigma(x)$ and $c_i = \sigma(y_i)$ for all $i < n$. By the argument in the preceding paragraph, $b \in (\triangle\sigma)(x)$. Say $x = z_0 \dot{+} \cdots \dot{+} z_{m-1}$ and $b = \langle \sigma(z_0), \ldots, \sigma(z_{m-1})\rangle$. Then

$$\begin{aligned} 1 &= z_0 \dot{+} \cdots \dot{+} z_{m-1} \dot{+} y_0 \dot{+} \cdots \dot{+} y_{n-1} \quad \text{and} \\ b^\frown c &= \langle \sigma(z_0), \ldots, \sigma(z_{m-1}), \sigma(y_0), \ldots, \sigma(y_{m-1}\rangle \end{aligned}$$

□

If M is an m-monoid, a countable subset α of ${}^{<<\omega}M$ has the *local refinement property*, L.P., iff the following conditions hold:

(L1) $\forall a \in M[\Phi_a(\alpha) = \emptyset$ or $\Phi_a(\alpha)$ has R.P.].

(L2) $\forall a \in M \forall c \in {}^{<<\omega}M[\langle a\rangle^\frown c \in \alpha \to \forall b \in \Phi_a(\alpha)[b^\frown c \in \alpha]]$.

Proposition 13.8. *If σ is a stable M measure, then $(\triangle\sigma)(1)$ satisfies L.P.*

Proof. By Proposition 13.7. □

Proposition 13.9. *Suppose that α is a countable subset of ${}^{<<\omega}M$ which satisfies C.P., R.P., and L.P. Also suppose that $a \in \alpha$ has domain n, $a \in \alpha$, $q \le n$, $c = \sum_{i<q} a_i$, b has domain m, $b \in \Phi_c(\alpha)$, and $p \ge m$.*

Then there is a d with domain $n \cdot p$ such that $d \in \alpha$, $\forall i < n[a_i = \sum_{j<p} d_{f_{np}(i,j)}]$, and $\langle d_{f_{np}(i,j)} : i < q, j < p\rangle \prec b$.

Proof. Let $e = \sum_{q \le i < n} a_i$.

13.9(1) $\langle c, e\rangle \in \alpha$.

For, let $u = \langle c, e\rangle$. Define $n \to 2$ by setting, for $i < n$,

$$\lambda(i) = \begin{cases} 0 & \text{if } i < q, \\ 1 & \text{if } q \le i < n. \end{cases}$$

Then $c = u_0 = \sum\{a_j : \lambda(j) = 0\}$ and $e = u_1 = \sum\{a_j : \lambda(j) = 1\}$. So $a \prec u$, and 13.9(1) follows by C.P.

Now $\Phi_c(\alpha) \neq \emptyset$ since $b \in \Phi_c(\alpha)$. So $\Phi_c(\alpha)$ has R.P. since α has L.P. Now $a = \langle a_i : i < q\rangle^\frown\langle a_i : q \leq i < n\rangle$, and $a \in \alpha$, so by Proposition 8.3, $\langle a_i : q \leq i < n\rangle^\frown\langle a_i : i < q\rangle \in \alpha$, and so $\langle a_i : q \leq i < n\rangle \in \Phi_e(\alpha)$. Hence $\Phi_e(\alpha) \neq \emptyset$, so $\Phi_e(\alpha)$ has R.P. since α has L.P. By Proposition 13.5, $\Phi_c(\alpha)$ and $\Phi_e(\alpha)$ also have C.P. We now apply Proposition 11.2, with α, m, n, a, b, p replaced by $\Phi_c(\alpha), q, m, a \restriction q, b, p$ to get $c' \in {}^{q\cdot p}M$ such that $c' \in \Phi_c(\alpha)$, $\forall i < q[\sum_{j<p} c'_{f_{qp}(i,j)} = a_i]$ and $c' \prec b$. Now let $g_i = a_{q+i}$ for all $i < n - q$. We now apply Proposition 11.2 with α, m, n, a, b, p replaced by $\Phi_e(\alpha), n - q, m, g, g, p$. This gives $c'' \in {}^{(n-q)\cdot p}M$ such that $c'' \in \Phi_e(\alpha)$ $\forall i < n - q[\sum_{j<p} c''_{f_{(n-q)p}(i,j)} = g_i]$ and $c'' \prec g$.

Now define d with domain $n \cdot p$ by setting, for $i < n$ and $j < p$,

$$d_{f_{np}(i,j)} = \begin{cases} c'_{f_{qp}(i,j)} & \text{if } i < q, \\ c''_{f_{(n-q)p}(i-q,j)} & \text{if } q \leq i < n. \end{cases}$$

Then for all $i < n$ we have

$$\sum_{j<p} d_{f_{np}(i,j)} = \begin{cases} \sum_{j<p} c'_{f_{qp}(i,j)} = a_i & \text{if } i < q, \\ \sum_{j<p} c''_{f_{(n-q)p}(i-q,j)} = g_{i-q} = a_i & \text{if } q \leq i < n. \end{cases}$$

We have $c' \in \Phi_c(\alpha)$, so by 13.9(1) and (L2), $c'^\frown\langle e\rangle \in \alpha$. Hence by Proposition 8.3, $\langle e\rangle^\frown c' \in \alpha$. Since $c'' \in \Phi_e(\alpha)$, it follows by 13.9(1) that $c''^\frown c' \in \alpha$. Then by Proposition 8.3 again we get $d \in \alpha$. Finally, if $i < q$ and $j < p$, then $d_{f_{np}(i,j)} = c'_{f_{qp}(i,j)}$, so $\langle d_{f_{np}(i,j)} : i < q, j < p\rangle = c' \prec b$. □

Proposition 13.10. *Assume the hypotheses of Proposition 13.9, and assume also that α satisfies S.P., and $\forall i < n[a_i \neq 0]$. Then in the conclusion we may assert that also $\forall i < n \forall j < p[d_{f_{np}(i,j)} \neq 0]$.*

Proof. Assume the hypotheses of Proposition 13.9, and assume also that α satisfies S.P., and $\forall i < n[a_i \neq 0]$. By Proposition 13.6, $\Phi_c(\alpha)$ and $\Phi_e(\alpha)$ have S.P. Then by Proposition 11.3, in the proof of Proposition 13.9 we may assume that $\forall i < q \forall j < p[c'_{f_{qp}(i,j)} \neq 0]$ and $\forall i < n - q \forall j < p[c''_{f_{(n-q)p}(i,j)} \neq 0]$. Hence $\forall i < n \forall j < p[d_{f_{np}(i,j)} \neq 0]$. □

Let α be a countable subset of ${}^{<<\omega}M$. An α-tree a is *uniformly dense* iff

$$\forall n \in \omega \forall G \subseteq \mathscr{D}_n \forall b \in \Phi_{T(a(G))}(\alpha) \exists m \geq n[a(G \restriction m) \prec b].$$

Proposition 13.11. *If α is a countable subset of ${}^{<<\omega}M$ and a is a uniformly dense α-tree, then a is dense.*

Proof. Assume that α is a countable subset of ${}^{<<\omega}M$ and a is a uniformly dense α-tree. Let $b \in \alpha$. Then $T(b) = T(\alpha) = T(a_0)$, so $b \in \Phi_{T(a(\mathscr{D}_0))}$, hence there is an $m \geq 0$ such that $a(\mathscr{D}_0 \restriction m) \prec b$. Since $\mathscr{D}_0 \restriction m = \mathscr{D}_m$, this means that $a_m \prec b$. Thus a is dense. □

Lemma 13.12. *If α is a countable subset of ${}^{<<\omega}M$ that satisfies C.P., R.P., and L.P., then there is a uniformly dense α-tree a.*

Proof. Recall from Proposition 8.12 that $\langle T(\alpha)\rangle \in \alpha$. Let $\{b_0, b_1, \ldots\}$ be an enumeration of $\Phi(\alpha)$ with $b_0 = \langle T(\alpha)\rangle$. We now define $m_n \in \omega$ and $a \restriction \bigcup_{r\leq m_n} \mathscr{D}_r$ by induction on n so that the following conditions hold:

13.12(1) $n < m_n$.

13.12(2) $\forall r \leq m_n[a_r \in \alpha]$.

13.12(3) $\forall r, s \leq m_n \forall f \in \mathscr{D}_r[r \leq s \to a(f) = \sum_{f\leq g\in\mathscr{D}_s} a(g)]$.

13.12(4) $\forall k \leq n \forall G \subseteq \mathscr{D}_n[b_k \in \Phi_{T(a(G))}(\alpha) \to a(G \restriction m_n) \prec b_k]$.

Let $m_0 = 1$, $a(\emptyset) = T(\alpha)$, $a(\langle 0\rangle) = 0$, $a(\langle 1\rangle) = T(\alpha)$. Now $\langle a_{lex(i)} : i < 1\rangle = \langle T(\alpha)\rangle \in \alpha$ by Proposition 8.12, and $\langle a(lex(i)) : i < 2\rangle = \langle 0, T(\alpha)\rangle \in \alpha$ by Proposition 8.4 and C.P. Hence 13.12(1)-13.12(4) hold for $n = 0$. Now suppose that 13.12(1)-13.12(4) hold for n. List the set $\{(k, G) : k \leq n+1, G \subseteq \mathscr{D}_{n+1}, b_k \in \Phi_{T(a(G))}(\alpha)\}$ as $\{(k_0, G_0), (k_1, G_1), \ldots, (k_{i-1}, G_{i-1})\}$. Then

13.12(5) There is a $m_0' > m_n$ and an extension of a to $\bigcup_{r\leq m_0'} \mathscr{D}_r$ such that
(a) $\forall r \leq m_0'[\langle a(lex(i)) : i < 2^r\rangle \in \alpha]$.
(b) $\forall r, s \leq m_0' \forall f \in \mathscr{D}_r[r \leq s \to a(f) = \sum_{f\leq g\in\mathscr{D}_s} a(g)]$.
(c) $a(G_0 \restriction m_0') \prec b_{k_0}$.

To prove 13.12(5), first note that $b_{k_0} \in \Phi_{T(a(G_0))}(\alpha)$. Now by 13.12(3) for m_n, since $n < m_n$, $a_{n+1} \in \alpha$. Note that $T(b_{k_0}) = T(a(G_0)) = T(\langle a(lex(i)) : i < |G_0|\rangle$. Let l be such that $n+1 \leq l$ and $\mathrm{dmn}(b_{k_0}) \leq 2^l$. By adjoining 0s to a_{m_n} we can obtain a sequence $e \in \alpha$ of length 2^l. By a permutation of e we obtain $e' \in \alpha$ such that $\langle a(lex(i)) : i < |G_0|\rangle$ is an initial segment of e'. Now we apply Proposition 13.9 with a, n, α, q, p, m, b replaced by $e', 2^l, \alpha, |G_0|, 2^l, \mathrm{dmn}(b_{k_0}), b_{k_0}$ to obtain $d \in \alpha$ of length $2^l \cdot 2^l$ such that $\forall i < 2^l[e_i' = \sum_{j<2^l} d_{f_{2^l 2^l}(i,j)}]$ and $\langle d_{f_{2^l 2^l}(i,j)} : i < |G_0|, j < 2^l\rangle \prec b_{k_0}$. Let $m_0' = 2l$. Let $g : 2^{2l} \to 2^l \times 2^l$ be a bijection. For each $i < 2^{2l}$ let $a(lex(i)) = d_{f_{2^l 2^l}((g(i))_0,(g(i))_1)}$. Now for s such that $m_n < s \leq m_0'$ and $f \in \mathscr{D}_s$ let $a(f) = \sum\{a(k) : f \subseteq k \in \mathscr{D}_{2l}\}$.

Clearly (b) holds, and (a) holds for $r = m_0'$. Now suppose that $m_n < r < m_0'$. For $i < 2^{2l}$ let $\lambda(i) = lex^{-1}(lex(i) \restriction r)$. Then for any $j < 2^r$, $a(lex(j)) = \sum_{lex(i)\subseteq h\in\mathscr{D}_{2l}} a(lex(i)) = \sum\{a(lex(i)) : i < 2^{2l}, \lambda(i) = j\}$. Hence $\langle a(lex(i)) : i < 2^{2i}\rangle \prec \langle a(lex(j)) : j < 2^r\rangle$ and so $\langle a(lex(j)) : j < 2^r\rangle \in \alpha$. Thus (a) holds. Now we claim that $a(G_0 \restriction m_0') \prec \langle d_{f_{2^l 2^l}(i,j)} : i < |G_0|, j < 2^l\rangle$, hence $a(G_0 \restriction m_0') \prec b_{k_0}$, giving (c). For, if $i < |G_0|$ and $j < 2^l$, then

$$d_{f_{2^l 2^l}(i,j)} = \sum\{a(lex(k)) : lex^{-1}(lex(k) \restriction (n+1)) = i\}.$$

Now we repeat this argument for $(k_1, G_1), \ldots, (k_{i-1}, G_{i-1})$, ending up with m_{n+1} satisfying 13.12(1)-13.12(4) for $n+1$.

Finally we get $a : \mathscr{D} \to M$, and this is clearly the desired uniformly dense α-tree. $\square$

Lemma 13.13. *If $\alpha \neq \emptyset$ is a countable subset of ${}^{<<\omega}M$ that satisfies C.P., R.P., L.P. and S.P., then there is a uniformly dense homogeneous α-tree a.*

Proof. We modify the proof of Lemma 13.12 by adding the condition that $a(f) \neq 0$ for all f. Instead of adding 0s in the proof, we split the last element. We apply Proposition 13.10 instead of 13.9. $\square$

Proposition 13.14. *Let $\alpha \in \triangle M$. Then the following are equivalent:*
(i) There is a stable M-measure σ on $\mathrm{Fr}(\omega)$ such that $((\triangle\sigma))(1) = \alpha$.
(ii) α has L.P.

Proof. (i)$\Rightarrow$(ii): Assume (i). Then (ii) holds by Proposition 13.8.

(ii)$\Rightarrow$(i): Assume (ii). Thus α satisfies C.P., R.P., S.P., and L.P. Hence by Proposition 13.13 there is a uniformly dense homogeneous α-tree a.

For each $n \in \omega$ and each $f \in \mathscr{D}_n$ let

13.14(1) $x(f) = \{g \in {}^{\omega}2 : f \subseteq g\}$.

Then $\{x(f) : f \in \mathscr{D}\}$ is a base for $\mathrm{clop}({}^{\omega}2)$, which is a denumerable atomless BA.

13.14(2) If $k \leq i \in \omega$ and $f \in \mathscr{D}_k$, then $x(f) = \bigcup\{x(g) : g \in \mathscr{D}_i, f \subseteq g\}$, and this union is pairwise disjoint.

13.14(3) If $1 = y_0 \dot{+} y_1 \dot{+} \cdots \dot{+} y_{n-1}$ with each y_i clopen, then there is a $k \in \omega$ and a partition $\mathscr{D}_k = G_0 \dot{\cup} G_1 \dot{\cup} \ldots \dot{\cup} G_{n-1}$ such that $\forall j < n[y_j = \sum\{x(f) : f \in G_j\}]$.

In fact, since $\{x(f) : f \in \mathscr{D}\}$ is a base for $\mathrm{clop}({}^{\omega}2)$ and each y_i is clopen and hence compact, there exist finite subsets $\mathscr{D}^i$ of $\mathscr{D}$ for $i < n$ such that $\forall i < n[y_i = \bigcup\{x(f) : f \in \mathscr{D}^i\}]$. Let k be greater than the domain of f, for all $f \in \bigcup_{i<n} \mathscr{D}^i$. By 13.14(2) we may assume that each $\mathscr{D}^i$ is a subset of $\mathscr{D}_k$. Clearly the sets $\mathscr{D}^i$ are pairwise disjoint. For any $f \in {}^{k}2$ the set $x(f)$ is a nonempty set, and hence $x(f) \cap y_i \neq \emptyset$ for some $i < n$. So we must actually have $f \in \mathscr{D}^i$. This proves 13.14(3).

Now let $z \in \mathrm{clop}({}^{\omega}2)$. Then by compactness there is a finite $X \subseteq \mathscr{D}$ such that $z = \bigcup\{x(f) : f \in X\}$. Hence by 13.14(2) there is a $k \in \omega$ and a $G \subseteq \mathscr{D}_k$ such that $z = \sum\{x(f) : f \in G\}$. (disjoint sum) We then define

$$\sigma(z) = \sum\{a(f) : f \in G\}.$$

This definition does not depend on the particular choice of k and G. For, suppose that also $l \in \omega$, $H \subseteq \mathscr{D}_l$, and $z = \sum\{x(f) : f \in H\}$; we claim that $\sum\{a(f) : f \in G\} = \sum\{a(f) : f \in H\}$. For, suppose that $k \leq l$, and let $K = \{f \in \mathscr{D}_l : \exists g \in G[g \subseteq f\}$. Then $z = \sum\{x(f) : f \in K\}$ by (88). Since $x(f) \cap x(g) = \emptyset$ for distinct $f, g \in \mathscr{D}_l$, it follows that $K = H$. Then by 13.14(2) we get

$$\sum\{a(f) : f \in G\} = \sum\{a(g) : g \in K\} = \sum\{a(f) : f \in H\}.$$

Now to show that σ is additive, suppose that $z, z' \in \mathrm{clop}({}^{\omega}2)$ and $z \cdot z' = 0$. Choose k, k', G, G' such that $G \subseteq \mathscr{D}_k$, $G' \subseteq \mathscr{D}_{k'}$, $z = \sum\{x(f) : f \in G\}$, and $z' = \sum\{x(f) : f \in G'\}$; thus $\sigma(z) = \sum\{a(f) : f \in G\}$ and $\sigma(z') = \sum\{a(f) : f \in G'\}$. Wlog $k \leq k'$. Let $H = \{f \in \mathscr{D}_{k'} : \exists g \in G[g \subseteq f]\}$. Then by 13.14(2), $z = \sum\{x(f) : f \in H\}$ and $\sigma(z) =$

$\sum\{a(f) : f \in H\}$. Since $z \cdot z' = 0$ we have $H \cap G' = \emptyset$. Now $z + z' = \sum\{x(f) : f \in H \cup G'\}$, so

$$\sigma(z+z') = \sum\{a(f) : f \in H \cup G'\} = \sum\{a(f) : f \in H\} + \sum\{a(f) : f \in G'\} = \sigma(z) + \sigma(z').$$

Since a is homogeneous, clearly $\sigma(z) = 0$ iff $z = 0$. So $\sigma \in \mathscr{M}(M)$.

Next we show that $(\triangle\sigma)(1) = \alpha$. Suppose that $b \in (\triangle\sigma)(1)$. Say $n \in \omega$, $1 = z_0 \dot{+} z_1 \dot{+} \cdots \dot{+} z_{n-1}$ and $b = \langle \sigma(z_0), \sigma(z_1), \ldots, \sigma(z_{n-1}) \rangle$. (89) gives $k \in \omega$ and $G_0, \ldots, G_{n-1}$ such that $\mathscr{D}_k = G_0 \dot{\cup} G_1 \dot{\cup} \ldots \dot{\cup} G_{n-1}$ and $\forall j < n[z_j = \sum\{x(f) : f \in G_j\}$. So $\forall j < n[\sigma(z_j) = \sum\{a(f) : f \in G_j\}$. Define $\lambda : 2^k \to n$ by $\lambda(m) = j$ iff $lex(m) \in G_j$. Then $\forall j < n[\sigma(z_j) = \sum\{a(lex(m)) : m < 2^\kappa, \lambda(m) = j\}$. Hence $a_k \prec b$, hence $b \in \alpha$ by (63) and C.P. This shows that $(\triangle\sigma)(1) \subseteq \alpha$.

For the other inclusion, first note that

13.14(4) $\forall f \in \mathscr{D}[\sigma(x(f)) = a(f)]$

In fact, let $f \in \mathscr{D}$. Let $z = x(f)$ and $G = \{f\}$. Then by definition $\sigma(x(f)) = a(f)$.

13.14(5) $\forall k \in \omega[a_k \in (\triangle\sigma)(1)]$.

For, let $k \in \omega$. Then

$$a_k = a(\mathscr{D}_k) = \langle a(lex(i)) : i < 2^k \rangle = \langle \sigma(x(lex(i))) : i < 2^k \rangle \in (\triangle\sigma)(1)$$

since $1 = \sum\{x(lex(i)) : i < 2^k\}$, a disjoint sum. So 13.14(5) holds.

Now take any $b \in \alpha$. Since a is dense there is a $k \in \omega$ such that $a_k \prec b$. Now $(\triangle\sigma)(1) \in \triangle M$ by Proposition 10.1, so $b \in (\triangle\sigma)(1)$ by 13.14(5) and C.P.

So $(\triangle\sigma)(1) = \alpha$.

It remains only to show that σ is stable. Suppose that $y, z \in \mathrm{Fr}(\omega)$ and $\sigma(y) = \sigma(z)$. Choose $n \in \omega$ and $G \subseteq \mathscr{D}_n$ so that $y = \sum_{f \in G} x(f)$. Then $\sigma(y) = \sum_{f \in G} \sigma(x(f)) = \sum_{f \in G} a(f)$ by 13.14(4). Thus $\sigma(y) = T(a(G))$. Now suppose that $b \in ((\triangle\sigma))(z)$. Then $T(b) = T(((\triangle\sigma)(z))) = \sigma(z)$ (by Proposition 10.2) $= \sigma(y) = T(a(G))$. Since a is uniformly dense, there is an $m \geq n$ such that $a(G \uparrow m) \prec b$. Now $\sum_{f \in G \uparrow m} x(f) = \sum_{f \in G} x(f) = y$, so $a(G \uparrow m) = \langle \sigma(x(lex(i))) : i < |G \uparrow m| \rangle \in ((\triangle\sigma))(y)$. Hence $b \in ((\triangle\sigma))(y)$ by C.P. This proves that $((\triangle\sigma))(z) \subseteq ((\triangle\sigma))(y)$. The other inclusion follows by symmetry, so $((\triangle\sigma))(z) = ((\triangle\sigma))(y)$. Thus σ is stable. $\square$

14. Iterated derivatives

Proposition 14.1. *Let M be an m-monoid. Then there are functions $\langle \triangle^\zeta M : \zeta < \omega_1 \rangle$ and $\langle T^\zeta_\eta : \eta \leq \zeta < \omega_1 \rangle$ with the following properties, for any $\zeta < \omega_1$ and $\eta \leq \zeta$:*

14.1(1) If $\zeta = 0$, then $\triangle^\zeta M = M$.
14.1(2) T^ζ_ζ is the identity on $\triangle^\zeta M$.
14.1(3) If $\zeta = \eta + 1$, then $\triangle^\zeta M = \triangle(\triangle^\eta M)$,
14.1(4) If $\zeta = \eta + 1$ and $\xi \leq \eta$, then $T^\zeta_\xi = T^\eta_\xi \circ T$, with $T : \triangle(\triangle^\xi M) \to \triangle^\xi M$.

14.1(5) If $\xi \leq \eta \leq \zeta$, then $T^\eta_\xi \circ T^\zeta_\eta = T^\zeta_\xi$
14.1(6) $\forall \eta \leq \zeta[\triangle^\eta M$ is an m-monoid$]$.
14.1(7) If ζ is a limit ordinal, then

$$\triangle^\zeta M = \{a : \mathrm{dmn}(a) = \zeta, \forall \eta < \zeta[a_\eta \in \triangle^\eta M], \forall \xi, \eta < \zeta[\xi < \eta \to T^\eta_\xi(a_\eta) = a_\xi]\}.$$

14.1(8) If ζ is a limit ordinal, then $\forall \eta < \zeta \forall a \in \triangle^\zeta M[T^\zeta_\eta(a) = a_\eta]$.
14.1(9) If $\eta \leq \zeta < \omega_1$ then $T^\zeta_\eta : \triangle^\zeta M \to \triangle^\eta M$.
14.1(10) if ζ is a limit ordinal, $x, y \in \triangle^\zeta M$, and $T^\zeta_\eta(x) = T^\zeta_\eta(y)$ for all $\eta < \zeta$, then $x = y$.

Proof. We construct $\triangle^\zeta M$ and T^ζ_η by recursion on ζ. For $\zeta = 0$ we take 14.1(1) and 14.1(2) as definitions. Clearly the other conditions hold. For the successor case we take 14.1(1)–14.1(4) as definitions. For 14.1(5), with $\zeta = \eta + 1$, assume that $\xi \leq \rho \leq \zeta$. If $\rho = \zeta$ the conclusion is clear. If $\rho < \zeta$, then $T^\rho_\xi \circ T^\zeta_\rho = T^\rho_\xi \circ T^\eta_\rho \circ T = T^\eta_\xi \circ T = T^\zeta_\xi$. 14.1(6) and 14.1(7) are clear.

Now for ζ limit, we use 14.1(2), 14.1(7), and 14.1(8) as definitions. Then 14.1(9) is clear. To check 14.1(5), suppose that $\xi \leq \eta < \zeta$. Then $\forall a \in \triangle^\zeta M[T^\eta_\xi \circ T^\zeta_\eta(a) = T^\eta_\xi(a_\eta) = a_\xi = T^\zeta_\xi(a)]$. To check 14.1(6), suppose $a, b \in \triangle^\zeta M$. Then $\forall \eta < \zeta[(a+b)_\eta = a_\eta + b_\eta \in \triangle^\eta M]$. Also, if $\xi < \eta < \zeta$, then $T^\eta_\xi((a+b)_\eta) = T^\eta_\xi(a_\eta + b_\eta) = T^\eta_\xi(a_\eta) + T^\eta_\xi(b_\eta) = a_\xi + b_\xi = (a+b)_\xi$. So this shows that $a + b \in \triangle^\zeta M$. Now the monoid conditions are clear.

It remains to check 14.1(10). So assume that ζ is a limit ordinal, $x, y \in \triangle^\zeta M$, and $T^\zeta_\eta(x) = T^\zeta_\eta(y)$ for all $\eta < \zeta$. Then for all $\eta < \zeta$, $x_\eta = T^\zeta_\eta(x) = T^\zeta_\eta(y) = y_\eta$. So $x = y$. □

Proposition 14.2. *Let M be an m-monoid and $\sigma \in \mathscr{M}(M)$. Then for each $\zeta < \omega_1$ there is a unique $\triangle^\zeta \sigma \in \mathscr{M}(\triangle^\zeta M)$ such that the following conditions hold:*

14.2(1) For $\zeta = 0$, $\triangle^\zeta \sigma = \sigma$.
14.2(2) For $\zeta = \xi + 1$, $\triangle^\zeta \sigma = \triangle(\triangle^\xi \sigma)]$.
14.2(3) $\forall \eta \leq \zeta < \omega_1[T^\zeta_\eta \circ \triangle^\zeta \sigma = \triangle^\eta \sigma]$.
14.2(4) For ζ limit, $\forall x \in \mathrm{Fr}(\omega)[(\triangle^\zeta \sigma)(x) = \langle (\triangle^\eta \sigma)(x) : \eta < \zeta \rangle$.

Proof. We define $\triangle^\zeta \sigma$ by recursion on ζ. For $\zeta = 0$ we take 14.2(1) as a definition. For $\zeta = \xi + 1$ we take 14.2(2) as a definition. Then clearly 14.2(2) holds for $\eta = \zeta$, while for $\eta < \zeta$, $T^\zeta_\eta \circ \triangle^\zeta \sigma = T^\xi_\eta \circ T \circ \triangle(\triangle^\xi \sigma) = T^\xi_\eta \circ \triangle^\xi \sigma$ (by Proposition 10.2) $= \triangle^\eta \sigma$. For ζ limit we take 14.2(3) as a definition. We check that $\triangle^\zeta \sigma \in \mathscr{M}(\triangle^\zeta M)$. Take any $x \in \mathrm{Fr}(\omega)$, and let $a = \langle (\triangle^\eta \sigma)(x) : \eta < \zeta \rangle$. Then a has domain ζ, and for any $\eta < \zeta$, $a_\eta = \triangle^\eta \sigma(x) \in \triangle^\eta M$. If $\xi < \eta < \zeta$, then $T^\eta_\xi(a_\eta) = T^\eta_\xi((\triangle^\eta \sigma)(x)) = (T^\eta_\xi \circ \triangle^\eta \sigma)(x) = (\triangle^\xi \sigma)(x) = a_\xi$. This checks that $(\triangle^\zeta \sigma)(x) \in \triangle^\zeta M$. Hence clearly $\triangle^\zeta \sigma \in \mathscr{M}(\triangle^\zeta M)$. Finally we check 14.2(2). Suppose that $\eta < \zeta$. Then for any $x \in \mathrm{Fr}(\omega)$,

$$(T^\zeta_\eta \circ \triangle^\zeta \sigma)(x) = T^\zeta_\eta((\triangle^\zeta \sigma)(x)) = T^\zeta_\eta(\langle (\triangle^\rho \sigma)(x) : \rho < \zeta \rangle) = (\triangle^\eta \sigma)(x) \qquad \square$$

Proposition 14.3. *If k is an automorphism of $\mathrm{Fr}(\omega)$, $\sigma = \tau \circ k$, and ξ is any ordinal, then $\triangle^\xi \sigma = (\triangle^\xi \tau) \circ k$.*

Proof. Induction on ξ. It is clear for $\xi = 0$. Assume that it holds for ξ. Then by Proposition 14.2(2), $\triangle^{\xi+1}\sigma = \triangle(\triangle^{\xi}\sigma) = \triangle(\triangle^{\xi}\tau) \circ k$ (by Proposition 10.4) $= \triangle^{\xi+1}\tau \circ k$. Now suppose that ξ is a limit ordinal and it holds for all $\eta < \xi$. Then for any $x \in \mathrm{Fr}(\omega)$,

$$(\triangle^{\xi}\sigma)(x) = \langle(\triangle^{\eta}\sigma)(x) : \eta < \xi\rangle = \langle(\triangle^{\eta}\tau)(k(x)) : \eta < \xi\rangle = (\triangle^{\xi}\tau)(k(x)). \qquad \square$$

15. The depth of measures

Recall the definition of stable measure from section 12.

Proposition 15.1. *If M is an m-monoid and $\sigma \in \mathscr{M}(M)$, then there is a smallest ordinal $d(\sigma) < \omega_1$ such that $\forall\zeta \geq d(\sigma)[\triangle^{\zeta}\sigma$ is stable$]$.*

Proof. For each $\zeta < \omega_1$ let $E_\zeta = \{(x, y) \in \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega) : \triangle^{\zeta}\sigma(x) = \triangle^{\zeta}(y)\}$. Clearly E_ζ is an equivalence relation on $\mathrm{Fr}(\omega)$.

15.1(1) If $\eta \leq \zeta < \omega_1$, then $E_\zeta \subseteq E_\eta$.

In fact, suppose that $\eta \leq \zeta < \omega_1$ and $(x, y) \in E_\zeta$. By Proposition 14.2(3), $T^{\zeta}_{\eta} \circ \triangle^{\zeta}\sigma = \triangle^{\eta}\sigma$. Hence $(\triangle^{\eta}\sigma)(x) = T^{\zeta}_{\eta}((\triangle^{\zeta}\sigma)(x)) = T^{\zeta}_{\eta}((\triangle^{\zeta}\sigma)(y)) = (\triangle^{\eta})(y)$, so that $(x, y) \in E_\eta$. Since $\mathrm{Fr}(\omega)$ is countable, there is a smallest $d(\sigma) < \omega_1$ such that $E_\zeta = E_{d(\sigma)}$ for all $\zeta \geq d(\sigma)$. Hence if $\zeta \geq d(\sigma)$, $x, y \in \mathrm{Fr}(\omega)$, and $(\triangle^{\zeta}\sigma)(x) = (\triangle^{\zeta}\sigma)(y)$, then $\triangle((\triangle^{\zeta}\sigma)(x) = (\triangle^{\zeta+1}\sigma)(x) = (\triangle^{\zeta+1}\sigma)(y) = \triangle((\triangle^{\zeta}\sigma)(y)$. $\square$

$d(\sigma)$ is the *depth* of σ.

Proposition 15.2. *If M is an m-monoid and $\sigma, \tau \in \mathscr{M}(M)$, then the following are equivalent:*

(i) $\sigma = \tau \circ k$ for some automorphism k of $\mathrm{Fr}(\omega)$.
(ii) $\forall\zeta < \omega_1[(\triangle^{\zeta}\sigma)(1) = (\triangle^{\zeta}\tau)(1)]$.
(iii) There is a countable ordinal $\zeta \geq d(\sigma), d(\tau)$ such that $(\triangle^{\zeta+1}\sigma)(1) = (\triangle^{\zeta+1}\tau)(1)$.

Proof. (i)⇒(ii): Assume that k is an automorphism of $\mathrm{Fr}(\omega)$ and $\sigma = \tau \circ k$.

15.2(1) $\forall\zeta < \omega_1[(\triangle^{\zeta}\sigma) = (\triangle^{\zeta}\tau) \circ k]$,

We prove 15.2(1) by induction on ζ. $(\triangle^{0}\sigma) = \sigma = \tau \circ k = (\triangle^{0}\tau) \circ k$. Assume that $(\triangle^{\zeta}\sigma) = (\triangle^{\zeta}\tau) \circ k$. Then by Proposition 10.4, $(\triangle^{\zeta+1}\sigma) = (\triangle(\triangle^{\zeta}\sigma)) = (\triangle(\triangle^{\zeta}\tau)) \circ k = (\triangle^{\zeta+1}\tau) \circ k$. Now assume that ζ is a limit ordinal, and $(\triangle^{\eta}\sigma) = (\triangle^{\eta}\tau) \circ k$ for every $\eta < \zeta$. Then for every $x \in \mathrm{Fr}(\omega)$ and every $\eta < \zeta$, $(\triangle^{\eta}\sigma)(x) = ((\triangle^{\eta}\tau) \circ k)(x) = (\triangle^{\eta}\tau(k(x))$. Thus

$$(\triangle^{\zeta}\sigma)(x) = \langle(\triangle^{\eta}\sigma)(x) : \eta < \zeta\rangle = \langle(\triangle^{\eta}\tau)(k(x)) : \eta < \zeta\rangle = (\triangle^{\zeta}(k(x)).$$

Thus 15.2(1) holds. Hence (ii) follows.

(ii)⇒(iii): obvious.

(iii)⇒(i): Assume (iii). By Proposition 12.2 applied to $(\triangle(\triangle^{\zeta}\sigma))(1) = (\triangle^{\zeta+1}\sigma)(1) = (\triangle^{\zeta+1}\tau)(1) = (\triangle(\triangle^{\zeta}\tau))(1)$, there is an automorphism k of $\mathrm{Fr}(\omega)$ such that $(\triangle^{\zeta}\sigma) = (\triangle^{\zeta}\tau) \circ k$. Hence by Proposition 14.2(3), $\sigma = T^{\zeta}_{0} \circ \triangle^{\zeta}\sigma = T^{\zeta}_{0} \circ (\triangle^{\zeta}\tau) \circ k = \tau \circ k$. $\square$

16. The Boolean hierarchy

Recall that $\mathscr{W}$ is ω_1 with a new o added, and $\mathscr{M}$ is $\mathscr{M}(\mathscr{W})$.

For each ordinal ζ, let $\mathscr{K}^\zeta = \{(\triangle^{\zeta+1}\sigma)(1) : \sigma \in \mathscr{M}, d(\sigma) \leq \zeta\}$. The system of sets $\langle \mathscr{K}^\zeta : \zeta < \omega_1 \rangle$ is called the *Boolean heirarchy.*

Proposition 16.1. *Define $\sigma \equiv \tau$ iff $\sigma, \tau \in \mathscr{M}$ and there is an automorphism k of* $\mathrm{Fr}(\omega)$ *such that $\sigma = \tau \circ k$. Then $\equiv$ is an equivalence relation on $\mathscr{M}$.* □

Proposition 16.2. *For all $\sigma, \tau \in \mathscr{M}$ the following are equivalent:*

(i) $\sigma \equiv \tau$.

(ii) $\forall \zeta > d(\sigma), d(\tau)[(\triangle^\zeta \sigma)(1) = (\triangle^\zeta \tau)(1)]$.

Proof. By Proposition 15.2. □

Proposition 16.3. *For $\eta \leq \zeta < \omega_1$ there is an injection $\Sigma^\zeta_\eta : \mathscr{K}^\eta \to \mathscr{K}^\zeta$ such that*

16.3(1) If $\xi \leq \eta \leq \zeta < \omega_1$, then $\Sigma^\zeta_\eta \circ \Sigma^\eta_\xi = \Sigma^\zeta_\xi$.

16.3(2) If $\eta \leq \zeta$, then $T^{\zeta+1}_{\eta+1} \circ \Sigma^\zeta_\eta$ is the identity on $\mathscr{K}_\eta$.

16.3(3) If $\rho \leq \eta \leq \zeta$, then $\mathrm{rng}(\Sigma^\zeta_\rho) \subseteq \mathrm{rng}(\Sigma^\zeta_\eta)$.

Proof. Assume that $\eta \leq \zeta < \omega_1$. Take any $\sigma \in \mathscr{M}$ with $d(\sigma) \leq \eta$. Define $\Sigma^\zeta_\eta((\triangle^{\eta+1}\sigma)(1)) = (\triangle^{\zeta+1}\sigma)(1)$. Then Σ^ζ_η is well-defined. For, suppose that also $\tau \in \mathscr{M}$, $d(\tau) \leq \eta$, and $(\triangle^{\eta+1}\sigma)(1) = (\triangle^{\eta+1}\tau)(1)$. Then by Proposition 16.2, $(\triangle^{\zeta+1}\sigma)(1) = (\triangle^{\zeta+1}\tau)(1)$.

Σ^ζ_η is one-one. For suppose that $\sigma, \tau \in \mathscr{M}$, $d(\sigma), d(\tau) \leq \eta$, and $(\triangle^{\zeta+1}\sigma)(1) = (\triangle^{\zeta+1}\tau)(1)$. Then by Proposition 14.2,

$$(\triangle^{\eta+1}\sigma)(1) = T^{\zeta+1}_{\eta+1}((\triangle^{\zeta+1}\sigma)(1)) = T^{\zeta+1}_{\eta+1}((\triangle^{\zeta+1}\tau)(1)) = (\triangle^{\eta+1}\tau)(1).$$

Now (16.3(1) is obvious.

For (16.3(2), suppose that $\eta \leq \zeta$, $\sigma \in \mathscr{M}$, and $d(\sigma) \leq \eta$. Then, using Proposition 14.2,

$$T^{\zeta+1}_{\eta+1}(\Sigma^\zeta_\eta((\triangle^{\eta+1}\sigma)(1))) = T^{\zeta+1}_{\eta+1}((\triangle^{\zeta+1}\sigma)(1)) = (\triangle^{\eta+1}\sigma)(1).$$

For 16.3(3), $\mathrm{rng}(\Sigma^\zeta_\rho) = \{(\triangle\sigma)(1) : d(\sigma) \leq \rho\} \subseteq \{(\triangle\sigma)(1) : d(\sigma) \leq \eta\} = \mathrm{rng}(\Sigma^\zeta_\eta)\}$. □

Now let

$$\begin{aligned}\mathscr{K} = \{a : \exists \eta < \omega_1 [&a \text{ is a function with domain } [\eta, \omega_1) \text{ and} \\ &\forall \zeta \in [\eta, \omega_1)[a_\zeta \in \mathscr{K}^\zeta] \text{ and } \forall \xi, \zeta \in [\eta, \omega_1)[\xi \leq \zeta \to \Sigma^\zeta_\xi(a_\xi) = a_\zeta]\}\end{aligned}$$

For any $\eta < \omega_1$ define Σ'^η with domain $\mathscr{K}^\eta$: for each $a \in \mathscr{K}^\eta$, $\Sigma'^\eta(a)$ is the function with domain $[\eta, \omega_1)$ such that for any $\zeta \in [\eta, \omega_1)$, $(\Sigma'^\eta(a))(\zeta) = \Sigma^\zeta_\eta(a)$.

Proposition 16.4. *For each $\eta < \omega_1$, Σ'^η is an injection from $\mathscr{K}^\eta$ into $\mathscr{K}$, and $\mathscr{K} = \bigcup_{\eta<\omega_1} \mathrm{rng}(\Sigma'^\eta)$. Moreover, if $\xi \leq \eta < \omega_1$ then $\Sigma'^\eta \circ \Sigma^\eta_\xi = \Sigma'^\xi$.*

Proof. Suppose that $a \in \mathscr{K}^\eta$. Then $\Sigma'^\eta(a)$ is a function with domain $\mathscr{K}^\eta$,

$$\forall \zeta \in [\eta, \omega_1)[(\Sigma'^\eta(a))(\zeta) = \Sigma^\zeta_\eta(a) \in \mathscr{K}^\zeta] \text{ and}$$
$$\forall \xi, \rho \in [\eta, \omega_1)[\xi \leq \rho \to \Sigma^\rho_\xi((\Sigma'^\eta(a))(\xi) = \Sigma^\rho_\xi(\Sigma^\xi_\eta(a)) = \Sigma^\rho_\eta(a) = (\Sigma'^\eta(a))(\rho).$$

Thus $\Sigma'^\eta(a) \in \mathscr{K}$.

If a, b are distinct elements of $\mathscr{K}^\eta$, then $(\Sigma'^\eta(a))(\eta) = \Sigma^\eta_\eta(a) = a \neq b = \Sigma^\eta_\eta(b) = (\Sigma'^\eta(b))(\eta)$. So $\Sigma'^\eta(a) \neq \Sigma'^\eta(b)$. Thus Σ'^η is an injection from $\mathscr{K}^\eta$ into $\mathscr{K}$. Now suppose that $a \in \mathscr{K}$. Choose $\eta < \omega_1$ such that a is a function with domain $[\eta, \omega_1)$ and $\forall \zeta \in [\eta, \omega_1)[a_\zeta \in \mathscr{K}^\zeta]$ and $\forall \xi, \zeta \in [\eta, \omega_1)[\xi \leq \zeta \to \Sigma^\zeta_\xi(a_\xi) = a_\zeta]$. Let $b = a_\eta$. Thus $b \in \mathscr{K}^\eta$. We claim that $a = \Sigma'^\eta(b)$. In fact, take any $\zeta \in [\eta, \omega_1)$. Then $(\Sigma'^\eta(b))(\zeta) = \Sigma^\zeta_\eta(b) = \Sigma^\zeta_\eta(a_\eta) = a_\zeta$. $\square$

Now we define $\langle L^\zeta : \zeta < \omega_1 \rangle$ and $\langle \rho^\zeta : \zeta < \omega_1 \rangle$ by recursion, so that:

(Z1) $L^0 \subseteq L^1 \subseteq \cdots$;
(Z2) $\forall \zeta < \omega_1[\rho^\zeta$ is a bijection from $\mathscr{K}^\zeta$ onto $L^\zeta]$;
(Z3) $\forall \eta, \zeta < \omega_1[\eta < \zeta \to \rho^\zeta \circ \Sigma^\zeta_\eta = \rho^\eta]$.

Let $L^0 = \mathscr{K}^0$ and let ρ^0 be the identity on $\mathscr{K}^0$. Suppose that L^ζ and ρ^ζ have been defined so that (Z1)–(Z3) hold. Note that $\rho^\zeta \circ (\Sigma^{\zeta+1}_\zeta)^{-1}$ is a bijection from $\mathrm{rng}(\Sigma^{\zeta+1}_\zeta)$ onto L^ζ. Let $M = \{(L^\zeta, x) : x \in \mathscr{K}^{\zeta+1} \backslash \mathrm{rng}(\Sigma^{\zeta+1}_\zeta)\}$. Thus $|M| = |\mathscr{K}^{\zeta+1} \backslash \mathrm{rng}(\Sigma^{\zeta+1}_\zeta)|$ and $M \cap L^\zeta = \emptyset$. Let $L^{\zeta+1} = L^\zeta \cup M$ and $\rho^{\zeta+1} = \rho^\zeta \circ (\Sigma^{\zeta+1}_\zeta)^{-1} \cup k$, where k is a bijection from $\mathscr{K}^{\zeta+1} \backslash \mathrm{rng}(\Sigma^{\zeta+1}_\zeta)$ onto M. Thus $L^\zeta \subseteq L^{\zeta+1}$ and $\rho^{\zeta+1}$ is a bijection from $\mathscr{K}^{\zeta+1}$ onto $L^{\zeta+1}$. Moreover, if $\eta \leq \zeta$, then

$$\rho^{\zeta+1} \circ \Sigma^{\zeta+1}_\eta = \rho^{\zeta+1} \circ \Sigma^{\zeta+1}_\zeta \circ \Sigma^\zeta_\eta = \rho^\zeta \circ \Sigma^\zeta_\eta = \rho^\eta.$$

This is illustrated by the following commutative diagram:

$$\begin{tikzcd}
L^\zeta \arrow[r, phantom, "\subseteq"] & L^{\zeta+1} \\
\mathscr{K}^\zeta \arrow[u, "\rho^\zeta"] \arrow[r, "\Sigma^{\zeta+1}_\zeta"'] & \mathscr{K}^{\zeta+1} \arrow[u, "\rho^{\zeta+1}"']
\end{tikzcd}$$

Now suppose that $\zeta < \omega_1$ is a limit ordinal. Then $\mathscr{K}^\zeta = \bigcup_{\eta \leq \zeta} \mathrm{rng}(\Sigma^\zeta_\eta)$. In fact, let $\sigma \in \mathscr{M}$ with $d(\sigma) \leq \zeta$; so $(\triangle^{\zeta+1}\sigma)(1) \in \mathscr{K}^\zeta$. Then $\Sigma^\zeta_{d(\sigma)}((\triangle^{d(\sigma)+1})(1)) = (\triangle^{\zeta+1}\sigma)(1)$.

Now let $N = \bigcup_{\eta<\zeta} L^\eta$, and let $M = \{(N, x) : x \in \mathscr{K}^\zeta \backslash \bigcup_{\eta<\zeta} \mathrm{rng}(\Sigma^\zeta_\eta)\}$. Let $L^\zeta = N \cup M$. We now define ρ^ζ with domain $\mathscr{K}^\zeta$. Take any $x \in \mathscr{K}^\zeta$. Say $x = (\triangle^{\zeta+1}\sigma)(1)$ with $d(\sigma) \leq \zeta$. If $d(\sigma) < \zeta$, note that $\Sigma^\zeta_{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1)) = (\triangle^{\zeta+1}\sigma)(1) = x$, and let

$\rho^\zeta(x) = \rho^{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1))$. If $d(\sigma) = \zeta$ let $\rho^\zeta(x) = k(x)$, where k is a bijection from $\mathscr{K}^\zeta \setminus \bigcup_{\eta<\zeta} \mathrm{rng}(\Sigma^\zeta_\eta)$ onto M.

Now we check (Z3) for ζ. Suppose that $\eta < \zeta$, and let $x \in \mathscr{K}^\eta$. Say $x = (\triangle^{\eta+1}\sigma)(1)$ with $d(\sigma) \leq \eta$. Then

$$\begin{aligned}\rho^\zeta(\Sigma^\zeta_\eta(x)) &= \rho^\zeta(\Sigma^\zeta_\eta((\triangle^{\eta+1}\sigma)(1))) = \rho^\zeta((\triangle^{\zeta+1}\sigma)(1)) = \rho^{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1)) \\ &= \rho^\eta(\Sigma^\eta_{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1))) = \rho^\eta((\triangle^{\eta+1}\sigma)(1)) = \rho^\eta(x).\end{aligned}$$

Next, ρ^ζ is one-one. For, suppose that $x, y \in \mathscr{K}^\zeta$, $\rho^\zeta(x) = \rho^\zeta(y)$, and $x \neq y$. Then we must have $x, y \in \bigcup_{\eta<\zeta} \mathrm{rng}(\Sigma^\zeta_\eta)$. Say $x = \Sigma^\zeta_\eta((\triangle^{\eta+1}\sigma)(1))$ and $y = \Sigma^\zeta_\tau((\triangle^{\tau+1}\tau)(1))$, with $\eta, \tau < \zeta$. Then $x = (\triangle^{\zeta+1}\sigma)(1)$ and $y = (\triangle^{\zeta+1}\tau)(1)$. Hence $\rho^\zeta(x) = \rho^{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1))$ and $\rho^\zeta(y) = \rho^{d(\tau)}((\triangle^{d(\tau)+1}\tau)(1))$. Say $d(\sigma) \leq d(\tau)$. Then

$$\begin{aligned}\rho^{d(\tau)}(\Sigma^{d(\tau)}_{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1))) &= \rho^{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1))) \\ &= \rho^\zeta(x) = \rho^\zeta(y) \\ &= \rho^{d(\tau)}((\triangle^{d(\tau)+1}\tau)(1))\end{aligned}$$

Hence $(\triangle^{d(\tau)+1}\sigma)(1) = \Sigma^{d(\tau)}_{d(\sigma)}((\triangle^{d(\sigma)+1}\sigma)(1)) = (\triangle^{d(\tau)+1}\tau)(1)$. So

$$y = (\triangle^{\zeta+1}\tau)(1) = \Sigma^\zeta_{d(\tau)}((\triangle^{d(\tau)+1}\tau)(1) = \Sigma^\zeta_{d(\tau)}((\triangle^{d(\tau)+1}\sigma)(1)) = (\triangle^{\zeta+1}\sigma(1)) = x,$$

contradiction.

Next, ρ^ζ maps onto L^ζ. In fact, if $x \in L^\zeta$ then there are two cases.

Case 1. $x \in N$. Say $\eta < \zeta$ and $x \in L^\eta$. Then there is a $y \in K^\eta$ such that $\rho^\eta(y) = x$. Say $y = (\triangle^{\eta+1}\sigma)(1)$ with $d(\sigma) \leq \eta$. Then $\Sigma^\zeta_\eta(y) \in \mathscr{K}^\zeta$, and

$$\rho^\zeta(\Sigma^\zeta_\eta(y)) = \rho^\zeta(\Sigma^\zeta_\eta((\triangle^{\eta+1}\sigma)(1)) = \rho^\eta((\triangle^{\eta+1}\sigma)(1)) = \rho^\eta(y) = x.$$

Case 2. $x \in M$. Obviously there is a $y \in K^\zeta$ such that $\rho^\zeta(y) = x$.

Thus ρ^ζ is a bijection L^ζ.

Hence for ζ limit we have the following diagram:

$$\begin{tikzcd} L^\eta \arrow[r, phantom, "\subseteq \cdots"] & L^\zeta \\ \mathscr{K}^\eta \arrow[u, "\rho^\eta"] \arrow[r, "\Sigma^\zeta_\eta"'] & \mathscr{K}^\zeta \arrow[u, "\rho^\zeta"'] \end{tikzcd}$$

Finally, define $\tau : \mathscr{K} \to \bigcup_{\eta<\omega_1} L^\eta$ by defining, for any $a \in \mathscr{K}$, with $\mathrm{dmn}(a) = [\eta, \omega_1)$, $\tau(a) = \rho^\eta(a_\eta)$. Then for any $a \in \mathscr{K}^\eta$ we have

$$\tau(\Sigma'^\eta(a)) = \tau(\langle \Sigma^\zeta_\eta(a) : \zeta \in [\eta, \omega_1)\rangle) = \rho^\eta(\Sigma^\zeta_\eta(a)).$$

This gives the following diagram:

$$\begin{tikzcd}
L^\eta & \subseteq \cdots & \bigcup_{\eta<\omega_1} L^\eta \\
\mathscr{K}^\eta \arrow[u, "\rho^\eta"] \arrow[rr, "\Sigma'^\eta"'] & & \mathscr{K} \arrow[u, "\tau"']
\end{tikzcd}$$

17. The heirarchy property

Lemma 17.1. $\mathscr{K}^\zeta \subseteq \triangle(\triangle^\zeta W)$.

Proof. Let $x \in \mathscr{K}^\zeta$. Then there is a $\sigma \in \mathscr{M}(W)$ such that $d(\sigma) \leq \zeta$ and $x = (\triangle^{\zeta+1}\sigma)(1)$. By Proposition 14.2, $x \in \triangle^{\zeta+1}W$, and by Proposition 14.2(103), $\triangle^{\zeta+1}W = \triangle(\triangle^\zeta W)$. $\square$

Lemma 17.2. *Let $\alpha \in \mathscr{K}^\zeta$, and suppose that $a \in \triangle^\zeta W$ with $\langle a\rangle \in \Phi(\alpha)$. Also suppose that $\eta < \zeta$ and $c \in T^\zeta_{\eta+1}(a)$. Then there is a $b \in \Phi_a(\alpha)$ such that $c = T^\zeta_\eta \circ b$.*

Proof. By definition of $\mathscr{K}^\zeta$, there is a $\sigma \in \mathscr{M}$ with $d(\sigma) \leq \zeta$ such that $\alpha = (\triangle^{\zeta+1}\sigma)(1)$. By Proposition 15.1, $\triangle^\zeta\sigma$ is stable. Now by Proposition 17.1 $\alpha \in \triangle(\triangle^\zeta W)$, so by definition $\alpha \subseteq {}^{<<\omega}(\triangle^\zeta W)$ Now $\langle a\rangle \in \Phi(\alpha)$, so there is a $b \in {}^{<<\omega}(\triangle^\zeta W)$ such that $\langle a\rangle^\frown b \in \alpha = (\triangle^{\zeta+1}\sigma)(1) = (\triangle(\triangle^\zeta\sigma))(1)$. So if b has domain m we can write $1 = x + y_0 + \cdots + y_{m-1}$ with $(\triangle^\zeta\sigma)(x) = a$ and $(\triangle^\zeta\sigma)(y_i) = b_i$ for all $i < m$. Now $c \in T^\zeta_{\eta+1}(a) = T^\zeta_{\eta+1}((\triangle^\zeta\sigma)(x)) = (\triangle^{\eta+1}\sigma)(x) = (\triangle(\triangle^\eta\sigma))(x)$. Hence we can write $x = d_0 + \cdots + d_{n-1}$ with $c = \langle(\triangle^\eta\sigma)(d_0), \ldots, (\triangle^\eta\sigma)(d_{n-1})\rangle$. Let $b_i = (\triangle^\zeta\sigma)(d_i)$ for all $i < n$. Then $T(b) = \sum_{i<n}(\triangle^\zeta\sigma)(d_i) = (\triangle^\zeta\sigma)(x) = a$. Now $1 = d_0 + \cdots + d_{n-1} + y_0 + \cdots + y_{m-1}$, so $\langle(\triangle^\zeta\sigma)(d_0), \ldots, (\triangle^\zeta\sigma)(d_{n-1}), (\triangle^\zeta\sigma)(y_0), \ldots, ((\triangle\sigma))(y_{m-1})\rangle \in (\triangle(\triangle^\zeta\sigma))(1) = (\triangle^{\zeta+1}\sigma)(1) = \alpha$. Thus $b \in \Phi_a(\alpha)$. Finally, for all $i < n$, $T^\zeta_\eta(b_i) = T^\zeta_\eta((\triangle^\zeta\sigma)(d_i)) = (\triangle^\eta\sigma)(d_i) = c_i$. $\square$

Now let M be an m-monoid and $\zeta < \omega_1$. A set $\alpha \in \triangle(\triangle^\zeta M)$ has the *heirarchy property* (H.P.) iff $\forall a \in \triangle^\zeta M[\langle a\rangle \in \Phi(\alpha) \to \forall\eta < \zeta\forall c \in T^\zeta_{\eta+1}(a)\exists b \in \Phi_a(\alpha)[c = T^\zeta_\eta \circ b]]$. Note that this condition holds vacuously if $\zeta = 0$.

Proposition 17.3. *If $\zeta < \omega_1$ and $\alpha \in \triangle^{\zeta+1}\mathscr{W}$, then the following conditions are equivalent:*

(i) $\alpha \in \mathscr{K}^\zeta$.

(ii) α *satisfies L.P. and H.P.*

Proof. (i)⇒(ii): Assume (i). Then by the definition of $\mathscr{K}^\zeta$ and $d(\sigma)$ and Proposition 13.8, α has L.P. By Lemma 17.2, α has R.P.

(ii)⇒(i): Assume (ii). By Proposition 14.1(94) we have $\alpha \in \triangle(\triangle^\zeta \mathscr{W})$. Hence by Proposition 13.14 there is a stable $\triangle^\zeta \mathscr{W}$-measure τ such that $(\triangle\tau)(1) = \alpha$.

17.3(1) $\forall x \in \mathrm{Fr}(\omega)\forall\eta < \zeta\forall c \in (T^\zeta_{\eta+1} \circ \tau)(x)\forall m \in \omega[\mathrm{dmn}(c) = m \to \exists y \in {}^m\mathrm{Fr}(\omega)[x = y_0 + \cdots + y_{m-1}$ and $\forall j < m[c_j = (T^\zeta_\eta)(\tau(y_j))]]]$.

In fact, suppose that $x \in \mathrm{Fr}(\omega)$, $\eta < \zeta$, $c \in (T^\zeta_{\eta+1} \circ \tau)(x)$, $m \in \omega$, and $\mathrm{dmn}(c) = m$. Now $x + -x = 1$, so $\langle \tau(x), \tau(-x)\rangle \in (\triangle\tau)(1) = \alpha$, and hence $\langle \tau(x)\rangle \in \Phi(\alpha)$. Since $c \in T^{\zeta+1}_\eta(\tau(x))$, we can apply H.P. to $\tau(x)$ in place of a to get $b \in \Phi_{\tau(x)}(\alpha)$ such that $c = T^\zeta_\eta \circ b$. Now there is a d such that $b^\frown d \in \alpha = (\triangle\tau)(1)$, so we can write $1 = y_0 + \cdots + y_{n-1}$ with $b^\frown d = \langle \tau(y_0), \ldots, \tau(y_{n-1})\rangle$. Let $z = y_0 + \cdots + y_{m-1}$. Then $b = \langle \tau(y_0), \ldots, \tau(y_{m-1}\rangle$ and so $b \in (\triangle\tau)(z)$. Then $\tau(z) = \tau(y_0) + \cdots + \tau(y_{m-1}) = T(b) = \tau(x)$ since $b \in \Phi_{\tau(x)}(\alpha)$. Since τ is stable it follows that $\triangle\tau(z) = \triangle\tau(x)$, so $b \in \triangle\tau(x)$. Hence we can write $x = d_0 \dot{+} \cdots \dot{+} d_{m-1}$ with $b = \langle \tau(d_0), \ldots, \tau(d_{m-1})\rangle$. So for any $j < m$, $c_j = T^\zeta_\eta(b_j) = T^\zeta_\eta(\tau(d_j))$. This proves 17.3(1).

Now let $\sigma = T^\zeta_0 \circ \tau$. Note that $\tau : \mathrm{Fr}(\omega) \to \triangle^\zeta \mathscr{W}$ and $T^\zeta_0 : \triangle^\zeta \mathscr{W} \to \mathscr{W}$. So $\sigma : \mathrm{Fr}(\omega) \to \mathscr{W}$, i.e., $\sigma \in \mathscr{M}$. Now we claim

17.3(2) $\forall\eta \leq \zeta[T^\zeta_\eta \circ \tau = \triangle^\eta\sigma]$.

Note that with $\eta = \zeta$ in 17.3(2) we get $\tau = \triangle^\zeta\sigma$. Since $(\triangle\tau)(1) = \alpha$, this gives $\alpha = (\triangle^{\zeta+1}\sigma)(1)$. Since τ is stable and $\tau = \triangle^\zeta\sigma$, we have $d(\sigma) \leq \zeta$. Hence $\alpha \in \mathscr{K}^\zeta$, finishing the proof.

We prove 17.3(2) by induction on η. It is obvious for $\eta = 0$. Now suppose that it holds for η. Suppose that $c \in (T^\zeta_{\eta+1} \circ \tau)(x)$. By (114) write $x = y_0 + \cdots + y_{m-1}$ with $\forall j < m[c_j = T^\zeta_\eta(\tau(x_j))]$. By the inductive hypothesis $\forall j < m[T^\zeta_\eta(\tau(x_j)) = (\triangle^\eta\sigma)(x_j)$. So $c \in (\triangle(\triangle^\eta\sigma)(x)$. This proves $\subseteq$ in (115) for $\eta + 1$.

Conversely suppose that $c \in (\triangle^{\eta+1}\sigma)(x) = (\triangle(\triangle^\eta\sigma))(x)$. Say c has length m. Then we can write $x = y_0 + \cdots + y_{m-1}$ with $\forall j < m[c_j = (\triangle^\eta\sigma)(y_j)]$. By the inductive hypothesis, $(\triangle^\eta\sigma(y_j) = T^\zeta_\eta(\tau(y_j))$. Now note by Proposition 14.1 that $T^{\eta+1}_\eta = T^\eta_\eta \circ T = T$. Hence by Proposition 14.1, $T \circ T^\zeta_{\eta+1} = T^{\eta+1}_\eta \circ T^\zeta_{\eta+1} = T^\zeta_\eta$. So $c_j = T(T^\zeta_{\eta+1}(\tau(y_j)))$. Now $T^\zeta_{\eta+1}(\tau(y_j) \in \triangle^{\eta+1}W = (\triangle(\triangle^\eta))W$, so by Proposition 8.12, $\langle c_j\rangle \in T^\zeta_{\eta+1}(\tau(y_j))$. Hence

$$\begin{aligned} c &= (c_0, 0, 0, \ldots, 0) + (0, c_1, 0, 0, \ldots, 0) + \cdots + (0, 0, \ldots, c_{m-1}) \\ &\in T^\zeta_{\eta+1}(\tau(y_0)) + \cdots + T^\zeta_{\eta+1}(\tau(y_{m-1}) \\ &= T^\zeta_{\eta+1}(\tau(x)). \end{aligned}$$

This proves (17.3(2) for $\eta + 1$.

Now suppose that $\xi < \zeta$ is a limit ordinal and 17.3(2) holds for all $\eta < \xi$. If $\eta < \xi$, then $T^\xi_\eta \circ \triangle^\xi\sigma = \triangle^\eta\sigma$ (by Proposition 14.2) $= T^\zeta_\eta \circ \tau$ (induction hypothesis) $= T^\xi_\eta \circ T^\zeta_\xi \circ \tau$ (by Proposition 14.1). Hence by Proposition 14.1, $\triangle^\xi\sigma = T^\zeta_\xi \circ \tau$. □

18. The monoid of isomorphism types

Recall from page 29 the definition of **BA** and **SBA**, which are m-monoids.

Proposition 18.1. **BA** *has the refinement property. That is, if* $m, n \in \omega\backslash\{0\}$, $a \in {}^m\mathbf{BA}$, $b \in {}^n\mathbf{BA}$, *and* $\sum_{i<m} a_i = \sum_{j<n} b_j$, *then there exist* $c_{ij} \in \mathbf{BA}$ *for* $i < m$ *and* $j < n$ *such that* $\forall i < m[a_i = \sum_{j<n} c_{ij}]$ *and* $\forall j < n[b_j = \sum_{i<m} c_{ij}]$.

Proof. Assume the hypotheses. Say $\forall i < m[a_i = [A_i]]$ and $\forall j < n[b_j = [B_j]]$. Let $D = \prod_{i<m} A_i$ and $E = \prod_{j<n} B_j$. Let $C_{ij} = A_i \times B_j$ for all $i < m$ and $j < n$. For each $i < m$ define $c_i \in \prod_{i<m} A_i$ by

$$c_i(k) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for each $j < n$ define $d_j \in \prod_{j<n} B_j$ by

$$d_j(k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise} \end{cases}$$

Let f be an isomorphism from D onto E. Then for any $i < m$,

$$f(c_i) = f(c_i) \cdot \sum_{j<n} d_j = \sum_{j<n} (f(c_i) \cdot d_j).$$

Hence

$$A_i \cong (D \restriction c_i) \cong (E \restriction f(c_i)) \cong \prod_{j<n} (E \restriction (f(c_i) \cdot d_j)) \cong \prod_{j<n} C_{ij}.$$

Hence $a_i = \sum_{j<n} [C_{ij}]$. Similarly, $\forall j < n[b_j = \sum_{i<m} [C_{ij}]$. □

NBA is the set of isomorphism classes of normalized BAs. **MBA** is the set of isomorphism classes of BAs which are neither superatomic nor normalized.

Proposition 18.2. $\mathbf{BA} = \mathbf{SBA} \dot\cup \mathbf{NBA} \dot\cup \mathbf{MBA}$. □

Proposition 18.3. **SBA** *is a submonoid of* **BA**.

Proof. By Proposition 5..13. □

Proposition 18.4. **NBA** *is a subsemigroup of* **BA**.

Proof. By Proposition 6.29. □

19. Refinement monoids

Let M be an m-monoid. Define $a \leq b$ iff $\exists c \in M[b = a + c]$.

Proposition 19.1. *Let* M *be an* m*-monoid.*

(i) $\leq$ *is reflexive and transitive.*

(ii) $\forall a \in M[0 \leq a]$.
(iii) $\forall a, b, c \in M[a \leq b \rightarrow a + c \leq b + c]$.

Proof. (i): $a = a + 0$.

(ii): Suppose that $a \leq b \leq c$. Choose d, e so that $b = a + d$ and $c = b + e$. Then $c = (a + d) + e = a + (d + e)$.

(iii): Assume that $a, b, c \in M$ and $a \leq b$. Choose d so that $b = a + d$. Then $b + c = a + c + d$. □

An element a of an m-monoid M has the *Schröder-Bernstein property* (S.-B.) iff $\forall b \in M[a \leq b \leq a \rightarrow a = b]$.

Proposition 19.2. *a has the Schröder-Bernstein property iff* $\forall c, d \in M[a = a + c + d \rightarrow a = a + c]$.

Proof. $\Rightarrow$: Assume that a has the Schröder-Bernstein property, and suppose that $c, d \in M$ with $a = a + c + d$. Then $a \leq a + c \leq a$, so $a = a + c$.

$\Leftarrow$: Assume that $\forall c, d \in M[a = a + c + d \rightarrow a = a + c]$ and $a \leq b \leq a$. Choose c, d so that $b = a + c$ and $a = b + d$. Then $a = a + c + d$, so $a = a + c = b$. □

If $a \in M$ we define $M \restriction a = \{b \in M : b \leq a\}$. M is *locally countable* iff $\forall a \in M[M \restriction a$ is countable$]$.

Proposition 19.3. **BA** *is locally countable.*

Proof. If $a \in \mathbf{BA}$, then $\mathbf{BA} \restriction a = \{b \in \mathbf{BA} : b \leq a\} = \{b \in \mathbf{BA} : \exists c[a = b + c]\}$. □

A submonoid N of M is a *hereditary submonoid* of M iff $\forall a \in M \forall b \in N[a \leq b \rightarrow a \in N]$.

Proposition 19.4. *N is a hereditary submonoid of M iff* $\forall a \in N[M \restriction a = N \restriction a]$. □

Proposition 19.5. *If N is a submonoid of M and M is locally countable, then so is N.* □

Proof. For $a, b \in N$, $a \leq b$ in the sense of N implies that $a \leq b$ in the sense of M. Hence $\forall a \in N[N \restriction a \subseteq M \restriction a]$. □

M is a *refinement monoid*, r-monoid, iff M is an m-monoid, and $\forall m, n \in \omega \backslash \{0\} \forall a \in {}^m M \forall b \in {}^n M[\sum_{i<m} a_i = \sum_{j<n} b_j \rightarrow \exists c \in {}^{m \times n} M[\forall i < m[a_i = \sum_{j<n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i<m} c_{ij}]]]$.

Proposition 19.6. *Every hereditary submonoid of a refinement monoid is a refinement monoid.*

Proof. Suppose that N is a hereditary submonoid of a refinement monoid M. Suppose that $m, n \in \omega$, $a \in {}^m N$, $b \in {}^n N$, and $\sum_{i<m} a_i = \sum_{j<n} b_j$. Choose $c \in {}^{m \times n} M$ such that $\forall i < m[a_i = \sum_{j<n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i<m} c_{ij}]]$. Each c_{ij} is $\leq a_i$, and hence $c_{ij} \in N$. □

M has the $(2, 2)$-*refinement property* iff the above definition holds for $m = n = 2$.

Proposition 19.7. *An m-monoid is an r-monoid iff M has the $(2,2)$-refinement property.*

Proof. $\Rightarrow$: obvious. $\Leftarrow$: Assume that M has the $(2,2)$-refinement property. First we obviously have

19.7(1) $\forall n \in \omega\backslash\{0\}\forall a \in {}^1M\forall b \in {}^nM[a_0 = \sum_{j<n} b_j \to \exists c \in {}^{1\times n}M[a_0 = \sum_{j<n} c_{0j}$ and $\forall j < n[b_j = c_{0j}]]$.

19.7(2) $\forall m \in \omega\backslash\{0\}\forall a \in {}^mM\forall b \in {}^1M[\sum_{i<m} a_i = b_0 \to \exists c \in {}^{m\times 1}M[\forall i < m[a_i = c_{i0}$ and $b_0 = \sum_{i<m} c_{i0}]]$.

Now we prove the refinement property for $m, n \geq 2$ by induction on $m + n$. The case $m + n = 4$ is given. Now suppose that $m + n > 4$ and the property is true for $2 \leq m', n'$ with $m' + n' < m + n$. Wlog $m \leq n$. Thus $2 < n$. Suppose that $a \in {}^nM$, $b \in {}^mM$, and $\sum_{i<m} a_i = \sum_{j<n} b_j$. Define $b' \in {}^{n-1}M$ by setting, for any $j < n - 1$,

$$b'_j = \begin{cases} b_j & \text{if } j < n-2, \\ b_{n-2} + b_{n-1} & \text{if } j = n-2. \end{cases}$$

Then $\sum_{i<m} a_i = \sum_{j<n-1} b'_j$. Hence by the inductive hypothesis there is a $c \in {}^{m\times(n-1)}M$ such that $\forall i < m[a_i = \sum_{j<n-1} c_{ij}]$ and $\forall j < n-1[b'_j = \sum_{i<m} c_{ij}]$. Now $b_{n-2} + b_{n-1} = b'_{n-2} = \sum_{i<m} c_{i,n-2}$.

Case 1. $m = 2$. We apply the 2,2 case to get

(*) $d \in {}^{m\times 2}M$ such that $\forall j < 2[b_{n-1-j} = \sum_{i<m} d_{ij}]$ and $\forall i < m[c_{i,n-2} = \sum_{j<2} d_{ij}]$.

Case 2. $m > 2$. We apply the inductive hypothesis to $2, m$ to again get (*).

Now we define $e \in {}^{m\times n}M$. For any $i < m$ and $j < n$,

$$e_{ij} = \begin{cases} c_{ij} & \text{if } j < n-2, \\ d_{i1} & \text{if } j = n-2, \\ d_{i0} & \text{if } j = n-1. \end{cases}$$

Then for any $i < m$,

$$a_i = \sum_{j<n-1} c_{ij} = \sum_{j<n-2} c_{ij} + c_{i,n-2} = \sum_{j<n-2} e_{ij} + \sum_{j<2} d_{ij} = \sum_{j<n} e_{ij}$$

and for any $j < n-2$, $b_j = b'_j = \sum_{i<m} c_{ij} = \sum_{i<m} e_{ij}$. Moreover, $b_{n-2} = \sum_{i<m} d_{i1} = \sum_{i<m} e_{i,n-2}$ and $b_{n-1} = \sum_{i<m} d_{i0} = \sum_{i<m} e_{i,n-1}$. □

Lemma 19.8. *Let M be an r-monoid, and suppose that $a \leq \sum_{j<n} b_j$. Then there is a $c \in {}^nM$ such that $a = \sum_{j<n} c_j$ and $\forall j < n[c_j \leq b_j]$.*

Proof. By the definition of $\leq$, there is an $a_1 \in M$ such that $\sum_{j<n} b_j = a_0 + a_1$, where $a_0 = a$. By the refinement property choose $c \in {}^{2\times n}M$ such that $\forall i < 2[a_i = \sum_{j<n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i<2} c_{ij}$. In particular, $a = a_0 = \sum_{j<n} c_{0j}$ and $\forall j < n[c_{0j} \leq b_j]$. □

An m-monoid M is *atomless* iff $\forall a \in M^*\exists b, c \in M^*[a = b + c]$.

Proposition 19.9. BA *is not atomless.*

Proof. If A is a 2-element BA, then $a = [A]$ is a counterexample. $\square$

If M is an m-monoid and $a \in M$, we let

$$\delta(a) = \delta_M(a) = \left\{ b : \exists n \in \omega\backslash\{0\} \left[b \in {}^nM \text{ and } a = \sum_{j<n} b_j \right] \right\}.$$

Proposition 19.10. *If M is a locally countable r-monoid and $a \in M$, then $\delta_M(a)$ is a countable subset of ${}^{<<\omega}M$ that satisfies C.P., R.P. and L.P.*

Proof. Assume that M is a locally countable r-monoid and $a \in M$. Clearly $\delta_M(a)$ is a countable subset of ${}^{<<\omega}M$.

C.P.: Suppose that $b, c \in {}^{<<\omega}M$, $b \in \delta_M(a)$, and $b \prec c$. Say $b \in {}^mM$ and $c \in {}^nM$. Let $\lambda : m \to n$ be such that $\forall j < n[c_j = \sum\{b_i : i < m, \lambda(i) = j\}$. Then $a = \sum_{i<m} b_i = \sum_{j<n} c_j$. Hence $c \in \delta_M(a)$.

R.P.: Assume that $m, n \in \omega\backslash\{0\}$, $b \in {}^mM$, $c \in {}^nM$, and $b, c \in \delta_M(a)$. Clearly then $\sum_{i<m} b_i = \sum_{j<n} c_j$. The desired conclusion follows since M is an r-monoid.

L.P.: First note that if $a \in M$ and $b \in \Phi_a(\delta_M(a))$, then, with $n = \mathrm{dmn}(b)$ we have $\sum_{i<n} b_i = a$ and so $b \in \Delta_M(a)$. Thus $\Phi_a(\delta_M(a)) \subseteq \delta_M(a)$. The converse holds by Proposition 13.2(i), so $\Phi_a(\delta_M(a)) = \delta_M(a)$. Hence (79) follows from the above. For (80), suppose that $b \in M$, $c \in {}^{<<\omega}M$, and $\langle b\rangle^\frown c \in \delta_M(a)$. Say $c \in {}^nM$. Then $b + \sum_{i<n} c_i = a$. If $d \in \Phi_b(\delta_M(a))$, clearly $d^\frown c \in \delta_M(a)$. $\square$

Proposition 19.11. *If M is an atomless locally countable r-monoid and $a \in M$, then $\delta_M(a)$ has S.P.*

Proof. Suppose that M is an atomless locally countable r-monoid and $a \in M$, and that $m \in \omega\backslash\{0\}$, $b \in {}^mM$, $b \in \delta_M(a)$, and $b_0 \neq o$. Since M is atomless, there are $c, d \in M^*$ such that $b_0 = c + d$. Clearly then $\langle c, d, b_1, \ldots, b_{m-1}\rangle \in \delta_M(a)$. $\square$

Proposition 19.12. *Let M is an atomless locally countable r-monoid. Define $\delta M = \{\delta_M(a) : a \in M\}$. Then δM is a submonoid of $\triangle M$. Moreover, δ_M is an isomorphism of M onto δM with inverse $T \restriction \delta M$.*

Proof. By Propositions 19.10 and 19.11, $\delta M \in \triangle M$. If $a, b \in M$, $c \in \delta_M(a)$, $d \in \delta_M(b)$, and $\mathrm{dmn}(c) = \mathrm{dmn}(d)$, then clearly $c + d \in \delta_M(a + b)$. Thus $\delta_M(a) + \delta_M(b) \subseteq \delta_M(a + b)$. Now suppose that $c \in \delta_M(a + b)$. Say $\mathrm{dmn}(c) = m \in \omega\backslash\{0\}$. Thus $\sum_{i<m} c_i = a + b$. Let $d_0 = a$, $d_1 = b$. By the refinement property there is an $e \in {}^{m\times 2}$ such that $\forall i < m[c_i = \sum_{j<2} e_{ij}]$ and $\forall j < 2[d_j = \sum_{i<m} e_{ij}]$. Let $u = \langle e_{i0} : i < m\rangle$ and $v = \langle e_{i1} : i < m\rangle$. Then $\sum_{i<m} e_{i0} = d_0 = a$ and $\sum_{i<m} e_{i1} = d_1 = b$. So $u \in \delta_M(a)$ and $v \in \delta_M(b)$ and so $a + b = \sum_{i<m} c_i = \sum_{i<m,j<2} e_{ij} = u + v$ and hence $c \in \delta_M(a) + \delta_M(b)$. Therefore $\delta_M(a) + \delta_M(b) \subseteq \delta_M(a + b)$.

Clearly $\delta_M(a) \neq \delta_M(b)$ if $a \neq b$. Obviously δ_M maps M onto δM. If $x \in \delta M$, say $x = \delta_M(a)$. Then $T(x) = a$. $\square$

20. The V-radical

Let M and N be m-monoids. A V-m-*relation* between M and N is a subset $R \subseteq M \times N$ such that the following conditions hold:

(Vm1) $\forall a \in M \forall b \in N[aRb \to (a = 0 \text{ iff } b = 0)]$.

(Vm2) $\forall a \in M \forall b \in N[aRb \to$
(Vm2a) $\forall a_0, a_1 \in M[a = a_0 + a_1 \to \exists b_0, b_1 \in N[b = b_0 + b_1 \text{ and } a_0Rb_0 \text{ and } a_1Rb_1]]$ and
(Vm2b) $\forall b_0, b_1 \in N[b = b_0 + b_1 \to \exists a_0, a_1 \in M[a = a_0 + a_1 \text{ and } a_0Rb_0 \text{ and } a_1Rb_1]]$.

A *weak V-m-relation* is a subset of $M \times N$ which satisfies (Vm2) but not necessarily (Vm1).

Proposition 20.1. *If M and N are m-monoids and $R \subseteq M \times N$, then R is a V-relation between M and N iff the following conditions hold:*

(Vm1) As above

20.1(1) $\forall a \in M \forall b \in N[aRb \to$
20.1(1a) $\forall m \in \omega \backslash \{0\} \forall c \in {}^mM[a = \sum_{i<m} c_i \to \exists d \in {}^mN[b = \sum_{i<m} d_i$ *and* $\forall i < m[c_iRd_i]]]$ *and*
20.1(1b) $\forall m \in \omega \backslash \{0\} \forall d \in {}^mM[b = \sum_{i<m} d_i \to \exists c \in {}^mN[a = \sum_{i<m} c_i$ *and* $\forall i < m[c_iRd_i]]]$.

Proof. 20.1(1)$\Rightarrow$(Vm2) is obvious. Now assume (Vm2). We prove 20.1(1) by induction on m. It is trivial for $m = 1$, and (Vm2) gives the case $m = 2$. Now assume 20.1(1) for m. By symmetry it suffices to prove 20.1(1a) for $m + 1$. Suppose that $c \in {}^{m+1}M$ and $a = \sum_{i \leq m} c_i$. By (Vm2a) there are $b_0, b_1 \in N$ such that $b = b_0 + b_1$ and $(\sum_{i<m} c_i)Rb_0$ and c_mRb_1. By the induction hypothesis there is a $d \in {}^mN$ such that $b_0 = \sum_{i<m} d_i$ and $\forall i < m[c_iRd_i]$. Since also c_mRb_1, this proves 20.1(1a) for $m + 1$. □

If $R \subseteq M \times N$ is a weak V-m-relation and is also a morphism, then it is called a V-m-*morphism*.

Proposition 20.2. *$R \subseteq M \times N$ is a V-m-morphism iff R is a morphism and $\forall a \in M \forall b_0, b_1 \in N$, if $R(a) = b_0 + b_1$ then there exist $a_0, a_1 \in M$ such that $a = a_0 + a_1$, $R(a_0) = b_0$, and $R(a_1) = b_1$.*

Proof. $\Rightarrow$: suppose that $R \subseteq M \times N$ is a V-morphism. Then it is a morphism. Suppose that $a \in M$, $b_0, b_1 \in N$, and $R(a) = b_0 + b_1$. thus $aR(b_0 + b_1)$, so by (119b) there are $a_0, a_1 \in M$ such that $a = a_0 + a_1$, a_0Rb_0, and a_1Rb_1. So $R(a_0) = b_0$ and $R(a_1) = b_1$.

$\Leftarrow$: Assume the indicated conditions. In particular, R is a morphism. To check (Vm2a), suppose that aRb and $a = a_0 + a_1$. Then $b = R(a) = R(a_0) + R(a_1)$. Let $b_0 = R(a_0)$ and $b_1 = R(a_1)$. Then $b = b_0 + b_1$ and a_0Rb_0 and a_1Rb_1. So (Vm2a) holds.

To check (Vm2b),suppose that aRb and $b = b_0 + b_1$. Then $R(a) = b = b_0 + b_1$, so there exist $a_0, a_1 \in M$ such that $a = a_0 + a_1$, $R(a_0) = b_0$, and $R(a_1) = b_1$. □

A V-m-*congruence* on M is a congruence R on M which is a weak V-m-relation.

Proposition 20.3. *If $R : M \to N$ is a V-morphism, then* $\ker(R) = \{(a_0, a_1) \in {}^2M : R(a_0) = R(a_1)\}$ *is a V-congruence on M.*

Proof. Assume that $R : M \to N$ is a V-morphism. To check (Vm2a) for $\ker(R)$, suppose that $a, a' \in M$, $a(\ker(R)a'$, and $a = a_0 + a_1$. Thus $R(a) = R(a')$. Hence $R(a_0) + R(a_1) = R(a')$. Let $b_i = R(a_i)$ for $i < 2$. Then $a'R(b_0 + b_1)$. Hence there exist $c_0, c_1 \in M$ such that $a' = c_0 + c_1$, c_0Rb_0, and c_1Rb_1. Then $R(c_0) = b_0 = R(a_0)$ and $R(c_1) = b_1 = R(a_1)$, hence $a_0(\ker(R))c_0$ and $a_1(\ker(R))c_1$. This checks (Vm2a for $\ker(R)$.

To check (Vm2b) for $\ker(R)$, suppose that $a, a' \in M$, $a(\ker(R)a'$, and $a' = a_0 + a_1$. Thus $R(a) = R(a')$. Hence $R(a_0) + R(a_1) = R(a)$. Let $b_i = R(a_i)$ for $i < 2$. Then $aR(b_0 + b_1)$. Hence there exist $c_0, c_1 \in M$ such that $a = c_0 + c_1$, c_0Rb_0, and c_1Rb_1. Then $R(c_0) = b_0 = R(a_0)$ and $R(c_1) = b_1 = R(a_1)$, hence $a_0(\ker(R))c_0$ and $a_1(\ker(R))c_1$. This checks (Vm2b) for $\ker(R)$. □

Proposition 20.4. *If $R : M \to N$ is a surjective morphism and* $\ker(R)$ *is a V-congruence, then R is a V-morphism.*

Proof. Suppose that $R : M \to N$ is a surjective morphism and $\ker(R)$ is a V-congruence. To check (Vm2a) for R, suppose that aRb, $a_0, a_1 \in M$, and $a = a_0 + a_1$. Let $b_0 = R(a_0)$ and $b_1 = R(a_1)$. Then $b = R(a) = R(a_0) + R(a_1) = b_0 + b_1$, as desired.

To check (Vm2b) for R, suppose that aRb, $b_0, b_1 \in N$, and $b = b_0 + b_1$. Since R is surjective, choose $a_0, a_1 \in M$ such that $R(a_0) = b_0$ and $R(a_1) = b_1$. Then $R(a) = b = b_0 + b_1 = R(a_0) + R(a_1) = R(a_0 + a_1)$. Thus $a(\ker(R))(a_0 + a_1)$. Hence there are $c_0, c_1 \in M$ such that $a = c_0 + c_1$, $c_0(\ker(R))a_0$, and $c_1(\ker(R))a_1$. Thus $R(c_0) = R(a_0) = b_0$ and $R(c_1) = R(a_1) = b_1$, as desired. □

Proposition 20.5. *Suppose that R is a (weak) V-relation between M and N and S is a (weak) V-relation between N and P. Then $R|S$ is a (weak) V-relation between M and P.*

Proof. Assume that R is a V-relation between M and N and S is a V-relation between N and P. Then $R \subseteq M \times N$ and $S \subseteq N \times P$, so $R|S \subseteq M \times P$. For (Vm1), suppose that $a(R|S)c$ and $a = 0$. Say $aRbSc$. Then $b = 0$ and so $c = 0$. The converse of (Vm1) is similar.

For (Vm2a), suppose that $a(R|S)c$, $a_0, a_1 \in M$, and $a = a_0 + a_1$. Say $aRbSc$. Choose $b_0, b_1 \in N$ such that $b = b_0 + b_1$, a_0Rb_0, and a_1Rb_1. Then choose $c_0, c_1 \in P$ such that $c = c_0 + c_1$, b_0Sc_0, and b_1Sc_1. Then $a_0(R|S)c_0$ and $a_1(R|S)c_1$, as desired.

(Vm2b) is treated similarly. □

Proposition 20.6. *If R is a (weak) V-relation between M and N, then R^{-1} is a (weak) V-relation between N and M.* □

Proposition 20.7. *If M is an m-monoid, then* $\mathrm{Id} \restriction M$ *is a V-relation.* □

Proposition 20.8. *If $\mathscr{A}$ is a collection of (weak) V-relations between M and N, then $\bigcup \mathscr{A}$ is a (weak) V-relation between M and N.* □

For any m-monoid M let

$$Y(M) = \bigcup\{R \subseteq M \times M : R \text{ is a weak } V\text{-relation}\}.$$

An m-monoid M is *V-simple* iff $Y(M) = \mathrm{Id} \restriction M$.

Proposition 20.9. *If M is an r-monoid, then $Y(M)$ is a V-congruence on M.*

Proof. By Proposition 20.8, $Y(M)$ is a weak V-relation. So it suffices to show that it is a congruence. By Propositions 20.5, 20.6, and 20.7 it is an equivalence relation on M. Now let

$$R = \{(a_0 + a_1, b_0 + b_1) : a_0(Y(M))b_0 \text{ and } a_1(Y(M))b_1\}.$$

We claim that R is a weak V-relation on M. (Hence $Y(M)$ is a congruence.) To prove this, by symmetry it suffices to prove (Vm2a). So suppose that aRb and $a = c_0 + c_1$. Say $a = a_0 + a_1$ and $b = b_0 + b_1$ with $a_0(Y(M))b_0$ and $a_1(Y(M))b_1$. By the refinement property let $d \in^{2\times 2} M$ be such that $\forall i < 2[c_i = d_{i0} + d_{i1}]$ and $\forall j < 2[a_i = d_{0j} + d_{1j}]$. Thus $d_{00} + d_{10} = a_0(Y(M))b_0$ so, since $Y(M)$ is a V-relation, there are $u_0, u_1 \in M$ such that $b_0 = u_0 + u_1$, $d_{00}(Y(M))u_0$ and $d_{10}(Y(M))u_1$. Similarly, $d_{01} + d_{11} = a_1(Y(M))b_1$, so there are $v_0, v_1 \in M$ such that $b_1 = v_0 + v_1$, $d_{01}(Y(M))v_0$, and $d_{11}(Y(M))v_1$. Then $b = b_0 + b_1 = u_0 + u_1 + v_0 + v_1 = u_0 + v_0 + u_1 + v_1$. Since $d_{00}(Y(M))u_0$ and $d_{01}(Y(M))v_0$ we have $(d_{00} + d_{01})R(u_0 + v_0)$. Since $d_{10}(Y(M))u_1$ and $d_{11}(Y(M))v_1$ we have $(d_{10} + d_{11})R(u_1 + v_1)$. Now $c_0 = d_{00} + d_{01}$ and $c_1 = d_{10} + d_{11}$, so this proves (Vm2a). □

Proposition 20.10. *If M is an r-monoid and R is a V-congruence on M, then M/R is an r-monoid.*

Proof. Assume that M is an r-monoid and R is a V-congruence on M. Suppose that $\sum_{i<m}[a_i] = \sum_{j<n}[b_j]$. Thus $[\sum_{i<m} a_i] = [\sum_{j<n} b_j]$, so $(\sum_{i<m} a_i)R(\sum_{j<n} b_j)$. By Proposition 20.1, choose $d \in {}^m M$ so that $\sum_{j<n} b_j = \sum_{i<m} d_i$ and $\forall i < m[a_i R d_i]$. Now by the refinement property choose $e \in {}^{m\times n}M$ so that $\forall j < n[b_j = \sum_{i<m} e_{ij}]$ and $\forall i < m[d_i = \sum_{j<n} e_{ij}]$. Now for each $i < m$ we have $a_i R d_i = \sum_{j<n} e_{ij}$, so by Proposition 20.1 choose $u_i \in {}^n M$ so that $a_i = \sum_{j<n} u_{ij}$ and $\forall j < n[u_{ij} R e_{ij}]$. Then $\forall i < m[[a_i] = \sum_{j<n}[u_{ij}] = \sum_{j<n}[e_{ij}]$ and $\forall j < n[[b_j] = \sum_{i<m}[e_{ij}]$. □

Proposition 20.11. *If M is an r-monoid, R is a V-congruence on M and S is a V-relation on M/R, let $T = \{(a, b) \in M \times M : [a]S[b]\}$. Then T is a V-relation on M.*

Proof. To check (Vm2a), suppose that aTb and $a = a_0 + a_1$. Then $[a]S[b]$. Now $[a] = [a_0] + [a_1]$ so, since S is a V-relation on M/R, there exist $c, d \in M$ such that $[b] = [c] + [d]$, $[a_0]S[c]$, and $[a_1]S[d]$. Thus $[b] = [c + d]$, so $bR(c + d)$. Since R is a V-relation, there are $u, v \in M$ such that $b = u + v$, uRc, and vRd. Then $[a_0]S[c] = [u]$ and $[a_1]S[d] = [v]$, so $a_0 T u$ and $a_1 T v$. This proves (Vm2a).

(Vm2b) is similar. □

Proposition 20.12. *If M is an r-monoid, then $M/Y(M)$ is a V-simple r-monoid.*

Proof. By Propositions 20.9 and 20.10, $M/Y(M)$ is an r-monoid. Now suppose that $[a](Y(M/Y(M)))[b]$. Let $T = \{(a,b) \in M \times M : [a](Y(M/Y(M)))[b]\}$. Then by Proposition 20.11, T is a V-relation on M. Hence $T \subseteq Y(M)$, so $a(Y(M))b$, hence $[a] = [b]$. This shows that $Y(M/Y(M)) \subseteq \mathrm{id} \restriction (M/Y(M))$. By Proposition 20.7, $\mathrm{id} \restriction (M/Y(M)) \subseteq Y(M/Y(M))$. So $M/Y(M)$ is V-simple. □

Proposition 20.13. *If M is an r-monoid, then the natural map from M onto $M/Y(M)$ is a V-morphism.*

Proof. By Propositions 20.9 and 20.4. □

The congruence relation $Y(M)$ on an r-monoid M is called the *V-radical* of M.

Proposition 20.14. *If $R : M \to N$ is a V-morphism of r-monoids, then* $\mathrm{rng}(R)$ *is a hereditary submonoid of N.*

Proof. Clearly $\mathrm{rng}(R)$ is a submonoid of N. Suppose that $a \in N$, $b \in \mathrm{rng}(R)$, and $a \leq b$. Choose $c \in N$ so that $b = a + c$. Say $b = R(d)$ with $d \in M$. Thus $R(d) = a + c$, i.e., $dR(a + c)$. By (Vm2b) there are $u, v \in M$ such that $b = u + v$, uRa, and vRc. Thus $R(v) = a$, as desired. □

Proposition 20.15. *If $R : M \to N$ is a V-morphism of r-monoids, then* $\ker(R) \subseteq Y(M)$.

Proof. By Proposition 20.3. □

Proposition 20.16. *If M is an hereditary submodel of N and R is a weak V-relation on M, then R is a weak V-relation on N.*

Proof. Assume the hypotheses. Suppose that aRb, $a_0, a_1 \in N$, and $a = a_0 + a_1$. Then $a_0, a_1 \leq a$, so $a_0, a_1 \in M$. Hence the desired conclusion follows. □

Proposition 20.17. *If M is an hereditary submodel of N and N is V-simple, then M is V-simple.*

Proof. Assume the hypotheses. Since $Y(M)$ is a V-relation on M, by Proposition 20.16 it is a weak V-relation on N. Hence $Y(M) \subseteq Y(N)$, and the desired conclusion follows. □

Proposition 20.18. *If $R : M \to N$ is a V-morphism of r-monoids, then there is an isomorphism T of $M/\ker(R)$ onto* $\mathrm{rng}(R)$ *such that $T([a]) = R(a)$ for all $a \in M$.* □

Proposition 20.19. *Suppose that R is a V-congruence on M. Let $S = ([a]_R, [b]_R) : aY(M)b\}$. Then S is a weak V-relation on M/R.*

Proof. Suppose that $u, v \in M/R$, uSv, $w_0, w_1 \in M/R$, and $u = w_0 + w_1$. Say $u = [a]$ and $v = [b]$, with $aY(M)b$, and $w_0 = [a_0]$, $w_1 = [a_1]$. Then $[a] = [a_0] + [a_1]$, so $(a_0 + a_1, a) \in R \subseteq Y(M)$. Now by Proposition 20.5, $Y(M)$ is closed under $|$. Hence $(a_0 + a_1, b) \in Y(M)$. Since $Y(M)$ is a V-relation, there are $c_0, c_1 \in M$ such that $b = c_0 + c_1$,

$c_0Y(M)a_0$, and $c_1Y(M)a_1$. So $a_0Y(M)c_0$ and $a_1Y(M)c_1$. So $v = [c_0] + [c_1]$, $w_0S[c_0]$, and $w_1S[c_1]$. This proves (Vm2a) for S.

Now suppose that $u, v \in M/R$, uSv, $w_0, w_1 \in M/R$, and $v = w_0 + w_1$. Say $u = [a]$ and $v = [b]$, with $aY(M)b$, and $w_0 = [a_0]$, $w_1 = [a_1]$. Then $[b] = [a_0]+[a_1]$, so $(b, a_0+a_1) \in R \subseteq Y(M)$. Now by Proposition 20.5, $Y(M)$ is closed under $|$. Hence $(a, a_0+a_1) \in Y(M)$. Since $Y(M)$ is a V-relation, there are $c_0, c_1 \in M$ such that $a = c_0 + c_1$, $c_0Y(M)a_0$, and $c_1Y(M)a_1$. So $u = [c_0] + [c_1]$, $[c_0]Sw_0$, and $[c_1]Sw_1$. This proves (Vm2b) for S. □

Proposition 20.20. *If R is a V-congruence on M, then $Y(M/R) = \{([a], [b]) : aY(M)b\}$.*

Proof. Let $T = \{(a, b) \in M \times M : [a]Y(M/R)[b]\}$. By Proposition 20.11, T is a V-relation on M, so $T \subseteq Y(M)$. So if $(u, v) \in Y(M/R)$ we can choose a, b so that $u = [a]$ and $v = [b]$. Then $(a, b) \in T$, so $aY(M)b$. Thus $\subseteq$ holds. Conversely, suppose that $aY(M)b$. Then $[a]S[b]$, where S is as in Proposition 20.19. Hence $([a], [b]) \in Y(M/R)$, proving $\supseteq$. □

Proposition 20.21. *If $R : M \to N$ is a V-morphism of r-monoids and N is V-simple, then* $\ker(R) = Y(M)$.

Proof. By Proposition 20.14, $\mathrm{rng}(R)$ is an hereditary submonoid of N. Hence by Proposition 20.17, $\mathrm{rng}(R)$ is V-simple. So by Proposition 20.18, $M/\ker(R)$ is V-simple. Now $\ker(R) \subseteq Y(M)$ by Proposition 20.15. Now suppose that $aY(M)b$. Then $[a]Y(M/\ker(R))[b]$ by Proposition 20.20, so $[a] = [b]$, i.e. $R(a) = R(b)$, i.e. $(a, b) \in \ker(R)$. □

Proposition 20.22. *If $R : M \to N$ is a V-morphism of r-monoids and N is V-simple, then there is only one V-morphism from M to N.*

Proof. Assume that $R, S : M \to N$ are V-morphisms of r-monoids and N is V-simple. Now $S^{-1}|R$ is a V-relation on N by Propositions 20.5 and 20.6. Hence $S^{-1}|R \subseteq Y(N) = (\mathrm{id} \restriction N)$. Now if $R(a) = b$ and $S(a) = c$, then $cS^{-1}aRb$, hence $c(S^{-1}|R)b$, hence $b = c$. □

Proposition 20.23. *If M and N are r-monoids and M is a hereditary submonoid of N, then the inclusion map from M to N is a V-morphism.*

Proof. By Proposition 20.2. □

21. Dobbertin's theorem

Lemma 21.1. *Let M be a locally countable r-monoid and $a \in M$. Then there exist a countable BA A uniquely determined up to isomorphism and a relation $R \subseteq A \times (M \restriction a)$ such that:*

(i) 1_ARa.

(ii) $\forall x \in A[xR0 \to x = 0]$ *and* $\forall b \in (M \restriction a)[0Rb \to b = 0]$.

(iii) $\forall x \in A\forall b_0, b_1 \in (M \restriction a)[xR(b_0 + b_1) \to \exists x_0.x_1 \in A[x = x_0\dot{+}x_1$ *and* x_0Rb_0 *and* $x_1Rb_1]]$ *and* $\forall x_0, x_1 \in A\forall b \in (M \restriction a)[(x_0\dot{+}x_1)Rb \to \exists b_0, b_1 \in (M \restriction a)[b = b_0 + b_1$ *and* x_0Rb_0 *and* $x_1Rb_1]]$.

Proof. Assume that M is a locally countable r-monoid and $a \in M$. If $a = 0$, clearly $A = \{0\}$ and $R = \{0, 0)\}$ works, and A is unique up to isomorphism. So assume that $a \neq 0$. Recall the definition of δ_M after Proposition 19.9. By Proposition 19.10, $\delta_M(a)$ satisfies C.P., R.P., and L.P. Hence by Proposition 13.12 there is a uniformly dense $\delta_M(a)$-tree b. Thus $b : \mathscr{D} \to M$. For each $f \in \mathscr{D}$ let $x(f) = \{p \in {}^\omega 2 : f \subseteq p\}$. Let J be the ideal in $\mathrm{clop}({}^\omega 2)$ generated by $\{x(f) : f \in \mathscr{D}, b(f) = 0\}$.

21.1(1) $\forall f \in \mathscr{D}[x(f) \in J$ iff $b(f) = 0]$.

In fact, $\Leftarrow$ is clear. Now suppose that $f \in \mathscr{D}$ and $x(f) \in J$. Then there is a finite $F \subseteq \mathscr{D}$ such that $\forall g \in F[b(g) = 0]$ and $x(f) \subseteq \bigcup_{g \in F} x(g)$. Hence $x(f) = \bigcup\{x(f \cup g) : g \in F, f \cup g$ is a function$\}$. For $g \in F$ and $f \cup g$ a function we have $b(f \cup g) = 0$, since $b(g) = 0$. Hence $b(f) = 0$, proving 21.1(1).

Let $A = \mathrm{clop}({}^\omega 2)/J$ and

$$R = \left\{ \left(\left[\bigcup_{f \in G} x(f) \right]_J, \sum_{f \in G} b(f) \right) : G \subseteq \mathscr{D}_n, n \in \omega \right\}.$$

We claim that $R \subseteq A \times (M \restriction a)$. For, suppose that $G \subseteq \mathscr{D}_n$ with $n \in \omega$. Then obviously $[\bigcup_{f \in G} x(f)]_J \in A$. By (t1), $\forall m \in \omega[b_m \in \delta_M(a)]$, where $b_m = b(\mathscr{D}_m) = \langle b(lex(p)) : p < |\mathscr{D}_m| \rangle$. By the definition of $\delta_M(a)$ after Proposition 19.9, $a = \sum\{b(lex(p)) : p < |\mathscr{D}_m\}$. Hence $\sum_{f \in G} b(f) \leq a$. This proves the claim.

Now we check (i)–(iii). For (i), let $\mathscr{D}_1 = \{f_0, f_1\}$. Then $[\bigcup_{f \in \mathscr{D}_n} x(f)] = 1_A$. Also, $b_1 \in \delta_M(a)$, so $\sum_{f \in \mathscr{D}_1} b(f) = a$. This proves (i).

For (ii), first suppose that $y \in A$ and $yR0$. Say $y = [\bigcup_{f \in G} x(f)]$ and $\sum_{f \in G} b(f) = 0$, where $G \subseteq \mathscr{D}_n$ with $n \in \omega$. Then $\forall f \in G[b(f) = 0]$, so by 21.1(1), $\forall f \in G[x(f) \in J]$. Hence $y = 0$.

Second, suppose that $c \in M \restriction a$ and $0Rc$. Say $[\bigcup_{f \in G} x(f)] = 0$ and $c = \sum_{f \in G} b(f)$, where $G \subseteq \mathscr{D}_n$ with $n \in \omega$. Then by 21.1(1), $c = 0$.

For (iii), first suppose that $y \in A$, $c_0, c_1 \in (M \restriction a)$, and $yR(c_0 + c_1)$. Note that $c_0 + c_1 \leq a$ by the above claim. Say $n \in \omega$, $G \subseteq \mathscr{D}_n$, $y = [\bigcup_{f \in G} x(f)]$, and $c_0 + c_1 = \sum_{f \in G} b(f)$. Then $\sum_{f \in G} b(f) = \sum_{j < |G|} b(lex(j)) = b(G)$. Thus $\langle c_0, c_1 \rangle \in \Phi_{T(b(G))}(\delta_M(a))$. Since b is a uniformly dense $\delta_M(a)$-tree, it follows that there is an $m \geq n$ such that $b(G \uparrow m) \prec \langle c_0, c_1 \rangle$. Thus $\langle b(lex(i)) : i < |G \uparrow m| \rangle \prec \langle c_0, c_1 \rangle$. Let $\lambda : |G \uparrow m| \to 2$ be such that $\forall j < 2[c_j = \sum\{b(lex(i)) : i < |G \uparrow m|, \lambda(i) = j\}]$. Let $H_j = \{lex(i) : i < |G \uparrow m|, \lambda(i) = j\}$ for $j < 2$. Thus $(G \uparrow m) = H_0 \dot{\cup} H_1$ and $\forall j < 2[c_j = T(b(H_j))]$. For each $j < 2$ let $x_j = [\sum_{f \in H_j} x(f)]$. Then $y = x_0 + x_1$, $x_0 R c_0$, and $x_1 R c_1$.

Second suppose that $y_0, y_1 \in A$, $c \in (M \restriction a)$, and $(y_0 \dot{+} y_1)Rc$. Then there exist $n \in \omega$ and $G \subseteq \mathscr{D}_n$ such that $y_0 \dot{+} y_1 = [\bigcup_{f \in G} x(f)]$ and $c = \sum_{f \in G} b(f)$. Then there exist $m \geq n$ and subsets $H_i \subseteq \mathscr{D}_m$ for $i < 2$ such that $y_i = [\bigcup_{g \in H_i} x(g)]$. Let $G_i = (G \uparrow m) \cap H_i$ for $i < 2$. Then $G_0 \cup G_1 = (G \uparrow m)$ and $y_i = [\bigcup_{g \in G_i} x(g)]$ for $i < 2$. Let $d_i = \sum_{f \in G_i} b(g)$ for $i < 2$. Then $c = d_0 + d_1$, $y_0 R d_0$, and $y_1 R d_1$. $\square$

Theorem 21.2. BA *is a locally countable V-simple r-monoid.*

Proof. **BA** is locally countable by Proposition 19.3. By Proposition 18.1 it is an r-monoid. Now let $S = Y(\mathbf{BA})$. Then S is a V-congruence on **BA** by Proposition 20.9. Define ARB iff $A, B \in \mathbf{BA}$ and $[A]S[B]$. Then R satisfies the conditions of Proposition 3.4, so ARB implies that A is isomorphic to B. Thus $[A]S[B]$ implies that $[A] = [B]$. So **BA** is V-simple. □

Theorem 21.3. *If M is a locally countable r-monoid, then there is a unique V-morphism $S : M \to \mathbf{BA}$; $\ker(S) = Y(M)$, and $\mathrm{rng}(S)$ is a hereditary submonoid of* **BA**.

Proof. Assume that M is a locally countable r-monoid. For each $a \in M$ let A_a and R_a be as in Lemma 21.1. For each $a \in M$ let $S(a) = [A_a]$. So $S : M \to \mathbf{BA}$. If $S(a) = 0$, then A_a is a one-element BA, hence by Lemma 21.1(i), $0R_a a$, and so by Lemma 21.1(ii), $a = 0$. On the other hand, if $a = 0$ then by Lemma 21.1(i), $1_{A_a} R 0$, and hence by Lemma 21.1(ii), $1_{A_a} = 0$, and so A_a is a one-element BA, and $S(a) = 0$.

To show that S is a morphism, suppose that $a_0, a_1 \in M$. Let $a = a_0 + a_1$. Then $1_{A_a} R_a a = a_0 + a_1$, so there are $x_0, x_1 \in A_a$ such that $1_{A_a} = x_0 \dot{+} x_1$, $x_0 R_a a_0$, and $x_1 R_a a_1$.

21.3(1) a_0, $M \restriction a_0$, $A_a \restriction x_0$, $R_a \cap ((A \restriction x_0) \times (M \restriction a_0))$ satisfy the conditions of Lemma 21.1, with these entries replacing a, M, A, R respectively.

For, let $T = R_a \cap ((A \restriction x_0) \times (M \restriction a_0))$. For (i), $1_{A_a \restriction x_0} = x_0 T a_0$, as desired.

For (ii), if $x \in (A_a \restriction x_0)$ and $xT0$, then $xR_a 0$ and hence $x = 0$. If $b \in (M \restriction a_0)$ and $0Tb$, then $b \in (M \restriction a)$ and $0R_a b$, so $b = 0$.

For (iii), first suppose that $x \in (A_a \restriction x_0)$, $c_0, c_1 \in (M \restriction a_0)$, and $xT(c_0 + c_1)$. Then $x \in A_a$, $c_0, c_1 \in (M \restriction a)$, and $xR_a(c_0 + c_1)$. Hence there are $y_0, y_1 \in A_a$ such that $x = y_0 \dot{+} y_1$, $y_0 R_a c_0$, and $y_1 R_a c_1$. Then $y_0, y_1 \in (A_a \restriction x_0)$, $x = y_0 \dot{+} y_1$, $y_0 T c_0$, and $y_1 T c_1$.

Second, suppose that $y_0, y_1 \in (A_a \restriction x_0)$, $c \in (M \restriction a_0)$. and $(y_0 \dot{+} y_1) T c$. Then $y_0, y_1 \in A_a$, $c \in (M \restriction a)$, and $(y_0 \dot{+} y_1) R_a c$. Hence there exist $d_0, d_1 \in (M \restriction a)$ such that $c = d_0 + d_1$, $y_0 R_a d_0$, and $y_1 R_a d_1$. Then $d_0, d_1 \in (M \restriction a_0)$, $c = d_0 + d_1$, $y_0 T d_0$, and $y_1 T d_1$. This proves 21.3(1).

Then by the uniqueness assertion of Lemma 21.1 we have $A_{a_0} \cong (A_a \restriction x_0)$

By symmetry we get

21.3(2) a_1, $M \restriction a_1$, $A_a \restriction x_1$, $R_a \cap ((A \restriction x_1) \times (M \restriction a_1))$ satisfy the conditions of Lemma 21.1, with these entries replacing a, M, A, R respectively.

By the uniqueness assertion of Lemma 21.1 we have $A_{a_1} \cong (A_a \restriction x_1)$.

Hence $S(a_0) + S(a_1) = [A_{a_0}] + [A_{a_1}] = [A_a \restriction x_0] + [A_a \restriction x_1] = [(A_a \restriction x_0) \times (A_a \restriction x_1)] = [A_a] = S(a) = S(a_0 + a_1)$. This proves that S is a morphism.

To check (Vm2a) for S, suppose that $a \in M$, $b \in \mathbf{BA}$, $S(a) = b$, and $a = a_0 + a_1$. Then since S is a morphism, $b = S(a) = S(a_0) + S(a_1)$, as desired.

To check (Vm2b) for S, suppose that $a \in M$, $b \in \mathbf{BA}$, $S(a) = b$, $b_0, b_1 \in \mathbf{BA}$, and $b = b_0 + b_1$. Say $b_0 = [C]$ and $b_1 = [D]$. Then $[A_a] = S(a) = b = [C] + [D]$, so that $A_a \cong C \times D$. Then there is an $x \in A_a$ such that $(A_a \restriction x) \cong C$ and $(A_a \restriction (-x)) \cong D$. Now $1 = (x + -x) R_a a$, so there are $c_0, c_1 \leq a$ such that $a = c_0 + c_1$, $xR_a c_0$, and $(-x) R_a c_1$. Now by 21.3(1) and 21.3(2), $A_{c_0} \cong (A_a \restriction x)$ and $A_{c_1} \cong (A_a \restriction (-x))$. Hence $S(c_0) = [A_{c_0}] = [A_a \restriction x] = [C] = b_0$ and similarly $S(c_1) = b_1$, as desired.

It follows that S is a V-morphism. Now by Theorem 21.2, **BA** is V-simple. Hence by Proposition 20.22, S is unique. By Proposition 20.21, $\ker(S) = Y(M)$. By Proposition 20.14, $\mathrm{rng}(S)$ is a hereditary submonoid of **BA**. □

Proposition 21.4. *Suppose that N is a monoid satisfying the following:*

(i) N is a locally countable V-simple r-monoid.

(ii) If M is any locally countable r-monoid, then there is a unique V-morphism $S : M \to N$ such that $\ker(S) = Y(M)$ and $\mathrm{rng}(S)$ is a hereditary submonoid of M.

Then N is isomorphic to **BA**.

Proof. Assume the hypotheses. By Theorem 21.3, let $S : N \to \mathbf{BA}$ be a V-morphism such that $\ker(S) = Y(N)$ and $\mathrm{rng}(S)$ is a hereditary submonoid of N. Since N is V-simple, S is an injection. By the hypotheses of the proposition, let $T : \mathbf{BA} \to N$ be a V-morphism such that $\ker(T) = Y(\mathbf{BA})$ and $\mathrm{rng}(T)$ is a hereditary submonoid of N. Since **BA** is V-simple, T is an injection. Then $S \circ T$ is a V-morphism from **BA** to **BA** such that $\ker(S \circ T) = Y(\mathbf{BA})$ and $\mathrm{rng}(S \circ T)$ is a hereditary submonoid of **BA**. The identity on **BA** also has these properties, so by the uniqueness assertion of Theorem 21.3, $S \circ T$ is the identity. Similarly, $T \circ S$ is the identity. □

22. Ketonen's theorem

Theorem 22.1. *Any countable commutative semigroup is isomorphic to a subsemigroup of* **BA**.

The proof of Theorem 22.1 will come after a series of lemmas.

Corollary 22.2. *$\mathbb{Z}_2$ can be isomorphically embedded into* **BA**.

Corollary 22.3. *There is a countable BA A such that $A \cong A \times A \times A$ but $A \ncong A \times A$.*

Proof. In $\mathbb{Z}_2$, $1 + 1 + 1 = 1$ but $1 + 1 \neq 1$. □

Corollary 22.4. *There are countable BAs A, B such that $A \cong A \times B \times B$ but $A \ncong A \times B$.* □

Corollary 22.5. *There are countable BAs A, B, C such that $A \cong A \times B \times C$ but $A \ncong A \times B$.* □

23. Products of measures

Recall the definition of $\mathscr{M}(M)$ from just before Proposition 7.2. Suppose that $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ is an isomorphism. For $\sigma, \tau \in \mathscr{M}(M)$ and $x \in \mathrm{Fr}(\omega)$ define

$$(\sigma \oplus_k \tau)(x) = \sigma(\pi_0(k(x))) + \tau(\pi_1(k(x))).$$

Thus $(\sigma \oplus_k \tau) : \mathrm{Fr}(\omega) \to M$.

Proposition 23.1. *If $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ is an isomorphism and $\sigma, \tau \in \mathscr{M}(M)$, then $(\sigma \oplus_k \tau) \in \mathscr{M}(M)$.*

Proof. Assume that $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ is an isomorphism and $\sigma, \tau \in \mathscr{M}(M)$. If x, y are disjoint elements of $\mathrm{Fr}(\omega)$, then $k(x), k(y)$ are disjoint. $\pi_0(k(x))$ and $\pi_0(k(y))$ are disjoint, and $\pi_1(k(x))$ and $\pi_1(k(y))$ are disjoint, hence

$$\begin{aligned}(\sigma \oplus_k \tau)(x+y) &= \sigma(\pi_0(k(x+y))) + \tau(\pi_1(k(x+y))) \\ &= \sigma(\pi_0(k(x)) + \pi_0(k(y))) + \tau(\pi_0(k(x)) + \pi_0(k(y))) \\ &= \sigma(\pi_0(k(x))) + \sigma(\pi_0(k(y))) + \tau(\pi_0(k(x))) + \tau(\pi_0(k(y))) \\ &= (\sigma \oplus_k \tau)(x) + (\sigma \oplus_k \tau)(y).\end{aligned}$$

Clearly $(\sigma \oplus_k \tau)(x) = 0$ iff $x = 0$. □

Proposition 23.2. *Assume:*

(i) k_0 and k_1 are automorphisms of $\mathrm{Fr}(\omega)$;
(ii) $\sigma, \sigma', \tau, \tau' \in \mathscr{M}(M)$;
(iii) $\forall x \in \mathrm{Fr}(\omega)[\sigma(x) = \sigma'(k_0(x))]$;
(iv) $\forall x \in \mathrm{Fr}(\omega)[\tau(x) = \tau'(k_1(x))]$;
(v) $l_0, l_1 : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ are isomorphisms;
(vi) $\forall x, y \in \mathrm{Fr}(\omega)[s(x, y) = (k_0(x), k_1(y))]$;
(vii) $\forall x \in \mathrm{Fr}(\omega)[k_2(x) = l_1^{-1}(s(l_0(x)))]$;

Then:

(viii) $s : \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ is an isomorphism;
(ix) k_2 is an automorphism of $\mathrm{Fr}(\omega)$;
(x) $\forall x \in \mathrm{Fr}(\omega)[(\sigma \oplus_{l_0} \tau)(x) = (\sigma' \oplus_{l_1} \tau')(k_2(x))]$.

Proof. Assume (i)–(vii). Then (viii) and (ix) are clear. For (x), suppose that $x \in \mathrm{Fr}(\omega)$. Then

$$\begin{aligned}(\sigma' \oplus_{l_1} \tau')(k_2(x)) &= \sigma'(\pi_0(l_1(k_2(x)))) + \tau'(\pi_1(l_1(k_2(x)))) \\ &= \sigma'(\pi_0(s(l_0(x)))) + \tau'(\pi_1(s(l_0(x)))) \\ &= \sigma'(\pi_0(s(\pi_0(l_0(x)), \pi_1(l_0(x))))) + \tau'(\pi_1(s(\pi_0(l_0(x)), \pi_1(l_0(x))))) \\ &= \sigma'(\pi_0(k_0(\pi_0(l_0(x))), k_1(\pi_1(l_0(x))))) + \tau'(\pi_1(k_0(\pi_0(l_0(x))), k_1(\pi_1(l_0(x))))) \\ &= \sigma'(k_0(\pi_0(l_0(x)))) + \tau'(k_1(\pi_1(l_0(x)))) \\ &= \sigma(\pi_0(l_0(x))) + \tau(\pi_1(l_0(x))) \\ &= (\sigma \oplus_{l_0} \tau)(x).\end{aligned}$$

□

Proposition 23.3. *If $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ is an isomorphism and $\xi < \omega_1$, then $\triangle^\xi(\sigma \oplus_k \tau) = \triangle^\xi\sigma \oplus_k \triangle^\xi\tau$.*

Proof. Assume that $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ is an isomorphism and $\xi < \omega_1$. We prove the indicated conclusion by induction on ξ. It is trivial for $\xi = 0$. Now assume

that it holds for ξ. Suppose that $a \in (\triangle^{\xi+1}(\sigma \oplus_k \tau))(x)$. By Proposition 14.2 (103), $a \in (\triangle(\triangle^{\xi}(\sigma \oplus_k \tau)))(x)$. Say $x = y_0 \dot{+} \cdots \dot{+} y_{n-1}$ and

$$\begin{aligned}
a &= \langle (\triangle^{\xi}(\sigma \oplus_k \tau))(y_0), \ldots, (\triangle^{\xi}(\sigma \oplus_k \tau))(y_{n-1}) \\
&= \langle (\triangle^{\xi}\sigma \oplus_k \triangle^{\xi}\tau)(y_0), \ldots, (\triangle^{\xi}\sigma \oplus_k \triangle^{\xi}\tau)(y_{n-1}) \rangle \\
&= \langle (\triangle^{\xi}\sigma)(\pi_0(k(y_0))) + (\triangle^{\xi}\tau)(\pi_1(k(y_0))), \ldots, \\
&\qquad (\triangle^{\xi}\sigma)(\pi_0(k(y_{n-1}))) + (\triangle^{\xi}\tau)(\pi_1(k(y_{n-1}))) \rangle \\
&= \langle (\triangle^{\xi}\sigma)(\pi_0(k(y_0))), \ldots, (\triangle^{\xi}\sigma)(\pi_0(k(y_{n-1}))) \rangle + \\
&\qquad \langle (\triangle^{\xi}\tau)(\pi_0(k(y_0))), \ldots, (\triangle^{\xi}\tau)(\pi_0(k(y_{n-1}))) \rangle \\
&\in (\triangle(\triangle^{\xi}\sigma))(\pi_0(k(x))) + (\triangle(\triangle^{\xi}\tau))(\pi_0(k(x))) \\
&= (\triangle^{\xi+1}\sigma \oplus_k \triangle^{\xi+1}\tau)(x).
\end{aligned}$$

Thus $(\triangle^{\xi+1}(\sigma \oplus_k \tau))(x) \subseteq (\triangle^{\xi+1}\sigma \oplus_k \triangle^{\xi+1}\tau)(x)$.

Now suppose that $a \in (\triangle^{\xi+1}\sigma \oplus_k \triangle^{\xi+1}\tau)(x)$. Thus

$$a \in (\triangle^{\xi+1}\sigma)(\pi_0(k(x))) + (\triangle^{\xi+1}\tau)(\pi_1(k(x))).$$

Say a has domain n. Then we can write $\pi_0(k(x)) = y_0 + \cdots + y_{n-1}$ and $\pi_1(k(x)) = z_0 + \cdots + z_{n-1}$ with

$$\begin{aligned}
a &= \langle (\triangle^{\xi}\sigma)(y_0), \ldots (\triangle^{\xi}\sigma)(y_{n-1}) \rangle + \langle (\triangle^{\xi}\tau)(z_0), \ldots (\triangle^{\xi}\tau)(z_{n-1}) \rangle \\
&= \langle (\triangle^{\xi}\sigma)(y_0) + (\triangle^{\xi}\tau)(z_0), \ldots, (\triangle^{\xi}\sigma)(y_{n-1}) + (\triangle^{\xi}\tau)(z_{n-1} \rangle.
\end{aligned}$$

Now let $x_i = k^{-1}(y_i, z_i)$ for all $i < n$. Then

$$k(x) = (y_0 + \cdots + y_{n-1}, z_0 + \cdots + z_{n-1}) = (y_0, z_0) + \cdots + (y_{n-1}, z_{n-1});$$

hence $x = x_0 + \cdots + x_{n-1}$. By the above,

$$\begin{aligned}
a &= \langle (\triangle^{\xi}\sigma)(y_0) + (\triangle^{\xi}\tau)(z_0), \ldots, (\triangle^{\xi}\sigma)(y_{n-1}) + (\triangle^{\xi}\tau)(z_{n-1} \rangle \\
&= \langle (\triangle^{\xi}\sigma)(\pi_0(k(x_0))) + (\triangle^{\xi}\tau)(\pi_1(k(x_0))), \ldots, \\
&\qquad (\triangle^{\xi}\sigma)(\pi_0(k(x_{n-1}))) + (\triangle^{\xi}\tau)(\pi_1(k(x_{n-1}))) \rangle \\
&= \langle (\triangle^{\xi}\sigma \oplus_k \triangle^{\xi}\tau)(x_0), \ldots . (\triangle^{\xi}\sigma \oplus_k \triangle^{\xi}\tau)(x_{n-1}) \rangle \\
&= \langle (\triangle^{\xi}(\sigma \oplus_k \tau))(x_0), \ldots, (\triangle^{\xi}(\sigma \oplus_k \tau))(x_{n-1}) \rangle \\
&\in (\triangle^{\xi+1}(\sigma \oplus_k \tau))(x).
\end{aligned}$$

This gives the conclusion for $\xi + 1$.

Now suppose the conclusion holds for all $\xi < \zeta$, with ζ a limit ordinal. Then

$$\begin{aligned}
(\triangle^{\zeta}(\sigma \oplus_k \tau))(x) &= \langle (\triangle^{\eta}(\sigma \oplus_k \tau))(x) : \eta < \zeta \rangle \\
&= \langle (\triangle^{\eta}\sigma \oplus_k \triangle^{\eta}\tau)(x) : \eta < \zeta \rangle \\
&= \langle (\triangle^{\eta}\sigma)(\pi_0(k(x))) + (\triangle^{\eta}\tau)(\pi_1(k(x))) : \eta < \zeta \rangle \\
&= \langle (\triangle^{\eta}\sigma)(\pi_0(k(x))) : \eta < \zeta \rangle + \langle (\triangle^{\eta}\tau)(\pi_1(k(x))) : \eta < \zeta \rangle \\
&= (\triangle^{\zeta}\sigma)(\pi_0(k(x))) + (\triangle^{\zeta}\tau)(\pi_1(k(x))) \\
&= (\triangle^{\zeta}\sigma \oplus_k \triangle^{\zeta}\tau)(x)
\end{aligned}$$

□

Proposition 23.4. *Suppose that* $0 < x < 1$ *in* $\mathrm{Fr}(\omega)$*, and* $k_1 : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \restriction x$ *and* $k_2 : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \restriction (-x)$ *are isomorphisms. Suppose that* $\rho \in \mathscr{M}(M)$*. Define* $\sigma = \rho \circ k_1$*,* $\tau = \rho \circ k_2$*, and* $\forall y \in \mathrm{Fr}(\omega)[k(y) = (k_1^{-1}(x \cdot y), k_2^{-1}((-x) \cdot y))]$*.*

Then k *is an isomorphism of* $\mathrm{Fr}(\omega)$ *onto* $\mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$*,* $\sigma, \tau \in \mathscr{M}(M)$*, and* $\rho = \sigma \oplus_k \tau$*.*

Proof. Clearly k is an isomorphism of $\mathrm{Fr}(\omega)$ onto $\mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ and $\sigma, \tau \in \mathscr{M}(M)$. Now suppose that $y \in \mathrm{Fr}(\omega)$. Then

$$\begin{aligned}(\sigma \oplus_k \tau)(y) &= \sigma(\pi_0(k(y))) + \tau(\pi_1(k(y))) \\ &= \sigma(k_1^{-1}(x \cdot y)) + \tau(k_2^{-1}((-x) \cdot y)) \\ &= \rho(x \cdot y) + \rho((-x) \cdot y) = \rho(y).\end{aligned}$$

□

Proposition 23.5. *Assume the hypotheses of Proposition 23.4, and also assume that* ρ *is stable. Then* σ *and* τ *are stable.*

Proof. By symmetry it suffices to show that σ is stable. So assume that $y, z \in \mathrm{Fr}(\omega)$ and $\sigma(y) = \sigma(z)$. Thus $\rho(k_1(y)) = \sigma(y) = \sigma(z) = \rho(k_1(z))$. Since ρ is stable, $(\triangle\rho)(k_1(y)) = (\triangle\rho)(k_1(z))$. Hence

$$\begin{aligned}((\triangle\sigma))(y) &= \{\langle \sigma(u_0), \ldots, \sigma(u_{n-1})\rangle : y = u_0 + \cdots + u_{n-1}\} \\ &= \{\langle \rho(k_1(u_0)), \ldots, \rho(k_1(u_{n-1}))\rangle : y = u_0 + \cdots + u_{n-1}\} \\ &= \{\langle \rho(v_0), \ldots, \rho(v_{n-1})\rangle : k_1(y) = v_0 + \cdots + v_{n-1}\} \qquad (*) \\ &= (\triangle\rho)(k_1(y)).\end{aligned}$$

To see the step $(*)$, clearly

$$\begin{aligned}&\{\langle \rho(k_1(u_0)), \ldots, \rho(k_1(u_{n-1}))\rangle : y = u_0 + \cdots + u_{n-1}\} \\ &\qquad \subseteq \{\langle \rho(v_0), \ldots, \rho(v_{n-1})\rangle : k_1(y) = v_0 + \cdots + v_{n-1}\}.\end{aligned}$$

On the other hand, suppose that $k_1(y) = v_0 + \cdots + v_{n-1}$. For each $i < n$ choose $u_i \in \mathrm{Fr}(\omega)$ such that $k_1(u_i) = v_i$. Then $k_1(u_0 + \cdots + u_{n-1}) = v_0 + \cdots + v_{n-1} = k_1(y)$, so $u_0 + \cdots + u_{n-1} = y$. This proves the other inclusion, so that $(*)$ holds.

Now by symmetry, $((\triangle\sigma))(z) = (\triangle\rho)(k_1(z))$, so $((\triangle\sigma))(y) = ((\triangle\sigma))(z)$. □

For the following proposition, recall the definition of B_s and g_s from just before Proposition 5.22.

Proposition 23.6. *If* s *and* t *are additive functions from* $\mathrm{Fr}(\omega)$ *into* ω_1 *and* $k : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ *is an isomorphism, then* $B_s \times B_t \cong B_{s \oplus_k t}$*.*

Proof. Recall g_s is an isomorphism from $B_s/I_{\mathrm{ar}(B_s)}(B_s)$ onto $\mathrm{Fr}(\omega)$ such that $r_{B_s g_s} = s$ and g_t is an isomorphism from $B_t/I_{\mathrm{ar}(B_t)}(B_t)$ onto $\mathrm{Fr}(\omega)$ such that $r_{B_t g_t} = t$. Now $I_{\mathrm{ar}(B_s)}(B_s) \times I_{\mathrm{ar}(B_t)}(B_t)$ is an ideal in $B_s \times B_t$. Clearly

23.6(1) There is an isomorphism h from $(B_s \times B_t)/(A \times A')$ onto $\mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ such that $\forall x \in B_s \forall y \in B_t[h([x, y]) = (g_s([x]), g_t([y])))$.

Then by definition, for any $u \in B_s$ and $v \in B_t$,

$$r_{B_s \times B_t, k^{-1} \circ h}(k^{-1}(h([u,v]))) = \alpha_*((B_s \times B_t) \restriction (u,v)).$$

Suppose that $w \in \mathrm{Fr}(\omega)$. Say $k(w) = (x,y)$. Choose $u \in B_s$ and $v \in B_t$ such that $g_s([u]) = x$ and $g_t([v]) = y$. Then

$$\begin{aligned} r_{B_{s \oplus_k t}, s \oplus_k t}(w) &= (s \oplus_k t)(w) \\ &= (s \oplus_k t)(k^{-1}(x,y)) \\ &= s(x) + t(y) = r_{B_s g_s}(g_s([u])) + r_{B_t g_t}(g_t([v])) \\ &= \alpha_*(B_s \restriction u) + \alpha_*(B_t \restriction v) = \alpha_*((B_s \times B_t) \restriction (u,v)) \\ &= r_{B_s \times B_t, k^{-1} \circ h}(k^{-1}(h([u,v]))) \\ &= r_{B_s \times B_t, k^{-1} \circ h}(k^{-1}(g_s([u], g_t([v])))) \\ &= r_{B_s \times B_t, k^{-1} \circ h}(w). \end{aligned}$$

By Proposition 5.8, $B_{s \oplus_k t} \cong B_s \times B_t$. □

24. The strict hierarchy property

Let M be an m-monoid and ξ a countable ordinal. We say that α has the *strict ξ-heirarchy property* iff $\alpha \in \triangle^{\xi+1} M$ and for all $\eta < \xi$, and all a and b with $\langle a \rangle^\frown b \in \alpha$, for all $n \in \omega$, and all e with domain n, if $e \in T^\xi_{\eta+1}(a)$ then there is a c with domain n such that $\sum_{i<n} c_i = a$, $c^\frown b \in \alpha$, and $T^\xi_\eta(c_i) = e_i$ for all $i < n$.

Proposition 24.1. *If α has the strict ξ-heirarchy property, then it has the heirarchy property with respect to ξ.*

Proof. Suppose that α has the strict ξ-heirarchy property, $a \in \triangle^\xi M$, $\langle a \rangle \in \Phi(\alpha)$, $\eta < \xi$, and $e \in T^\xi_{\eta+1}(a)$. Choose b so that $\langle a \rangle^\frown b \in \alpha$. Say e has domain n. Choose c with domain n such that $\sum_{i<n} c_i = a$, $c^\frown b \in \alpha$, and $T^\xi_\eta(c_i) = e_i$ for all $i < n$. Thus $c \in \Phi_a(\alpha)$ and $e = T^\xi_\eta \circ c$. □

Proposition 24.2. *Let M be an m-monoid and ξ a countable ordinal. If $\alpha \in \triangle^{\xi+1} M$ has L.P. and H.P. then it has the strict ξ-heirarchy property.*

Proof. Assume that M is an m-monoid, ξ a countable ordinal, $\alpha \in \triangle^{\xi+1} M$ has L.P. and H.P., $\eta < \xi$, a, b are such that $\langle a \rangle^\frown b \in \alpha$, $n \in \omega$, e has domain n, and $e \in T^\xi_{\eta+1}(\alpha)$. Then $a \in \triangle^\xi M$ and $\langle a \rangle \in \Phi(\alpha)$, so by H.P. there is a $c \in \Phi_a(\alpha)$ such that $e = T^\xi_\eta \circ c$. By L.P., $c^\frown b \in \alpha$. □

For every $\xi < \omega_1$ let

$$\mathscr{L}^\xi = \{\alpha \in \triangle^{\xi+1} \mathscr{W} : \alpha \text{ has the strict } \xi\text{-heirarchy property}\}.$$

Proposition 24.3. $\forall \xi < \omega_1 [\mathscr{K}^\xi \subseteq \mathscr{L}^\xi]$.

Proof. By Propositions 17.3 and 24.2. □

Proposition 24.4. *For all $\xi < \omega_1$, $\mathscr{L}^\xi$ is a submonoid of $\triangle^{\xi+1}\mathscr{W}$.*

Proof. First we deal with 0. Note:

24.4(1) For each $\xi < \omega_1$ let 0_ξ be the zero of $\triangle^\xi\mathscr{W}$. Then
24.4(2a) $0_{\xi+1} = \{\langle 0_\xi\rangle, \langle 0_\xi, 0_\xi\rangle \ldots\}$.
24.4(2b) For ξ limit, 0_ξ is the function with domain ξ such that $\forall\eta < \xi[0_\xi(\eta) = 0_\eta]$.

24.4(3) If $\eta \leq \xi$, then $T^\xi_\eta(0_\xi) = 0_\eta$.

24.4(4) $0_{\xi+1} \in \mathscr{L}_\xi$.

For, suppose that $\eta < \xi$, $\langle a\rangle^\frown b \in 0_{\xi+1}$, $n \in \omega$, e has domain n, and $e \in T^\xi_{\eta+1}(a)$. Then $a = 0_\xi$ and $b = \langle 0_\xi, 0_\xi, \ldots\rangle$. Also, $T^\xi_{\eta+1}(a) = 0_{\eta+1}$, so $e = \langle 0_\eta, 0_\eta, \ldots\rangle$. Let $c_i = 0_\xi$ for all $i < n$. Then $\sum_{i<n} c_i = a$, $c^\frown b \in \alpha$, and $T^\xi_\eta(c_i) = 0_\eta = e_i$. So 24.4(3) holds.

Now suppose that α and β are nonzero elements of $\mathscr{L}^\xi$. Suppose that $\eta < \xi$, $\langle a\rangle^\frown b \in \alpha + \beta$, $n \in \omega$, e has domain n, and $e \in T^\xi_{\eta+1}(a)$. Then there exist a_0, a_1, b_0, b_1 such that $a = a_0 + a_1$, $b = b_0 + b_1$, $\langle a_0\rangle^\frown b_0 \in \alpha$, and $\langle a_1\rangle^\frown b_1 \in \beta$. Then $e \in T^\xi_{\eta+1}(a) = T^\xi_\eta(a_0) + T^\xi_\eta(a_1)$, so there exist $e_0 \in T^\xi_{\eta+1}(a_0)$ and $e_1 \in T^\xi_{\eta+1}(a_1)$ such that $e = e_0 + e_1$. Since α and β have the strict ξ-hierarchy property, we get c^0, c^1 with domain n such that $\sum_{i<n} c^j_i = a_j$, $c^0{}^\frown b_0 \in \alpha$, $c^1{}^\frown b_1 \in \beta$, and $T^\xi_\eta \circ c^j = e_j$ for $j < 2$. Let $c = c^0 + c^1$. Then $\sum_{i<n} c_i = a_0 + a_1 = a$, $c^\frown b \in \alpha + \beta$, and $T^\xi_\eta \circ c = e$. □

Proposition 24.5. *Let M be a locally countable submonoid of $\triangle^{\xi+1}\mathscr{W}$ such that*
(i) M has the refinement property.
(ii) $\forall\alpha \in M\forall a \in \alpha\forall n \in \omega[\mathrm{dmn}(a) = n \to \exists\beta \in {}^nM[\forall i < n[T(\beta_i) = a_i]$ and $\sum_{i<n} \beta_i = \alpha]]$.

Then M^* is isomorphic to a subsemigroup of **NBA**.

Proof. First we note:

24.5(1) M is atomless.

For, $M \subseteq \triangle^{\xi+1}\mathscr{W}$, so M satisfies S.P. Hence M is atomless. By Proposition 19.12, δ_M is an isomorphism of M into $\triangle^{\xi+2}\mathscr{W}$. By Proposition 19.10, if $\alpha \in M$ then $\delta_M(\alpha)$ satisfies R.P. and L.P., and so by Proposition 24.2 $\delta_M(\alpha)$ has the strict $(\xi+1)$-hierarchy property. Then by Propositions 17.3, $\delta_M(\alpha) \in \mathscr{K}^{\xi+1}$. Hence by the definition of $\mathscr{K}^{\xi+1}$ there is a $\sigma_\alpha \in \mathscr{M}$ such that $\delta_M(\alpha) = (\triangle^{\xi+2}\sigma_\alpha)(1)$ and $d(\sigma_\alpha) \leq \xi + 1$.

24.5(2) $\forall\alpha \in M[(\triangle^{\xi+1}\sigma_\alpha)(1) = \alpha]$.

In fact, if $a \in \delta_M(\alpha)$ then we can write $1 = x_0 \dot{+} \cdots \dot{+} x_{n-1}$ with $\forall i < n[a_i = (\triangle^{\xi+1}\sigma_\alpha)(x_i)]$. Then $\alpha = \sum_{i<n} a_i = (\triangle^{\xi+1}\sigma_\alpha)(1)$.

If k is an automorphism of $\mathrm{Fr}(\omega)$ such that $\sigma_\alpha = \sigma_\beta \circ k$, then, using Proposition 14.3, $\delta_M(\alpha) = (\triangle^{\xi+2}\sigma_\alpha)(1) = ((\triangle^{\xi+2}(\sigma_\beta) \circ k))(1) = (\triangle^{\xi+2}\sigma_\beta)(1) = \delta_M(\beta)$, hence $\alpha = \beta$, since by Proposition 19.12 δ_M is one-one.

Suppose that $\alpha, \beta \in M^*$. Then

$$\begin{aligned}\langle(\triangle^{\xi+1}\sigma_\alpha)(1), (\triangle^{\xi+1}\sigma_\beta)(1)\rangle &= \langle(\triangle^{\xi+1}\sigma_\alpha)(1), 0\rangle + \langle 0, (\triangle^{\xi+1}\sigma_\beta)(1)\rangle \\ &\in (\triangle^{\xi+2}\sigma_\alpha)(1) + (\triangle^{\xi+2}\sigma_\beta)(1) \\ &= \delta_M(\alpha) + \delta_M(\beta) = \delta_M(\alpha+\beta) = \\ &= (\triangle^{\xi+2}\sigma_{\alpha+\beta})(1) = (\triangle(\triangle^{\xi+1}\sigma_{\alpha+\beta}))(1).\end{aligned}$$

Hence by the definition of $\triangle\sigma$ before Proposition 10.1, we can write $1 = x + (-x)$ with

$$\langle(\triangle^{\xi+1}\sigma_\alpha)(1), (\triangle^{\xi+1}\sigma_\beta)(1)\rangle = \langle(\triangle^{\xi+1}\sigma_{\alpha+\beta})(x), (\triangle^{\xi+1}\sigma_{\alpha+\beta})(-x)\rangle.$$

So $(\triangle^{\xi+1}\sigma_\alpha)(1) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(x)$ and $(\triangle^{\xi+1}\sigma_\beta)(1) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(-x)$. Let $k_1 : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \restriction x$ and $k_2 : \mathrm{Fr}(\omega) \to \mathrm{Fr}(\omega) \restriction (-x)$ be isomorphisms. Then $(\triangle^{\xi+1}\sigma_{\alpha+\beta})(k_1(1)) = (\triangle^{\xi+1}\sigma_\alpha)(1)$ and $(\triangle^{\xi+1}\sigma_{\alpha+\beta})(k_2(1)) = (\triangle^{\xi+1}\sigma_\beta)(1)$. Now $\triangle^{\xi+1}\sigma_{\alpha+\beta}$ is stable, so by Proposition 23.5, also $(\triangle^{\xi+1}\sigma_{\alpha+\beta}) \circ k_1$ and $(\triangle^{\xi+1}\sigma_{\alpha+\beta}) \circ k_2$ are stable.

24.5(3) $(\triangle^{\xi+2}\sigma_\alpha)(1) = (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x)$.

In fact, suppose that $1 = u_0 + \cdots + u_{n-1} = v_0 + \cdots + v_{m-1}$. Then

$$\langle(\triangle^{\xi+1}\sigma_\alpha)(u_0), \ldots, (\triangle^{\xi+1}\sigma_\alpha)(u_{n-1})\rangle \in (\triangle^{\xi+2}\sigma_\alpha)(1)$$

and

$$\langle(\triangle^{\xi+1}\sigma_\beta)(v_0), \ldots, (\triangle^{\xi+1}\sigma_\beta)(v_{m-1})\rangle \in (\triangle^{\xi+2}\sigma_\beta)(1),$$

so

$$\begin{aligned}&\langle(\triangle^{\xi+1}\sigma_\alpha)(u_0), \ldots, (\triangle^{\xi+1}\sigma_\alpha)(u_{n-1}), 0, 0, \ldots, 0\rangle + \\ &\quad \langle 0, 0, \ldots, 0, (\triangle^{\xi+1}\sigma_\beta)(v_0), \ldots, (\triangle^{\xi+1}\sigma_\beta)(v_{m-1})\rangle \\ &\in (\triangle^{\xi+2}\sigma_\alpha)(1) + (\triangle^{\xi+2}\sigma_\beta)(1) = \delta_M(\alpha) + \delta_M(\beta) = \delta_M(\alpha+\beta).\end{aligned}$$

Hence we can write $1 = w_0 \dotplus \cdots \dotplus w_{n-1} \dotplus w_n \dotplus \cdots \dotplus w_{n+m-1}$ so that $\forall i < n[(\triangle^{\xi+1}\sigma_\alpha)(u_i) = (\triangle^{\xi+1}\sigma_{\alpha+\beta}(w_i)]$ and $\forall i < m[(\triangle^{\xi+1}\sigma_\beta)(v_i) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(w_{n+i})]$. Thus

$$\begin{aligned}(\triangle^{\xi+1}\sigma_{\alpha+\beta})(x) = (\triangle^{\xi+1}\sigma_\alpha)(1) &= \sum_{i<n}(\triangle^{\xi+1}\sigma_\alpha)(u_i) \\ &= \sum_{i<n}(\triangle^{\xi+1}\sigma_{\alpha+\beta})(w_i) = (\triangle^{\xi+1}\sigma_{\alpha+\beta})\left(\sum_{i<n} w_i\right).\end{aligned}$$

Since $\triangle^{\xi+1}\sigma_{\alpha+\beta}$ is stable, it follows that $(\triangle^{\xi+2}\sigma_{\alpha+\beta})(x) = (\triangle^{\xi+2}\sigma_{\alpha+\beta})\left(\sum_{i<n} w_i\right)$. Now

$$\begin{aligned}\langle(\triangle^{\xi+1}\sigma_\alpha)(u_0), \ldots, (\triangle^{\xi+1}\sigma_\alpha)(u_{n-1})\rangle &= \langle(\triangle^{\xi+1}\sigma_{\alpha+\beta})(w_0), \ldots, (\triangle^{\xi+1}\sigma_{\alpha+\beta})(w_{n-1})\rangle \\ &\in (\triangle^{\xi+2}\sigma_{\alpha+\beta})\left(\sum_{i<n} w_i\right) = (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x).\end{aligned}$$

It follows that $(\triangle^{\xi+2}\sigma_\alpha)(1) \subseteq (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x)$.

Now suppose that $a \in (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x)$. Say $a_i = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(y_i)$ for all $i < n$ with $x = y_0 \dot{+} \cdots \dot{+} y_{n-1}$. Then $\sum_{i<n} a_i = (\triangle^{\xi+1}\sigma_{\alpha+\beta})(x) = (\triangle^{\xi+1}\sigma_\alpha)(1) = \alpha$, and so $a \in \delta_M(\alpha) = (\triangle^{\xi+2}\sigma_\alpha)(1)$. Thus $(\triangle^{\xi+2}\sigma_\alpha)(1) = (\triangle^{\xi+2}\sigma_{\alpha+\beta})(x)$. Thus (130) holds.

24.5(4) There is an automorphism l of $\mathrm{Fr}(\omega)$ such that $\triangle^{\xi+1}\sigma_{\alpha+\beta} \circ k_1 = \triangle^{\xi+1}\sigma_\alpha \circ l$.

In fact, by 24.5(3) we have $(\triangle^{\xi+2}\sigma_\alpha)(1) = (\triangle^{\xi+2}\sigma_{\alpha+\beta}) \circ k_1$. Hence 24.5(4) holds by Proposition 12.2.

By symmetry we have

24.5(5) There is an automorphism l' of $\mathrm{Fr}(\omega)$ such that $\triangle^{\xi+1}\sigma_{\alpha+\beta} \circ k_2 = \triangle^{\xi+1}\sigma_\alpha \circ l'$.

Now by Proposition 23.4 with ρ, σ, τ replaced by $\triangle^{\xi+1}\sigma_{\alpha+\beta}$, $\triangle^{\xi+1}\sigma_\alpha \circ l$ and $\triangle^{\xi+1}\sigma_\alpha \circ l'$, we get an isomorphism s of $\mathrm{Fr}(\omega)$ onto $\mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega)$ such that $\triangle^{\xi+1}\sigma_{\alpha+\beta} = (\triangle^{\xi+1}\sigma_\alpha \circ l) \oplus_s (\triangle^{\xi+1}\sigma_\beta \circ l')$. Now we apply Proposition 23.2(x) with $k_0, k_1, \sigma, \sigma', \tau, \tau', l_0, l_1$ replaced by $l, l', \triangle^{\xi+1}\sigma_\alpha \circ l, \triangle^{\xi+1}\sigma_\alpha, \triangle^{\xi+1}\sigma_\beta \circ l', \triangle^{\xi+1}\sigma_\beta, s, s$; we get an automorphism k_2 of $\mathrm{Fr}(\omega)$ such that $\triangle^{\xi+1}\sigma_{\alpha+\beta} = ((\triangle^{\xi+1}\sigma_\alpha) \oplus_s (\triangle^{\xi+1}\sigma_\beta)) \circ k_2$. By Proposition 23.3 we then have $\triangle^{\xi+1}\sigma_{\alpha+\beta} = (\triangle^{\xi+1}(\sigma_\alpha \oplus_s \sigma_\beta)) \circ k_2$. Then $(\triangle^{\xi+1}\sigma_{\alpha+\beta})(1) = (\triangle^{\xi+1}(\sigma_\alpha \oplus_s \sigma_\beta))(1)$. By Proposition 15.2 there is an automorphism k_3 of $\mathrm{Fr}(\omega)$ such that $\sigma_{\alpha+\beta} = (\sigma_\alpha \oplus_s \sigma_\beta) \circ k_3$.

Now for the proof of the proposition, for each $\alpha \in M$ let $f(\alpha) = B_{\sigma_\alpha}$; see Proposition 6.24. Then $f(\alpha+\beta) = B_{\sigma_{\alpha+\beta}} \cong B_{\sigma_\alpha \oplus_s \sigma_\beta}$ (by Proposition 16.1) $\cong B_{\sigma_\alpha} \times B_{\sigma_\beta}$ (by Proposition 23.6) $= f(\alpha) \times f(\beta)$. So f is a homomorphism. If $f(\alpha) \cong f(\beta)$, then by Proposition 16.1 there is an automorphism k_4 of $\mathrm{Fr}(\omega)$ such that $\sigma_\alpha = \sigma_\beta \circ k_4$, and so by a remark early in this proof, $\alpha = \beta$. □

25. Shifting monoids

Recall the definition of $\mathcal{N}$ from just before Proposition 9.3. For any $\theta \in \mathcal{N}$ let

$$\mathrm{supp}(\theta) = \{\xi < \omega_1 : \theta(\xi) > 0\}.$$

Proposition 25.1. *Suppose that $\theta \in \mathcal{N}$.*

(i) $\mathrm{supp}(\theta)$ *has a largest element.*

(ii) $\mathrm{supp}(\theta)$ *is countable and nonempty.*

(iii) $\theta(\min(\mathrm{supp}(\theta))) = \omega$. □

Now if $a \in {}^{<<\omega}W$ we define $\varphi_a : \omega_1 \to \omega$ by $\varphi_a(\xi) = |\{i < \mathrm{dmn}(a) : a_i = \xi\}|$. Then $\mathrm{supp}(\varphi_a) = \{\xi < \omega_1 : \varphi_a(\xi) > 0\}$. Then for any $\theta \in \mathcal{N}^*$ we let $\Psi(\theta)$ be $g(\theta)$ with g as in the proof of Proposition 9.6. We also set $\Psi(o) = o$.

Two mappings $\theta, \chi : \omega+1 \to \omega+1$ are *equal almost everywhere*, in symbols $\theta \equiv_{ae} \chi$, iff $\{k \leq \omega : \theta(k) \neq \chi(k)\}$ is finite. 0^c and ω^c are the constant mappings: $0^c(k) = 0$ and $\omega^c(k) = \omega$ for all $k \in \omega+1$. $\mathcal{N}^{0^*}$ is the set of all $\theta : \omega+1 \to \omega+1$ such that one of the following holds:

(AE1) $\theta \equiv_{ae} 0^c$.

(AE2) $\theta \equiv_{ae} \omega^c$ and $\theta(\omega) = \omega$.

(AE3) $\{k \in \omega+1 : \theta(k) = \omega\}$ is finite, and $\lim_{k\to\infty} \theta(k) = \omega = \theta(\omega)$.

Here this limit is with respect to the order topology on $\omega+1$. Thus $\lim_{k\to\infty} \theta(k) = \omega$ iff $\forall l < \omega \exists m \forall k \in (m,\omega)[l < \theta(k)]$.

Addition in $\mathcal{N}^{0^*}$ is defined like this: $\forall k \in \omega+1[(\theta+\chi)(k) = \theta(k)+\chi(k)]$, where $\omega + k = k + \omega = \omega$ for any $k \in \omega+1$. We also adjoin a new element o to $\mathcal{N}^{0^*}$ and define $o + k = k + o = k$ for any $k \in (\omega+1) \cup \{o\}$. Finally, $\mathcal{N}^0 = \mathcal{N}^{0^*} \cup \{o\}$.

We also define $x \equiv'_{ae} y$ iff one of the following holds:

(AE4) $x, y \in \mathcal{N}^{0^*}$ and $x \equiv_{ae} y$.

(AE5) $x \in \mathcal{N}^{0^*}$, $y = o$, and $x \equiv_{ae} 0^c$.

(AE6) $x = o$, $y \in \mathcal{N}^{0^*}$, and $y \equiv_{ae} 0^c$.

(AE7) $x = y = o$.

We also define $k + l$ for $k, l \in \omega+1$ by

$$k + l = \begin{cases} k+l & \text{if } k,l \in \omega, \\ \omega & \text{if } k = \omega \text{ or } l = \omega. \end{cases}$$

Proposition 25.2. $(\omega+1, +)$ *is an r-monoid.*

Proof. Clearly $(\omega+1,+)$ is an m-monoid. To show that it is an r-monoid we apply Proposition 19.7. So, suppose that $a, b \in {}^2(\omega+1)$ and $a_0 + a_1 = b_0 + b_1$. We want to find $c \in {}^{2\times 2}(\omega+1)$ such that $\forall i < 2[a_i = \sum_{j<2} c_{ij}]$ and $\forall j < 2[b_j = \sum_{i<2} c_{ij}]$.

Case 1. $a_0 = a_1 = b_0 = b_1 = \omega$. Let $c_{ij} = \omega$ for all $i, j < 2$.

Case 2. $a_0 = a_1 = \omega = b_0$ and $b_1 < \omega$. Let $c_{01} = 0$, $c_{11} = b_1$, and $c_{ij} = \omega$ otherwise.

Case 3. $a_0 = b_0 = b_1 = \omega$ and $a_1 < \omega$. Symmetric to Case 2.

Case 4. $a_0 = b_0 = \omega$ and $a_1, b_1 < \omega$. Let $c_{10} = a_1$, $c_{11} = 0$, $c_{01} = b_1$, and $c_{ij} = \omega$ otherwise.

Case 5. Other cases with one or more of a_0, a_1, b_0, b_1 equal to ω. This is symmetric to above cases.

Case 6. $a_0, a_1, b_0, b_1 < \omega$. Wlog $a_0 \le b_0$. Let $c_{00} = a_0$, $c_{01} = 0$, $c_{10} = b_0 - a_0$, and $c_{11} = b_1$. Then $c_{00} + c_{01} = a_0$, $c_{10} + c_{11} = b_0 - a_0 + b_1 = a_1$, $c_{00} + c_{10} = a_0 + b_0 - a_0 = b_0$, and $c_{01} + c_{11} = b_1$. □

Proposition 25.3. $\mathcal{N}^0$ *is an r-monoid.*

Proof. First we check that $\mathcal{N}^0$ is closed under $+$. Suppose that $x, y \in \mathcal{N}^0$. Obviously $x + y \in \mathcal{N}^0$ if one of x, y is o. Now suppose that $x, y \neq o$. So $x, y : \omega+1 \to \omega+1$.

Case 1. $x, y \equiv_{ae} 0^c$. Clearly $x + y \equiv_{ae} 0^c$.

Case 2. $x \equiv_{ae} 0^c$, $y \equiv_{ae} \omega^c$, and $y(\omega) = \omega$. Clearly $x + y \equiv_{ae} \omega$ and $(x+y)(\omega) = \omega$.

Case 3. $x \equiv_{ae} 0^c$, $\{k \in \omega+1 : y(k) = \omega\}$ is finite, $\lim_{k\to\infty} y(k) = \omega = y(\omega)$. Then clearly $\{k \in \omega+1 : (x+y)(k) = \omega\}$ is finite, $\lim_{k\to\infty}(x+y)(k) = \omega = (x+y)(\omega)$.

Case 4. $x \equiv_{ae} \omega^c$, $y \equiv_{ae} 0^c$. Symmetric to Case 2.

Case 5. $x \equiv_{ae} \omega^c$, $y \equiv_{ae} \omega^c$, $x(\omega) = \omega$, $y(\omega) = \omega$. Clearly $x + y \equiv_{ae} \omega^c$ and $(x+y)(\omega) = \omega$.

Case 6. $x \equiv_{ae} \omega^c$, $\{k \in \omega+1 : y(k) = \omega\}$ is finite, $\lim_{k\to\infty} y(k) = \omega = y(\omega)$. Clearly $x + y \equiv_{ae} \omega^c$ and $(x+y)(\omega) = \omega$.

Case 7. $\{k \in \omega+1 : x(k) = \omega\}$ is finite, $\lim_{k\to\infty} x(k) = \omega = x(\omega)$, $y \equiv_{ae} 0^c$. Symmetric to Case 3.

Case 8. $\{k \in \omega+1 : x(k) = \omega\}$ is finite, $\lim_{k\to\infty} x(k) = \omega = x(\omega)$, $y \equiv_{ae} \omega^c$. Symmetric to Case 6.

Case 9. $\{k \in \omega+1 : x(k) = \omega\}$ is finite, $\lim_{k\to\infty} x(k) = \omega = x(\omega)$, $\{k \in \omega+1 : y(k) = \omega\}$ is finite, $\lim_{k\to\infty} y(k) = \omega = y(\omega)$. Clearly $\{k \in \omega+1 : (x+y)(k) = \omega\}$ is finite, $\lim_{k\to\infty}(x+y)(k) = \omega = (x+y)(\omega)$,

So $\mathcal{N}^0$ is closed under $+$.

Clearly now $\mathcal{N}^0$ is an m-monoid.

To prove that $\mathcal{N}^0$ is an r-monoid we again apply Proposition 19.7. So, suppose that $a_0, a_1, b_0, b_1 \in \mathcal{N}^0$ and $a_0 + a_1 = b_0 + b_1$. We want to define $\theta \in {}^{2\times 2}\mathcal{N}^0$ so that $\forall i < 2[a_i = \sum_{j<2} \theta_{ij}]$ and $\forall j < 2[b_j = \sum_{i<2} \theta_{ij}]$.

Case 1. One or more of a_0, a_1, b_0, b_1 is equal to o. By symmetry say $a_0 = o$. Let $\theta_{00} = o = \theta_{01}$, $\theta_{10} = b_0$, and $\theta_{11} = b_1$.

Case 2. All of a_0, a_1, b_0, b_1 are different from o. For each $k \le \omega$, by Proposition 25.2 there is a $\langle \theta_{ij}(k) : i, j < 2\rangle$ such that each $\theta_{ij}(k) \in \omega+1$, $\forall i < 2[a_i(k) = \sum_{j<2} \theta_{ij}(k)]$, and $\forall j < 2[b_j(k) = \sum_{i<2} \theta_{ij}(k)]$.

Subcase 2.1. $a_0 \equiv_{ae} 0^c$ and $a_1 \equiv_{ae} 0^c$. Clearly then $b_0 \equiv_{ae} 0^c$ and $b_1 \equiv_{ae} 0^c$. Clearly also $\theta_{ij} \equiv_{ae} 0^c$ for all $i, j < 2$.

Subcase 2.2. $a_0 \equiv_{ae} 0^c$, $a_1 \equiv_{ae} \omega^c$, and $a_1(\omega) = \omega$. Then $a_0 + a_1 \equiv_{ae} \omega^c$, so also $b_0 + b_1 \equiv_{ae} \omega^c$.

Subsubcase 2.2.1. $b_0 \equiv_{ae} 0^c$. Then $b_1 \equiv_{ae} \omega^c$ and $b_1(\omega) = \omega$ since $b_0 + b_1 \equiv_{ae} \omega^c$. Now the set $M \stackrel{\text{def}}{=} \{k \in \omega+1 : a_0(k) \neq 0 \text{ or } a_1(k) \neq \omega \text{ or } b_0(k) \neq 0 \text{ or } b_1(k) \neq \omega\}$ is finite. For $k \in (\omega+1)\backslash M$ we have $0 = a_0(k) = \sum_{j<2} \theta_{0j}(k)$, hence $\theta_{00}(k) = 0$ and $\theta_{01}(k) = 0$. Also, $0 = b_0(k) = \sum_{i<2} \theta_{i0}(k)$, hence $\theta_{00}(k) = 0$ and $\theta_{10}(k) = 0$. Thus $\theta_{00}, \theta_{01}, \theta_{10}$ are in $\mathcal{N}^0$ by (133). Also $\omega = a_1(k) = \sum_{j<2} \theta_{1j}(k)$ while $\theta_{10}(k) = 0$, so $\theta_{11}(k) = \omega$. Let θ'_{11} be equal to θ_{11} except that $\theta_{11}(\omega) = \omega$. Then $\theta'_{11} \in \mathcal{N}^0$, $a_1(\omega) = \sum_{j<2} \theta'_{1j}(\omega)$, and $b_1(\omega) = \sum_{i<2} \theta'_{i1}(\omega)$, with $\theta'_{01} = \theta_{01}$.

Subsubcase 2.2.2. $b_1 \equiv_{ae} 0^c$. This is symmetric to Subsubcase 2.2.1.

Subsubcase 2.2.3. $b_0 \equiv_{ae} \omega^c \equiv_{ae} b_1$ and $b_0(\omega) = b_1(\omega) = \omega$. Then the set $M \stackrel{\text{def}}{=} \{k \in \omega+1 : a_0(k) \neq 0 \text{ or } a_1(k) \neq \omega \text{ or } b_0(k) \neq \omega \text{ or } b_1(k) \neq \omega\}$ is finite. For $k \in (\omega+1)\backslash M$ we have $0 = a_0(k) = \sum_{j<2} \theta_{0j}(k)$, hence $\theta_{00}(k) = \theta_{01}(k) = 0$. Hence $\theta_{00}, \theta_{01} \in \mathcal{N}^0$. Also, $\omega = b_0(k) = \sum_{i<2} \theta_{i0}(k)$ and $\theta_{00}(k) = 0$, so $\theta_{10}(k) = \omega$. Also $\omega = b_1(k) = \sum_{i<2} \theta_{i1}(k)$ and $\theta_{01}(k) = 0$, so $\theta_{11}(k) = \omega$. Let θ'_{10} be like θ_{10} except that $\theta'_{10}(\omega) = \omega$, and let θ'_{11} be like θ_{11} except that $\theta'_{11}(\omega) = \omega$, Then $\theta'_{10}, \theta'_{11} \in \mathcal{N}^0$, $a_1 = \sum_{j<2} \theta'_{1j}$, $b_0 = \sum_{i<2} \theta'_{i0}$, and $b_1 = \sum_{i<2} \theta'_{i1}$, with $\theta'_{00} = \theta_{00}$ and $\theta'_{01} = \theta_{01}$.

Subsubcase 2.2.4. $b_0 \equiv_{ae} \omega^c$, $b_0(\omega) = \omega$, $\{k \in \omega+1 : b_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} b_1(k) = \omega = b_1(\omega)$. Then $M \stackrel{\text{def}}{=} \{k \in \omega+1 : a_0(k) \neq 0 \text{ or } a_1(k) \neq \omega \text{ or } b_0(k) \neq \omega \text{ or } b_1(k) = \omega\}$ is finite. Take any $k \in (\omega+1)\backslash M$. Then as in Subsubcase 2.2.3, $\theta_{00}(k) = \theta_{01}(k) = 0$ and $\theta_{10}(k) = \omega$. Hence $\theta_{00}, \theta_{01} \in \mathcal{N}^0$. Also $b_1(k) = \sum_{i<1} \theta_{i1}(k)$ and $\theta_{01}(k) = 0$, so $b_1(k) = \theta_{11}(k)$. Let θ'_{10} be like θ_{10} except that $\theta'_{10}(\omega) = \omega$, and let θ'_{11} be

like θ_{11} except that $\theta'_{11}(\omega) = \omega$. Then $\theta'_{10}, \theta'_{11} \in \mathcal{N}^0$. $a_1 = \sum_{j<q} \theta'_{1j}$, $b_0 = \sum_{i<2} \theta'_{i0}$, and $b_1 = \sum_{i<2} \theta'_{i1}$, with $\theta'_{00} = \theta_{00}$ and $\theta'_{01} = \theta_{01}$.

Subsubcase 2.2.5. $b_1 \equiv_{ae} \omega^c$, $b_1(\omega) = \omega$, $\{k \in \omega + 1 : b_0(k) = \omega\}$ is finite, and $\lim_{k\to\infty} b_0(k) = \omega = b_0(\omega)$. This is symmetric to Subsubcase 2.2.4.

Subcase 2.3. $a_0 \equiv_{ae} 0^c$, $\{k \in \omega + 1 : a_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} a_1(k) = \omega = a_1(\omega)$. Then (134) does not hold for b_0 or b_1. Also b_0 and b_1 are not both $\equiv_{ae} 0^c$.

Subsubcase 2.3.1. $b_0 \equiv_{ae} 0^c$, $\{k \in \omega + 1 : b_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} b_1(k) = \omega = b_1(\omega)$. Then $M \stackrel{\text{def}}{=} \{k \in \omega + 1 : a_0(k) \neq 0$ or $a_1(k) = \omega$ or $b_0(k) \neq 0$ or $b_1(k) = \omega\}$ is finite. Suppose that $k \in (\omega + 1)\backslash M$. Then $0 = a_0(k) = \sum_{j<2} \theta_{0j}(k)$, so $\theta_{00}(k) = \theta_{01}(k) = 0$. Also, $0 = b_0(k) = \sum_{i<2} \theta_{i0}(k)$, so $\theta_{10}(k) = 0$. Thus $\theta_{00}, \theta_{01}, \theta_{10} \in \mathcal{N}^0$. Now $a_1(k) = \sum_{j<2} \theta_{1j}(k)$ and $\theta_{10}(k) = 0$, so $a_1(k) = \theta_{11}(k)$. Let θ'_{11} be like θ_{11} except that $\theta'_{11}(\omega) = \omega$). Then $\theta'_{11} \in \mathcal{N}^0$, $\forall i < 2[a_i = \sum_{j<2} \theta'_{ij}]$, and $\forall j < 2[b_j = \sum_{i<2} \theta'_{ij}]$, with $\theta'_{00} = \theta_{00}$, $\theta'_{01} = \theta_{01}$, $\theta'_{10} = \theta_{10}$.

Subsubcase 2.3.2. $\{k \in \omega + 1 : b_0(k) = \omega\}$ is finite, $\lim_{k\to\infty} b_0(k) = \omega = b_0(\omega)$, $\{k \in \omega + 1 : b_1(k) = \omega\}$ is finite, $\lim_{k\to\infty} b_1(k) = \omega = b_1(\omega)$, Then $M \stackrel{\text{def}}{=} \{k \in \omega + 1 : a_0(k) \neq 0$ or $a_1(k) = \omega$ or $b_0(k) = \omega$ or $b_1(k) = \omega\}$ is finite. Suppose that $k \in (\omega + 1)\backslash M$. Then $\theta_{00}(k) = \theta_{01}(k) = 0$ as in Subsubcase 2.3.1. Hence $\theta_{00}, \theta_{01} \in \mathcal{N}^0$. $b_0(k) = \sum_{i<2} \theta_{i0}(k) = \theta_{10}(k)$. $b_1(k) = \sum_{i<2} \theta_{i1}(k) = \theta_{11}(k)$. Let θ'_{10} be like θ_{10} except that $\theta'_{10}(\omega) = \omega$, and let θ'_{11} be like θ_{11} except that $\theta'_{11}(\omega) = \omega$. Then $\theta'_{10}, \theta'_{11} \in \mathcal{N}^0$, $\forall i < 2[a_i = \sum_{j<2} \theta'_{ij}]$, and $\forall j < 2[b_j = \sum_{i<2} \theta'_{ij}]$, where $\theta'_{00} = \theta_{00}$ and $\theta'_{01} = \theta_{01}$.

Subcase 2.4. $a_0 \equiv_{ae} \omega^c \equiv_{ae} a_1$ and $a_0(\omega) = \omega = a_1(\omega)$. Then $b_0 \equiv_{ae} \omega^c$ and $b_0(\omega) = \omega$, or $b_1 \equiv_{ae} \omega^c$ and $b_1(\omega) = \omega$.

Subsubcase 2.4.1 $b_0 \equiv_{ae} \omega^c \equiv_{ae} b_1$, $b_0 = \omega$, and $b_1(\omega) = \omega$. Then $M = \{k \in \omega + 1 : a_0(k) \neq \omega$ or $a_1(k) \neq \omega$ or $b_0(k) \neq \omega$ or $b_1(k) \neq \omega\}$ is finite. For $k \in (\omega + 1)\backslash M$ we have $\theta_{ij}(k) = \omega$ for all $i, j < 2$. Defining θ'_{ij} as above, the desired conclusion follows.

Subsubcase 2.4.2 $b_0 \equiv_{ae} \omega^c$, $b(\omega) = \omega$, $\{k \in \omega + 1 : b_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} b_1(k) = \omega = b_1(\omega)$. Then $M = \{k \in \omega + 1 : a_0(k) \neq \omega$ or $a_1(k) \neq \omega$ or $b_0(k) \neq \omega$ or $b_1(k) = \omega\}$ is finite. Suppose that $k \in (\omega + 1)\backslash M$. Then $b_1(k) = \sum_{i<2} \theta_{i1}(k)$, so $\theta_{01}(k) \neq \omega \neq \theta_{11}(k)$. $a_0(k) = \sum_{j<2} \theta_{0j}(k)$, so $\theta_{00}(k) = \omega$. $a_1(k) = \sum_{j<2} \theta_{1j}(k)$, so $\theta_{10}(k) = \omega$. Thus $\theta_{00}, \theta_{10} \in \mathcal{N}^0$. Now we define for $k \in \omega + 1$

$$\theta'_{00}(k) = \begin{cases} \theta_{00}(k) & \text{if } k < \omega, \\ \omega & \text{if } k = \omega \end{cases} \quad \theta'_{10}(k) = \begin{cases} \theta_{10}(k) & \text{if } k < \omega, \\ \omega & \text{if } k = \omega \end{cases};$$

$$\theta'_{01}(k) = \begin{cases} \theta_{01}(k) & \text{if } k \in M,\ k \neq \omega, \\ b_1(k) & \text{if } k \notin M,\ k \neq \omega, \\ \omega & \text{if } k = \omega, \end{cases} \quad \theta'_{11}(k) = \begin{cases} \theta_{11}(k) & \text{if } k \in M, \\ 0 & \text{if } k \notin M \text{ or } k = \omega. \end{cases}$$

Then each $\theta'_{ij} \in \mathcal{N}^0$, $\forall i < 2[a_i = \sum_{j<2} \theta'_{ij}]$, and $\forall j < 2[b_j = \sum_{i<2} \theta'_{ij}]$.

Subcase 2.5. $a_0 \equiv_{ae} \omega^c$, $a_0(\omega) = \omega$, $\{k \in \omega + 1 : a_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} a_1(k) = \omega = a_1(\omega)$. Then (134) holds for b_0 or b_1. If it holds for both, we have a subcase symmetric to Subcase 2.4; and if (133) holds for one and (134) for the other we have a subcase symmetric to Subcase 2.2. So we may assume that $b_0 \equiv_{ae} \omega^c$, $b_0(\omega) = \omega$, $\{k \in \omega + 1 : b_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} b_1(k) = \omega = b_1(\omega)$. Then $M = \{k \in \omega + 1 : a_0(k) \neq \omega$ or $a_1(k) = \omega$ or $b_0(k) \neq \omega$ or $b_1(k) = \omega\}$ is finite. Suppose

that $k \in (\omega+1)\backslash M$. Now $a_1(k) = \sum_{j<2} \theta_{1j}(k)$, so $\theta_{10}(k) \neq \omega \neq \theta_{11}(k)$. Also $b_1(k) = \sum_{i<2} \theta_{i1}(k)$, so $\theta_{01}(k) \neq \omega \neq \theta_{11}(k)$. Since $a_0(k) = \sum_{j<2} \theta_{0j}(k)$, we have $\theta_{00}(k) = \omega$. Define

$$\theta'_{00}(k) = \begin{cases} \theta_{00}(k) & \text{if } k < \omega, \\ \omega & \text{if } k = \omega, \end{cases} \quad \theta'_{11}(k) = \begin{cases} \theta_{11}(k) & \text{if } k \in M,\ k \neq \omega, \\ \lfloor \min(a_1(k), b_1(k))/2 \rfloor & \text{if } k \notin M,\ k \neq \omega, \\ \omega & \text{if } k = \omega, \end{cases}$$

$$\theta'_{10}(k) = \begin{cases} a_1(k) - \theta'_{11}(k) & \text{if } k \notin M,\ k \neq \omega, \\ \theta_{10}(k) & \text{if } k \in M,\ k \neq \omega, \\ \omega & \text{if } k = \omega \end{cases}$$

$$\theta'_{01}(k) = \begin{cases} b_1(k) - \theta'_{11}(k) & \text{if } k \notin M,\ k \neq \omega, \\ \theta_{01}(k) & \text{if } k \in M,\ k \neq \omega, \\ \omega & \text{if } k = \omega. \end{cases}$$

Clearly $\forall i < 2[a_i = \sum_{j<2} \theta'_{ij}]$ and $\forall j < 2[b_j = \sum_{i<2} \theta'_{ij}]$; and also each $\theta'_{ij} \in \mathcal{N}^0$.

Subcase 2.6. $\{k \in \omega+1 : a_0(k) = \omega\}$ is finite, $\lim_{k\to\infty} a_0(k) = \omega = a_0(\omega)$, $\{k \in \omega+1 : a_1(k) = \omega\}$ is finite, $\lim_{k\to\infty} a_1(k) = \omega = a_1(\omega)$, $\{k \in \omega+1 : b_0(k) = \omega\}$ is finite, $\lim_{k\to\infty} b_0(k) = \omega = b_0(\omega)$, $\{k \in \omega+1 : b_1(k) = \omega\}$ is finite, and $\lim_{k\to\infty} b_1(k) = \omega = b_1(\omega)$. Then $M \stackrel{\text{def}}{=} \{k \in \omega+1 : a_0(k) = \omega \text{ or } a_1(k) = \omega \text{ or } b_0(k) = \omega \text{ or } b_1(k) = \omega\}$ is finite.

(*) Suppose that $k \notin M \cup \{\omega\}$ and $a_0(k) \min(a_0(k), a_1(k), b_0(k), b_1(k))$. Then there are c^k_{ij} for $i, j < 2$ such that $\forall i < 2[a_i(k) = \sum_{j<2} c^k_{ij}]$, $\forall j < 2[b_j(k) = \sum_{i<2} c^k_{ij}]$, and $\forall i, j < 2[\lfloor a_0(k)/2 \rfloor \leq c^k_{ij}]$.

For, let $c^k_{01} = \lfloor a_0(k)/2 \rfloor$, $c^k_{00} = a_0(k) - c^k_{01}$, $c^k_{10} = b_0(k) - c^k_{00}$, $c^k_{11} = b_1(k) - c^k_{01}$. Then $c^k_{00} + c^k_{01} = a_0(k)$, $c^k_{00} + c^k_{10} = b_0(k)$, $c^k_{01} + c^k_{11} = b_1(k)$, and $c^k_{10} + c^k_{11} = b_0(k) - c^k_{00} + b_1(k) - c^k_{01} = a_0(k) - c^k_{00} + a_1(k) - c^k_{01} = a_1(k)$. Moreover,

$$c^k_{00} = a_0(k)/2 + a(0)/2 - \lfloor a_0(k)/2 \rfloor \geq \lfloor a_0(k)/2 \rfloor;$$

$$c^k_{10} \geq a_0(k) - (a_0(k) - c^k_{01}) = c^k_{01} = \lfloor a_0(k)/2 \rfloor;$$

$$c^k_{11} \geq a_0(k) - \lfloor a_0(k)/2 \rfloor \geq \lfloor a_0(k)/2 \rfloor.$$

Thus (*) holds.

We obtain c^k_{ij} similarly for other cases of $\min(a_0(k), a_1(k), b_0(k), b_1(k))$. Now for each $i, j < 2$ and $k \in \omega+1$ define

$$\theta'_{ij}(k) = \begin{cases} \theta_{ij}(k) & \text{if } k \in M\backslash\{\omega\}, \\ c^k_{ij} & \text{if } k \notin M \cup \{\omega\}, \\ \omega & \text{if } k = \omega. \end{cases}$$

Clearly the desired conditions hold. □

Proposition 25.4. *Suppose that $a_0, a_1, b_0, b_1 \in \omega$, one of them is 0, and $a_0 + a_1 = b_0 + b_1$. Then there exists $c \in {}^{2\times 2}\omega$ such that $\forall i < 2[a_i = \sum_{j<2} c_{ij}]$, and $\forall j < 2[b_j = \sum_{i<2} c_{ij}]$.*

Proof. Say wlog $a_0 = 0$. Let $c_{00} = c_{01} = 0$, $c_{10} = b_0$, and $c_{11} = b_1$. Then $a_0 = c_{00} + c_{01}$, $a_1 = c_{10} + c_{11}$, $b_0 = c_{00} + c_{10}$, and $b_1 = c_{01} + c_{11}$. $\square$

Proposition 25.5. *Suppose that* $a_0, a_1, b_0, b_1 \in \omega\backslash\{0\}$, $0 < m < \min\{a_0, a_1, b_0, b_1\}/2$, $a_0 + a_1 = b_0 + b_1$. *Then there exists* $c \in {}^{2\times 2}\omega$ *such that* $\min\{c_{ij} : i, j < 2\} = m$, $c_{00} = m$ *or* $c_{11} = m$, $\forall i < 2[a_i = \sum_{j<2} c_{ij}]$, *and* $\forall j < 2[b_j = \sum_{i<2} c_{ij}]$.

Proof. Assume the hypotheses.

Case 1. $\min\{a_0, a_1, b_0, b_1\} = b_0$. Let $c_{00} = m$, $c_{01} = a_0 - m$, $c_{10} = b_0 - m$, $c_{11} = a_1 - b_0 + m$. So $c_{00} + c_{01} = a_0$, $c_{10} + c_{11} = a_1$, $c_{00} + c_{10} = b_0$, $c_{01} + c_{11} = b_1$.

Case 2. $\min\{a_0, a_1, b_0, b_1\} = b_1$. Let $c_{11} = m$, $c_{01} = b_1 - m$, $c_{10} = a_1 - m$, $c_{00} = a_0 - b_1 + m$. So $c_{00} + c_{01} = a_0$, $c_{10} + c_{11} = a_1$, $c_{00} + c_{10} = b_0$, $c_{01} + c_{11} = b_1$.

Case 3. $\min\{a_0, a_1, b_0, b_1\} = a_0$. Let $c_{00} = m$, $c_{01} = a_0 - m$, $c_{10} = b_0 - m$, $c_{11} = b_1 - a_0 + m$. So $c_{00} + c_{01} = a_0$, $c_{10} + c_{11} = a_1$, $c_{00} + c_{10} = b_0$, $c_{01} + c_{11} = b_1$.

Case 4. $\min\{a_0, a_1, b_0, b_1\} = a_1$. Let $c_{11} = m$, $c_{01} = b_1 - m$, $c_{10} = a_1 - m$, $c_{00} = b_0 - a_1 + m$. So $c_{00} + c_{01} = a_0$, $c_{10} + c_{11} = a_1$, $c_{00} + c_{10} = b_0$, $c_{01} + c_{11} = b_1$. $\square$

Proposition 25.6. *If* $\theta_0 + \theta_1 = \psi_0 + \psi_1$ *in* $\mathcal{N}^{0*}$, $\theta_0 \not\equiv_{ae} 0^c$, $\theta_1 \not\equiv_{ae} 0^c$, $\psi_0 \not\equiv_{ae} 0^c$, $\psi_1 \not\equiv_{ae} 0^c$, *and not*$(\theta_0 \equiv_{ae} \theta_1 \equiv_{ae} \psi_0 \equiv_{ae} \psi_1 \equiv_{ae} \omega^c)$, *then there are uncountably many* $\chi \in {}^{2\times 2}(\mathcal{N}^{0*})$ *such that* $\forall i < 2[\theta_i = \sum_{j<2} \chi_{ij}]$ *and* $\forall j < 2[\psi_j = \sum_{i<2} \chi_{ij}]$.

Proof. For each $k \in \omega + 1$ let $\mu(k) = \min\{\theta_0(k), \theta_1(k), \psi_0(k), \psi_1(k)\}$. Then $\mu \in \mathcal{N}^{0*}$. In fact, this is clear since one of $\theta_0, \theta_1, \psi_0, \psi_1$ satisfies satisfies (135) and so clearly μ does also. Clearly also $\mu \not\equiv_{ae} 0^c, \omega^c$. Let $F \subseteq \omega$ be a finite set such that $\forall k \in \omega\backslash F[0 < \mu(k) < \omega]$. For each $E \subseteq \omega\backslash F$ we define $\chi^E \in {}^{2\times 2}(\mathcal{N}^{0*})$. For $k \in F$ take $\chi^E_{ij}(k)$ by Proposition 25.2 so that $\forall i < 2[\theta_i(k) = \sum_{j<2} \chi^E_{ij}(k)]$ and $\forall j < 2[\psi_j = \sum_{i<2} \chi^E_{ij}(k)]$. For $k \in E$ take $\chi^E_{ij}(k)$ by Proposition 25.5 so that $\forall i < 2[\theta_i(k) = \sum_{j<2} \chi^E_{ij}(k)]$, $\forall j < 2[\psi_j = \sum_{i<2} \chi^E_{ij}(k)]$, and $\min\{\chi^E_{ij}(k) : i, j < 2\} = \mu(k)/3$. For $k \in \omega\backslash(F \cup E)$ take $\chi^E_{ij}(k)$ by Proposition 25.5 so that $\forall i < 2[\theta_i(k) = \sum_{j<2} \chi^E_{ij}(k)]$, $\forall j < 2[\psi_j = \sum_{i<2} \chi^E_{ij}(k)]$, and $\min\{\chi^E_{ij}(k) : i, j < 2\} = \mu(k)/4$. Finally, for any $i, j < 2$ we let $\chi^E_{ij}(k) = \omega$. $\square$

For each $\theta \in \mathcal{N}^{0^*}$ define $(\Omega\theta) : \omega_1 \to \omega + 1$ by:

$$(\Omega\theta)(\alpha) = \begin{cases} \omega & \text{if } \alpha = 0, \\ \theta(\alpha - 1) & \text{if } 0 < \alpha < \omega, \\ \theta(\omega) & \text{if } \alpha = \omega, \\ 0 & \text{if } \omega < \alpha < \omega_1. \end{cases}$$

Proposition 25.7. $(\Omega\theta) \in \mathcal{N}^*$. $\square$

We also define $(\Omega o) = o$. So $(\Omega x) \in \mathcal{N}$ for any $x \in \mathcal{N}^0$.

Proposition 25.8. $\Omega : \mathcal{N}^0 \to \mathcal{N}$ *is a morphism of m-monoids.*

Proof. $(\Omega(o + o)) = (\Omega o) = o = o + o = (\Omega o) + (\Omega o)$. For any $\theta \in \mathcal{N}^{0^*}$ we have $(\Omega(\theta + o)) = (\Omega\theta) = (\Omega\theta) + o = (\Omega\theta) + (\Omega o)$ and similarly for $o = \theta$. Finally, suppose that

$\theta, \psi \in \mathcal{N}^{0^*}$. Then for any $\alpha \in \omega_1$,

$$
\begin{aligned}
(\Omega(\theta+\psi))(\alpha) &= \begin{cases} \omega & \text{if } \alpha = 0 \\ (\theta+\psi)(\alpha-1) & \text{if } 0 < \alpha < \omega \\ (\theta+\psi)(\omega) & \text{if } \alpha = \omega \\ 0 & \text{if } \omega < \alpha < \omega_1. \end{cases} \\
&= \begin{cases} \omega + \omega & \text{if } \alpha = 0 \\ \theta(\alpha-1) + \psi(\alpha-1) & \text{of } 0 < \alpha < \omega \\ \theta(\omega) + \psi(\omega) & \text{if } \alpha = \omega \\ 0 + 0 & \text{if } \omega < \omega_1 \end{cases} \\
&= (\Omega\theta)(\alpha) + (\Omega\psi)(\alpha) \\
&= ((\Omega\theta) + (\Omega\psi))(\alpha).
\end{aligned}
$$

□

Proposition 25.9. *(i)* $T(\Psi((\Omega o))) = 0$.
(ii) $T(\Psi((\Omega 0^c))) = 0$.
(iii) $\forall \theta \in \mathcal{N}^* \backslash \{0^c\}[\theta \equiv_{ae} 0^c \to T(\Psi((\Omega\theta))) = 1 + \max \operatorname{supp}(\theta)]$.
(iv) $\forall \theta \in \mathcal{N}^*[\theta \not\equiv_{ae} 0^c \to T(\Psi((\Omega\theta))) = \omega]$.

Proof. (i): $T(\Psi((\Omega o))) = T(\Psi(o)) = T(o) = o$.
(ii): For any $\alpha < \omega_1$ we have

$$
(\Omega 0^c)(\alpha) = \begin{cases} \omega & \text{if } \alpha = 0, \\ 0 & \text{if } 0 < \alpha < \omega \\ 0 & \text{if } \alpha = \omega \\ 0 & \text{if } \omega < \alpha < \omega_1. \end{cases}
$$

Thus $\operatorname{supp}((\Omega 0^c)) = \{0\}$. So $\Psi((\Omega 0^c)) = \{a \in {}^{<<\omega}W : T(a) = 0 \text{ and } \forall \alpha \in \omega_1 \backslash \{0, \omega\}[a_\alpha = 0]\}$. Hence $T(\Psi((\Omega 0^c))) = 0$.

(iii): Suppose that $\theta \in \mathcal{N}^* \backslash \{0^c\}$ and $\theta \equiv_{ae} 0^c$. Then there is a $k \leq \omega$ such that $\theta(k) \neq 0$. Let $k = \max(\operatorname{supp}(\theta)$. Then $k+1$ is the "η" for $(\Omega\theta)$, and so $T(\Psi((\Omega\theta))) = 1 + \max \operatorname{supp}(\theta)]$.

(iv): Suppose that $\theta \in \mathcal{N}^*$ and $\theta \not\equiv_{ae} 0^c$. Then $\theta(\omega) = \omega$, and so ω is the "η" for $(\Omega\theta)$. Hence $T(\Psi((\Omega\theta))) = \omega$. □

We extend $\equiv'_{ae}$ to ${}^{<<\omega}\mathcal{N}^0$ by defining $a \equiv'_{ae} b$ iff $\operatorname{dmn}(a) = \operatorname{dmn}(b)$ and $\forall i < \operatorname{dmn}(a)[a_i \equiv'_{ae} b_i]$.

$\mathcal{N}^{1^*}$ is the set of all $\alpha \in \triangle\mathcal{N}^0$ such that:

(AE8) $\forall n \in \omega \forall a, b[\operatorname{dmn}(a) = \operatorname{dmn}(b) = n \text{ and } a \equiv'_{ae} b \in \alpha \text{ and } \sum_{i<n} a_i = \sum_{i<n} b_i \to a \in \alpha]$.

(AE9) $\forall a, b[a^\frown b \in \alpha \text{ and } a \not\equiv_{ae} 0^c \to \exists c, d[a = c + d \text{ in } \mathcal{N}^0 \text{ and } c, d \not\equiv_{ae} 0^c \text{ and } c^\frown d^\frown b \in \alpha]]$.

Further, we let $\mathcal{N}^1 = \mathcal{N}^{1^*} \cup \{0\}$, where 0 is the 0 of $\triangle\mathcal{N}^0$.

Proposition 25.10. *In* $\omega+1$, *if* $\sum_{i<n} u_i = \sum_{i<n}(v_i + w_i)$, *then there exist* $v', w' \in {}^n(\omega+1)$ *such that* $\forall i < n[u_i = v'_i + w'_i]$, $\sum_{i<n} v'_i = \sum_{i<n} v_i$, *and* $\sum_{i<n} w'_i = \sum_{i<n} w_i$.

Proof. We may assume that $n > 1$. For $i < 2n$ let

$$b_i = \begin{cases} v_i & \text{if } i < n, \\ w_{i-n} & \text{if } n \leq i < 2n. \end{cases}$$

Then let $c \in {}^{n\times 2n}(\omega+1)$ be such that $\forall i < n[u_i = \sum_{j<2n} c_{ij}]$ and $\forall j < 2n[b_j = \sum_{i<n} c_{ij}]$. Then $\forall j < n[v_j = \sum_{i<n} c_{ij}]$ and $\forall j < n[w_j = \sum_{i<n} c_{i,j+n}]$. For each $i < n$ let $v'_i = \sum_{j<n} c_{ij}$ and $w'_i = \sum_{j<n} c_{i,n+j}$. Then $\forall i < n[u_i = v'_i + w'_i]$,

$$\sum_{i<n} v'_i = \sum_{i<n}\sum_{j<n} c_{ij} = \sum_{j<n}\sum_{i<n} c_{ij} = \sum_{j<n} v_j,$$

$$\sum_{i<n} w'_i = \sum_{i<n}\sum_{j<n} c_{i,n+j} = \sum_{j<n}\sum_{i<n} c_{i,n+j} = \sum_{j<n} w_j. \qquad \square$$

Proposition 25.11. *$\mathcal{N}^1$ is a submonoid of $\triangle\mathcal{N}^0$.*

Proof. Clearly $0 + 0 = 0$ and $\alpha + 0 = 0 + \alpha = \alpha$ for any $\alpha \in \mathcal{N}^1$. Now suppose that $\alpha, \beta \in \mathcal{N}^{1^*}$. We want to prove that $\alpha + \beta \in \mathcal{N}^{1^*}$. Suppose that $n \in \omega$, a, b have domain n, $a \equiv'_{ae} b \in (\alpha + \beta)$, and $\sum_{i<n} a_i = \sum_{i<n} b_i$. Choose c, d with domain n such that $\forall i < n[b_i = c_i + d_i]$, $c \in \alpha$, and $d \in \beta$. Let $F = \{k \in \omega + 1 : \exists i < n[a_i(k) \neq b_i(k)]\}$. Thus F is a finite subset of $\omega + 1$. For any $k \in (\omega + 1)\backslash F$ and any $i < n$ let $c'_i(k) = c_i(k)$ and $d'_i(k) = d_i(k)$. Now for any $k \in F$ we apply Proposition 25.10 with $a_i(k), c_i(k), d_i(k)$ in place of u_i, v_i, w_i to obtain $c'_i(k)$ and $d'_i(k)$ such that $\forall i < n[a_i(k) = c'_i(k) + d'_i(k)]$, $\sum_{i<n} c'_i(k) = \sum_{i<n} c_i(k)$, and $\sum_{i<n} d'_i(k) = \sum_{i<n} d_i(k)$. Then $\forall i < n[a_i = c'_i + d'_i]$. Now $c' \equiv_{ae} c \in \alpha$ and $d' \equiv_{ae} d \in \beta$, so $a \in (\alpha + \beta)$, as desired in (AE8).

For (AE9), suppose that $a^\frown b \in (\alpha+\beta)$ and $a \not\equiv_{ae} 0^c$. Write $a^\frown b = (a'^\frown b') + (a''^\frown b'')$ with $a'^\frown b' \in \alpha$ and $a''^\frown b'' \in \beta$. Then choose c', d', c'', d'' so that $a' = c' + d'$, $a'' = c'' + d''$, $c', d', c'', d'' \not\equiv_{ae} 0^c$, and $c'^\frown d'^\frown b' \in \alpha$ and $c''^\frown d''^\frown b'' \in \beta$. Let $c = c' + c''$ and $d = d' + d''$. Then $c^\frown d^\frown b \in (\alpha + \beta)$, as desired. $\square$

Proposition 25.12. *If $\Theta : M \to N$ is an injective morphism of m-monoids, for any $\alpha \in \triangle M$ let $(\triangle\Theta)(\alpha) = \{\Theta \circ a : a \in \alpha\}$. Then $\triangle\Theta$ is an injective morphism from $\triangle M$ to $\triangle N$.*

Proof. First we check that $\forall \alpha \in \triangle M[(\triangle\Theta)(\alpha) \in \triangle N]$. So suppose that $\alpha \in \triangle M$.

C.P.: Suppose that $a \in (\triangle\Theta)(\alpha)$ and $a \prec b$. Say $\mathrm{dmn}(a) = m$, $\mathrm{dmn}(b) = n$, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. Say $a = \Theta \circ c$ with $c \in \alpha$. Define $d_j = \sum\{c_i : i < m, \lambda(i) = j\}$ for all j, n. Then $c \prec d$, so $d \in \alpha$. Clearly $\Theta \circ d = b$. So $b \in (\triangle\Theta)(\alpha)$.

R.P.: Suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^mN$, $b \in {}^nN$, and $a, b \in (\triangle\Theta)(\alpha)$. Choose $a', b' \in \alpha$ such that $a = \Theta \circ a'$ and $b = \Theta \circ b'$. Then there is a $c \in {}^{m\cdot n}M$ such that $c \in \alpha$, $\forall i < m[a'_i = \sum\{c_{f(i,j)} : j < n\}]$, and $\forall j < n[b'_j = \sum\{c_{f(i,j)} : i < m\}]$. Then $(\Theta \circ c) \in (\triangle\Theta)(\alpha)$, $\forall i < m[a_i = \sum\{(\Theta \circ c)_{f(i,j)} : j < n\}]$, and $\forall j < n[b_j = \sum\{(\Theta \circ c)_{f(i,j)} : i < m\}]$.

S.P.: Suppose that $m \in \omega\backslash\{0\}$, $a \in {}^mN$, $a \in (\triangle\Theta)(\alpha)$, and $a_0 \neq 0$. Say $a = \Theta \circ a'$ with $a' \in \alpha$. Then $a'_0 \neq 0$. Hence there are $b, c \in M\backslash\{0\}$ such that

$a'_0 = b + c$ and $\langle b, c, a'_1, \ldots, a'_{m-1}\rangle \in \alpha$. Then since Θ is injective, $\Theta(b), \Theta(c) \neq o$, and $(\Theta \circ \langle b, c, a'_1, \ldots, a'_{m-1}\rangle) \in (\triangle\Theta)(\alpha)$.

Thus $(\triangle\Theta)(\alpha) \in \triangle N$.

We check that $\triangle\Theta$ preserves the operation:

$$\begin{aligned}(\triangle\Theta)(\alpha+\beta) &= \{\Theta \circ a : a \in \alpha+\beta\}\\ &= \{\Theta \circ a : \exists b, c[b \in \alpha \text{ and } c \in \beta \text{ and } a = b+c]\}\\ &= \{\Theta \circ (b+c) : b \in \alpha, c \in \beta\}\\ &= \{\Theta \circ b : b \in \alpha\} + \{\Theta \circ c : c \in \beta\}\\ &= (\triangle\Theta)(\alpha) + (\triangle\Theta)(\beta).\end{aligned}$$

Finally, if $\alpha, \beta \in \triangle M$ and $\alpha \neq \beta$, say $a \in \alpha \backslash \beta$. Then $\Theta \circ a \in (\triangle\Theta)(\alpha)\backslash(\triangle\Theta)(\beta)$. □

Proposition 25.13. *If $\Theta : M \to N$ is an isomorphism, then $\triangle\Theta : \triangle M \to \triangle N$ is an isomorphism.*

Proof. For any $\alpha \in \triangle M$,

$$\begin{aligned}(\triangle\Theta^{-1})((\triangle\Theta)(\alpha)) &= \{\Theta^{-1} \circ a : a \in (\triangle\Theta)(\alpha)\}\\ &= \{\Theta^{-1} \circ a : \exists b \in \alpha[a = \Theta \circ b]\}\\ &= \alpha.\end{aligned}$$

Thus $(\triangle\Theta^{-1}) \circ (\triangle\Theta)$ is the identity. Similarly, $(\triangle\Theta) \circ (\triangle\Theta^{-1})$ is the identity. □

Proposition 25.14. *If $\Theta : M \to N$ and $T : N \to P$ are morphisms of m-monoids, then $\triangle(T \circ \Theta) = (\triangle T) \circ (\triangle\Theta)$.*

Proof. Assume that $\Theta : M \to N$ and $T : N \to P$ are morphisms of m-monoids. Let $\alpha \in \triangle M$. Then

$$\begin{aligned}(\triangle(T \circ \Theta))(\alpha) &= \{T \circ \Theta \circ a : a \in \alpha\}\\ &= \{T \circ b : \exists a \in \alpha[b = \Theta \circ a]\}\\ &= \{T \circ b : b \in (\triangle\Theta)(\alpha)\}\\ &= (\triangle T)((\triangle\Theta)(\alpha)).\end{aligned}$$

□

Proposition 25.15. *$(\triangle(\Psi \circ \Omega)) \restriction \mathcal{N}^1$ is an injective morphism from $\mathcal{N}^1$ into $\mathscr{L}^1$.*

Proof. By Proposition 25.8, Ω is a morphism from $\mathcal{N}^0$ to $\mathcal{N}$, and by the proof of Proposition 9.6, Ψ is a morphism from $\mathcal{N}$ to $\triangle\mathscr{W}$; see the definition of Ψ following Propostition 25.1. Clearly both Ω and Ψ are injective, so by Proposition 25.12, $(\triangle(\Psi\circ\Omega)) \restriction \mathcal{N}^1$ is injective. Now $\Psi \circ \Omega$ is a morphism from $\mathcal{N}^0$ to $\triangle\mathscr{W}$. Hence by 25.13, $\triangle(\Psi \circ \Omega)$ is a morphism from $\triangle\mathcal{N}^0$ to $\triangle(\triangle\mathscr{W})$. By Proposition 25.11, $\mathcal{N}^1$ is a submonoid of $\triangle\mathcal{N}^0$. By Proposition 24.4, $\mathscr{L}^1$ is a submonoid of $\triangle(\triangle\mathscr{W})$.

Now let $\alpha \in \mathcal{N}^1$; we want to show that $(\triangle(\Psi \circ \Omega))(\alpha) \in \mathscr{L}^1$. Thus we want to show that $(\triangle(\Psi\circ\Omega))(\alpha)$ has the strict 1-hierarchy property. Suppose that $\langle a\rangle^\frown b \in (\triangle(\Psi\circ\Omega))(\alpha)$ and e, with domain n, is in a. We want to find c with domain n such that $\sum_{i<n} c_i = a$, $c^\frown b \in (\triangle(\Psi \circ \Omega))(\alpha)$ and $\forall i < n[T(c_i) = e_i]$. By the definition of $(\triangle(\Psi \circ \Omega))(\alpha)$, there exists $\langle a'\rangle^\frown b' \in \alpha$ such that $\langle a\rangle^\frown b = \Psi \circ \Omega \circ (\langle a'\rangle^\frown b'\rangle$. Thus $a = \Psi(\Omega(a'))$. Note that $a \subseteq {}^{<<\omega}\mathscr{W}$ and $a \in \triangle\mathscr{W}$. By C.P. for $\triangle\mathscr{W}$ we may assume that e has the form

$$\langle e_0, \ldots, e_{r-1}, e_r, \ldots, e_{s-1}, e_s, \ldots, e_{n-1}\rangle,$$

where

$$e_0 \geq e_1 \geq \cdots \geq e_{r-1} \geq 1 > e_r = \cdots = e_{s-1} = 0 > e_s = \cdots = e_{n-1} = o$$

Now for each $i < n$ and $k \in \omega + 1$ we define

$$c_i(k) = \begin{cases} 1 & \text{if } 1 \leq i < r \text{ and } k = e_i - 1 \\ 0 & \text{if } 1 \leq i < r \text{ and } k \neq e_i - 1 \\ 0 & \text{if } r \leq i < s, \end{cases}$$

with $c_i = o$ if $s \leq i < n$. Let $c_0 = a' - \sum_{1\leq i<n} c_i$, with $c_0 = \omega$ if $a' = \omega$. Then $\sum_{i<n} c_i = a'$. If $i < n$ and $k < \omega_1$, then

$$\begin{aligned}(\Omega c_i)(k) &= \begin{cases} \omega & \text{if } k = 0, \\ c_i(k-1) & \text{if } 0 < k < \omega, \\ c_i(\omega) & \text{if } k = \omega, \\ 0 & \text{if } \omega < k < \omega_1 \end{cases} \\ &= \begin{cases} \omega & \text{if } k = 0, \\ 1 & \text{if } 0 < k < \omega \text{ and } e_i = k, \\ 0 & \text{if } 0 < k < \omega \text{ and } e_i \neq k, \\ 1 & \text{if } k = \omega \text{ and } e_i = \omega, \\ 0 & \text{if } k = \omega \text{ and } e_i \neq \omega, \\ 0 & \text{if } \omega < k < \omega_1. \end{cases}\end{aligned}$$

The "η" for Ωc_i is the largest k such that $e_i = k$. Thus $\forall i < n[T(c_i) = e_i]$. Also, $\mathrm{supp}(c_i) = \{e_i - 1\}$, so by Proposition 25.9(iii), $T(\Psi(\Omega c_i)) = e_i$. Now for any $k < \omega_1$,

$$(\Omega a')(k) = \begin{cases} \omega & \text{if } k = 0, \\ a'(k-1) & \text{if } 0 < k < \omega, \\ a(\omega) & \text{if } k = \omega, \\ 0 & \text{if } \omega < k < \omega_1. \end{cases}$$

Since $e \in a = \Theta(\Omega(a'))$, it follows that for any $k \in \omega_1$, $|\{i < n : e_i = k\}| \leq (\Omega a')(k)$. Hence $c_0(k) \geq 0$. Thus $\Psi \circ \Omega \circ c$ is as desired. $\square$

26. The monoid $\mathscr{P}$

Clearly $\equiv_{ae}$ is a congruence relation on $\mathcal{N}^0$. The congruence class of an element θ of $\mathcal{N}^0$ will be denoted by $[\theta]_{ae}$. $\mathscr{P}$ is the quotient semigroup of all congruence classes. Thus $[\theta]_{ae} + [\psi]_{ae} = [\theta + \psi]_{ae}$. Π is the homomorphism of $\mathcal{N}^0 \to \mathscr{P}$ such that $\Pi(\theta) = [\theta]_{ae}$ for all $\theta \in \mathcal{N}^0$.

Proposition 26.1. *$\mathscr{P}$ is an m-monoid.*

Proof. It remains only to show that if $[\theta]_{ae}+[\psi]_{ae} = [0^c]_{ae}$, then $[\theta]_{ae} = [0^c]_{ae} = [\psi]_{ae}$. Suppose that $[\theta]_{ae} + [\psi]_{ae} = [0^c]_{ae}$. Then there is a finite set F such that $\forall x \notin F[\theta(x) + \psi(x) = 0]$, hence $\forall x \notin F[\theta(x) = 0]$, so $[\theta]_{ae} = [0^c]_{ae}$. Similarly $[\psi]_{ae} = [0^c]_{ae}$. $\square$

Proposition 26.2. $\forall \theta \in \mathcal{N}^0[[\theta]_{ae}$ *is countable*$]$. $\square$

Proposition 26.3. *The natural ordering of $\mathscr{P}$ is antisymmetric.*

Proof. Clearly $[\theta]_{ae} \leq [0^c]$ implies that $[\theta]_{ae} = [0^c]_{ae}$. So $\forall\theta[[0^c] \leq [\theta]_{ae} \leq [0^c] \to [0^c]_{ae} = [\theta]_{ae}]$.

26.3(1) $\forall\theta[[\omega^c]_{ae} \leq [\theta]_{ae} \to [\omega^c]_{ae} = [\theta]_{ae}]$.

In fact, suppose that $[\omega^c]_{ae} \leq [\theta]_{ae}$. Choose ψ so that $[\omega^c]_{ae} + [\psi]_{ae} = [\theta]_{ae}$. Then there is a finite set F such that $\forall x \notin F[\omega + \psi(x) = \theta(x)]$, hence $\forall x \notin F[\omega = \theta(x)]$. Thus $[\omega^c]_{ae} = [\theta]_{ae}$.

By 26.3(1) we have $\forall\theta[[\omega^c]_{ae} \leq [\theta]_{ae} \leq [\omega^c]_{ae} \to [\theta]_{ae} = [\omega^c]_{ae}]$.

Hence to prove the proposition it suffices to take θ, ψ each satisfying (AE3) and show that $[\theta]_{ae} \leq [\psi]_{ae} \leq [\theta]_{ae}$ implies that $[\theta]_{ae} = [\psi]_{ae}$. So, suppose that θ, ψ each satisfy (AE3) and $[\theta]_{ae} \leq [\psi]_{ae} \leq [\theta]_{ae}$. Say $[\theta]_{ae} + [\mu]_{ae} = [\psi]_{ae}$ and $[\psi]_{ae} + [\nu]_{ae} = [\theta]_{ae}$. Then there is a finite set F such that $\forall x \notin F[\theta(x) \neq \omega \neq \psi(x)$ and $\theta(x) + \mu(x) = \psi(x)$ and $\psi(x) + \nu(x) = \theta(x)]$. So $\forall x \notin F[\omega \neq \theta(x) = \theta(x) + \mu(x) + \nu(x)]$, hence $\forall x \notin F[\mu(x) = \nu(x) = 0]$, so $\forall x \notin F[\theta(x) = \psi(x)]$. $\square$

Proposition 26.4. *$\mathscr{P}\backslash\{[\omega^c]_{ae}\}$ is a submonoid of $\mathscr{P}$.* $\square$

Proposition 26.5. *$\mathscr{P}\backslash\{[\omega^c]_{ae}\}$ is isomorphic to a submonoid of the additive monoid of a vector space over $\mathbb{Q}$.*

Proof. Define $f \sim g$ iff $f, g \in {}^\omega\mathbb{Q}$ and $\{n \in \omega : f(n)) \neq g(n)\}$ is finite. Clearly $\sim$ is an equivalence relation on ${}^\omega\mathbb{Q}$. Note that ${}^\omega\mathbb{Q}$ can be considered to be a vector space over $\mathbb{Q}$. The following two statements are clear.

26.5(1) If $f, g, f', g' \in^\omega \mathbb{Q}$, $f \sim f'$, and $g \sim g'$, then $f + g \sim f' + g'$.

26.5(2) If $f, f' \in^\omega \mathbb{Q}$, $q \in \mathbb{Q}$, and $f \sim f'$, then $qf \sim qf'$.

Now let V be the collection of all $\sim$-classes, with $[f]_\sim + [g]_\sim = [f + g]_\sim$ and $q[f]_\sim = [qf]_\sim$; so V is a vector space over $\mathbb{Q}$.

For any $x \in \mathscr{P}\backslash\{[\omega^c]_{ae}\}$ choose $\theta \in \mathcal{N}^0$ such that $\forall k \in \omega[\theta(k) < \omega]$ and $x = [\theta]_{ae}$, and set $f(x) = [\theta \restriction \omega]_\sim$. Clearly f is well-defined and one-one and is an isomorphism of $\mathscr{P}\backslash\{[\omega^c]_{ae}\}$ into V. $\square$

Proposition 26.6. *$\mathscr{P}$ is an r-monoid.*

Proof. We apply Proposition 19.7. Suppose that $\theta, \psi \in {}^2\mathcal{N}^0$ and $[\theta_0]_{ae} + [\theta_1]_{ae} = [\psi_0]_{ae} + [\psi_1]_{ae}$. Thus $[\theta_0 + \theta_1]_{ae} = [\psi_0 + \psi_1]_{ae}$. Let F be a finite subset of $\omega + 1$ such that $\forall k \in (\omega+1)\backslash F[\theta_0(k) + \theta_1(k) = \psi_0(k) + \psi_1(k)]$. Define $\theta_0', \theta_1', \psi_0', \psi_1'$ by:

$$\theta_0'(k) = \begin{cases} \theta_0(k) & \text{if } k \notin F, \\ 0 & \text{otherwise;} \end{cases} \qquad \theta_1'(k) = \begin{cases} \theta_1(k) & \text{if } k \notin F, \\ 0 & \text{otherwise;} \end{cases}$$

$$\psi_0'(k) = \begin{cases} \psi_0(k) & \text{if } k \notin F, \\ 0 & \text{otherwise;} \end{cases} \qquad \psi_1'(k) = \begin{cases} \psi_1(k) & \text{if } k \notin F, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\theta_0' + \theta_1' = \psi_0' + \psi_1'$, so by Proposition 25.3 there is a $c \in {}^{2\times 2}\mathcal{N}^0$ such that $\forall i < 2[\theta_i' = \sum_{j<2} c_{ij}]$ and $\forall j < 2[\psi_j' = \sum_{i<2} c_{ij}]$. Then $\forall i < 2[[\theta_i]_{ae} = \sum_{j<2}[c_{ij}]_{ae}]$ and $\forall j < 2[[\psi_j]_{ae} = \sum_{i<2}[c_{ij}]_{ae}]$. $\square$

Proposition 26.7. *If $a \in \mathscr{P}^*\backslash\{[\omega^c]_{ae}\}$, then $\mathscr{P} \restriction a$ is uncountable.*

Proof. We may assume that $a = [\theta]_{ae}$, where θ satisfies (135) and $\forall k \in \omega[0 < \theta(k) < \omega]$. Define $\varepsilon \equiv \delta$ iff $\varepsilon, \delta \in {}^\omega 2$ and $\{i \in \omega : \varepsilon(i) \neq \delta(i)\}$ is finite. This is an equivalence relation, and each equivalence class is countable. Let X consist of one element from each equivalence class. So $|X| = 2^\omega$. Let $\langle k_n : n \in \omega\rangle$ be strictly increasing such that $\langle \theta(k_n); n \in \omega\rangle$ is also stictly increasing. For each $\varepsilon \in X$ define ψ_ε with domain $\omega + 1$ by

$$\psi_\varepsilon(m) = \begin{cases} |\lceil \frac{\theta(k_n)}{2} \rceil - \varepsilon(m)| & \text{if } m = k_n \text{ for some } n \in \omega, \\ |\theta(m) - \varepsilon(m)| & \text{if } m \in \omega\backslash \mathrm{rng}(k), \\ \omega & \text{if } m = \omega. \end{cases}$$

Then $\lim_{m\to\infty} \psi_\varepsilon(m) = \omega$. For, given $M > 0$ choose N so that $\forall m \geq N[\theta(m) \geq 2M + 2]$. Then if $m \geq N$ and $m = k_n$ for some n, then $\psi_\varepsilon(m) \geq \frac{\theta(m)}{2} - 1 \geq M$, while if $m \notin \mathrm{rng}(k)$ then $\psi_\varepsilon(m) = \theta(m) - 1 \geq 2M + 1$.

Next, note that for any $n \in \omega$, $\theta(k_n) \geq \lceil \frac{\theta(k_n)}{2} \rceil$. Now define χ_ε with domain $\omega + 1$ by:

$$\chi_\varepsilon(m) = \begin{cases} \theta(k_n) - \lceil \frac{\theta(k_n)}{2} \rceil + \varepsilon(m) & \text{if } m = k_n \text{ for some } n \in \omega, \\ \theta(m) + \varepsilon(m) & \text{if } m \in \omega\backslash \mathrm{rng}(k), \\ \omega & \text{if } m = \omega. \end{cases}$$

Clearly $\lim_{m\to\infty} \chi_\varepsilon(m) = \omega$. It follows that $[\psi_\varepsilon]_{ae} \leq [\theta]_{ae}$.

Clearly $[\psi_\varepsilon]_{ae} \neq [\psi_\delta]_{ae}$ for $\varepsilon \neq \delta$. $\square$

Proposition 26.8. *If K is a countable subset of $\mathscr{P}$, and $a \in \mathscr{P}$, then there exist $a_0, a_1 \in \mathscr{P}$ such that $a = a_0 + a_1$ and $a_0 \neq [\omega^c]_{ae}$ is not in the vector space span of $K\backslash\{[\omega^c]_{ae}\}$.*

Proof. If $a = [\omega^c]_{ae}$, then let a_0 be any member of $\mathscr{P}\backslash\{[\omega^c]_{ae}\}$ not in the span of $K\backslash\{[\omega^c]_{ae}\}$, using Proposition 26.7; and let $a_1 = [\omega^c]_{ae}$. If $a \neq [\omega^c]_{ae}$, then apply Proposition 26.7 to get the desired a_0, a_1. $\square$

27. The monoid $\mathscr{P}^1$

Let $\mathscr{P}^1 = \triangle\mathscr{P}$.

Proposition 27.1. *Suppose that $m, n \in \omega\backslash\{0\}$, $\theta \in {}^m\mathcal{N}^{0^*}$, $\chi \in {}^n\mathcal{N}^{0^*}$, $\psi \in {}^{m\times n}\mathscr{P}$, $\forall i < m[\Pi(\theta_i) = \sum_{j<n}\psi_{ij}]$, and $\forall j < n[\Pi(\chi_j) = \sum_{i<m}\psi_{ij}]$. Then there is a $\rho \in {}^{m\times n}\mathcal{N}^{0^*}$ such that $\forall i < m\forall j < n[\Pi(\rho_{ij}) = \psi_{ij}]$, $\forall i < m[\theta_i = \sum_{j<n}\rho_{ij}]$, and $\forall j < n[\chi_j = \sum_{i<m}\rho_{ij}]$.*

Proof. Assume the hypotheses. Since Π is surjective, there is a $\psi' \in {}^{m\times n}\mathcal{N}^{0^*}$ such that $\forall i < m\forall j < n[\Pi(\psi'_{ij}) = \psi_{ij}]$. Thus $\forall i < m[\Pi(\theta_i) = \sum_{j<n}\Pi(\psi'_{ij})]$, and $\forall j < n[\Pi(\chi_j) = \sum_{i<m}\Pi(\psi'_{ij})]$. Hence there is a finite subset F of $\omega+1$ such that $\forall k \in (\omega+1)\backslash F[\forall i < m[\theta_i(k) = \sum_{j<n}\psi'_{ij}(k)]$ and $\forall j < n[\chi_j(k) = \sum_{i<m}\psi'_{ij}(k)]]$. Now for each $k \in F$ let $\sigma^k \in {}^{m\times n}(\omega+1)$ be such that $\forall i < m[\theta_i(k) = \sum_{j<n}\sigma^k_{ij}]$ and $\forall j < n[\chi_j(k) = \sum_{i<m}\sigma^k_i]$. Now for any $k \in \omega+1$ and $i < m$, $j < n$ define

$$\rho_{ij}(k) = \begin{cases} \psi'_{ij} & \text{if } k \notin F, \\ \sigma^k_{ij} & \text{if } k \in F. \end{cases}$$

Clearly the desired conclusion holds. $\square$

For any $\alpha \in \mathcal{N}^1$ let

$$\Pi_1(\alpha) = \{\langle \Pi(\theta_i) : i < n\rangle : \theta \in \alpha\}.$$

Proposition 27.2. *Π_1 is a monoid morphism from $\mathcal{N}^1$ to $\mathscr{P}^1$.*

Proof. Let $\alpha \in \mathcal{N}^1$. Thus by definition, $\alpha \in \triangle\mathcal{N}^0$. If $\theta \in \alpha$, say θ has domain m. Then $\Pi \circ \theta \in {}^m\mathcal{P}$. Hence $\Pi_1(\alpha) \in [{}^{<<\omega}\mathcal{P}]^\omega$.

C.P.: Suppose that $a, b \in {}^{<<\omega}\mathscr{P}$, $a \in \Pi_1(\alpha)$, and $a \prec b$. Say $a = \Pi\circ\theta$ with $m \in \omega\backslash\{0\}$ and $\theta \in \alpha$, θ with domain m. Also say b has domain n, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. Say $b = \Pi \circ \psi$. Then $\forall j < n[\Pi(\psi_j)) = \sum\{\Pi(\theta(i)) : i < m, \lambda(i) = j\}$. So $\forall j < n[\psi_j \equiv_{ae} \sum\{\theta(i) : i < m, \lambda(i) = j\}$. Hence there is a ψ' with $\forall j < n[\psi_j \equiv_{ae} \psi'_j]$ and $\theta \prec \psi'$. So $\psi' \in \alpha$ and hence $b = \Pi \circ \psi = \Pi \circ \psi' \in \Pi_1(\alpha)$.

R.P.: Suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^m\mathscr{P}$, $b \in {}^n\mathscr{P}$, and $a, b \in \Pi_1(\alpha)$. Choose $\theta, \psi \in \alpha$ such that $a = \Pi \circ \theta$ and $b = \Pi \circ \psi$. Then choose $c \in {}^{m\times n}\mathcal{N}^0$ so that $\forall i < m[\theta_i = \sum_{j<n} c_{ij}]$ and $\forall j < n[\psi_j = \sum_{i<m} c_{ij}]$. Then $\forall i < m[\Pi(\theta_i) = \sum_{j<n}\Pi(c_{ij})]$ and $\forall j < n[\Pi(\psi_j) = \sum_{i<m}\Pi(c_{ij})]$.

S.P.: Suppose that $m \in \omega\backslash\{0\}$, $a \in {}^m\mathscr{P}$, $a \in \Pi_1(\alpha)$, $a_0 \neq o$. Say $a = \Pi(\theta)$ with $\theta \in \alpha$. Thus $\theta_0 \neq o$. By (141) there are c, d such that $\theta_0 = c+d$, $c, d \not\equiv_{ae} 0^c$, and $c^\frown d^\frown\langle\theta_1, \ldots\rangle \in \alpha$. Then $a_0 = \Pi(c) + \Pi(d)$, $\Pi(c), \Pi(d) \neq o$, and $\Pi(c)^\frown\Pi(d)^\frown\langle a_1, \ldots\rangle \in \Pi(\alpha)$.

This checks that $\Pi_1(\alpha) \in \mathcal{P}^1$. To check that it preserves the monoid operation, suppose that $\alpha, \beta \in \mathcal{N}^1$. Then

$$\Pi_1(\alpha + \beta) = \{\Pi \circ \theta : \theta \in \alpha + \beta\}$$

$$= \{\Pi \circ (\theta + \psi) : \theta, \psi \text{ have the same domain and } \theta \in \alpha, \psi \in \beta\}$$
$$= \{\mu + \rho : \mu, \rho \text{ have the same domain and } \mu \in \Pi_1(\alpha), \rho \in \Pi_1(\beta)\}$$
$$= \Pi_1(\alpha) + \Pi_1(\beta). \qquad \square$$

Proposition 27.3. *For any $\alpha \in \mathscr{P}^1$ and any $\chi \in \mathcal{N}^0$, if $T(\alpha) = \Pi(\chi)$. then there is a unique $\beta \in \mathcal{N}^1$ such that $\Pi_1(\beta) = \alpha$ and $T(\beta) = \chi$.*

Proof. Assume that $\alpha \in \mathscr{P}^1$, $\chi \in \mathcal{N}^0$, and $T(\alpha) = \Pi(\chi)$. Let

$$\beta = \left\{\theta \in {}^n\mathcal{N}^0 : n \in \omega\backslash\{0\}, \Pi \circ \theta \in \alpha, \sum_{i<n} \theta_i = \chi\right\}.$$

We claim that $\beta \in \mathcal{N}^1$. First we show that $\beta \in \triangle\mathcal{N}^0$. By the definition of $\mathscr{P}^1$, α is countable, and so by Proposition 26.2, β is countable.

C.P.: Suppose that $a, b \in {}^{<<\omega}\mathcal{N}^0$, $a \in \beta$, and $a \prec b$. Say a has domain m, b has domain n, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. Since $a \in \beta$, we have $\Pi \circ a \in \alpha$. Now $\forall j < n[\Pi(b_j) = \sum\{\Pi(a_i) : i < m, \lambda(i) = j\}$. Hence $\Pi \circ a \prec \Pi \circ b$, so $\Pi \circ b \in \alpha$. Also, $\sum_{j<n} b_j = \sum_{i<n} a_i = \chi$. So $b \in \beta$.

R.P.: Suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^m\mathcal{N}^0$, $b \in {}^n\mathcal{N}^0$, and $a, b \in \beta$. So $\Pi \circ a, \Pi \circ b \in \alpha$. By R.P. for α, there is a $c \in {}^{m\cdot n}\mathscr{P}$ such that $\forall i < n[\Pi(a_i) = \sum\{c_{f(i,j)} : j < n\}]$ and $\forall j < n[\Pi(b_j) = \sum\{c_{f(i,j)} : i < m\}]$. Let $c'(i,j) = c_{f(i,j)}$ for all $i < m$ and $j < n$. By Proposition 27.1 there is a $\rho \in {}^{m\times n}\mathcal{N}^0$ such that $\forall i < m \forall j < n[\Pi(\rho_{ij}) = c'_{ij}]$. $\forall i < m[a_i = \sum_{j<n} \rho_{ij}]$, and $\forall j < n[b_j = \sum_{i<m} \rho_{ij}]$. Let $\rho'_{f(i,j)} = \rho_{ij}$ for all $i < m$ and $j < n$. This proves R.P. for β.

S.P.: Suppose that $m \in \omega\backslash\{0\}$, a has domain m, $a \in \beta$, and $a_0 \neq o$. Then also $\Pi(a_0) \neq o$. By S.P. for α, choose $b, c \in \mathscr{P}$ such that $\Pi(a_0) = b + c$ and $\langle b, c, \Pi(a_1), \ldots, \Pi(a_{m-1})\rangle \in \alpha$. Say $b = \Pi(b')$ and $c = \Pi(c')$ Then $\Pi \circ \langle b', c', a_1, \ldots, a_{m-1}\rangle \in \alpha$. Hence $\langle b', c', a_1, \ldots, a_{m-1}\rangle \in \beta$.

This proves that $\beta \in \triangle\mathcal{N}^0$. Now for $\beta \in \mathcal{N}^1$, for (AE8), suppose that $n \in \omega$, $\mathrm{dmn}(a) = \mathrm{dmn}(b) = n$, $a \equiv_{ae} b \in \beta$, and $\sum_{i<n} a_i = \sum_{i<n} b_i$. Then $\Pi \circ a = \Pi \circ b \in \alpha$ and $\sum_{i<n} a_i = \sum_{i<n} b_i = \chi$. So $a \in \beta$.

For (AE9), suppose that $a^\frown b \in \beta$ and $a \not\equiv_{ae} 0^c$. Then $\Pi \circ (a^\frown b) \in \alpha$ and $\sum_{i<m} a_i + \sum_{j<n} b_j = \chi$. Then $\Pi \circ a \neq 0$. Wlog $\Pi(a_0) \neq 0$. By S.P. for α choose $c, d \neq o$ so that $\Pi(a_0) = c + d$ and $\langle c, d, \Pi(a_1), \ldots, \Pi(b_{n-1})\rangle \in \alpha$. Say $c = \Pi(c')$ and $d = \Pi(d')$. Then $\Pi \circ \langle c', d', a_1, \ldots, b_{n-1}\rangle \in \alpha$. We may assume that $c' + d' = a_0$. In fact, let $F = \{i \in \omega + 1 : c'(i) + d'(i) \neq a_0(i)\}$; so F is finite. let

$$c''(i) = \begin{cases} c'(i) & \text{if } i \notin F, \\ a_0(i) & \text{if } i \in F, \end{cases} \qquad d''(i) = \begin{cases} d'(i) & \text{if } i \notin F, \\ 0 & \text{if } i \in F. \end{cases}$$

Then $c'' + d'' = a_0$ and $\Pi(c'') = \Pi(c')$, $\Pi(d'') = \Pi(d')$. This proves (141).

So $\beta \in \mathcal{N}^1$. Clearly $\Pi_1(\beta) \subseteq \alpha$. If $c \in \alpha$, say $c \in {}^n\mathscr{P}$. Then there is a $b \in {}^n\mathcal{N}^0$ such that $c = \Pi \circ b$. Then $[\chi] = \Pi(\chi) = T(\alpha) = T(c) = [\sum_{i<n} b_i]$. Let F be a finite subset of

$\omega+1$ such that $\chi(m)=\sum_{i<n} b_i(m)$ for all $m \notin F$. Then we define

$$b'_i(m)=\begin{cases} b_i(m) & \text{if } m \notin F, \\ \chi(m) & \text{if } m \in F \text{ and } i=0, \\ 0 & \text{if } m \notin F \text{ and } i \neq 0. \end{cases}$$

Then $c=\Pi \circ b'$ and $\sum_{i<n} b'_i=\chi$. Hence $c \in \Pi_1(\beta)$. So $\Pi_1(\beta)=\alpha$. Clearly $T(\beta)=\chi$.

For uniqueness, suppose that $\Pi_1(\gamma)=\alpha$ and $T(\gamma)=\chi$. Take any $\theta \in \gamma$; say θ has domain n. Then $\Pi \circ \theta \in \Pi_1(\gamma)$, so $\Pi \circ \theta \in \Pi_1(\beta)$. Choose $\psi \in \beta$ so that $\Pi \circ \theta = \Pi \circ \psi$. Now $T(\gamma)=\sum_{i<n} \theta_i=\sum_{i<n} \psi_i$. Hence by (140), $\theta \in \beta$. By symmetry, $\gamma=\beta$. □

Let $\mathscr{Q}^1=\{\alpha \in \mathscr{P}^1 : T(\alpha)=[\omega^c]_{ae}\}$.

Proposition 27.4. *$\mathscr{Q}^1$ is a subsemigroup of $\mathscr{P}^1$.* □

Proposition 27.5. *There is a unique morphism $\Gamma_1 : \mathscr{Q}^1 \to \mathcal{N}^1$ such that $\forall \alpha \in \mathscr{Q}^1[\Pi_1(\Gamma_1(\alpha))=\alpha]$ and $T(\Gamma_1(\alpha))=\omega^c$.*

Proof. Apply Proposition 27.3 with $\chi=\omega^c$; for $\alpha \in \mathscr{Q}^1$ let $\Gamma_1(\alpha)$ be the β of 27.3. □

Proposition 27.6. *Π_1 maps $\mathcal{N}^1$ onto $\mathscr{P}^1$. In fact, if $\alpha \in \mathscr{P}^1$ and $T(\alpha)=\Pi(d)$ then there is a $\beta \in \mathcal{N}^1$ such that $\Pi_1(\beta)=\alpha$ and $T(\beta)=d$.*

Proof. By Proposition 27.3. □

28. One step up the heirarchy

Proposition 28.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\triangle^2\mathscr{W} \arrow[d, "T"'] & \mathcal{N}^1 \arrow[l, "\triangle(\Psi\circ\Omega)"'] \arrow[r, "\Pi_1"] \arrow[d, "T"'] & \mathscr{P}^1 \arrow[d, "T"'] \\
\triangle\mathscr{W} & \mathcal{N}^0 \arrow[l, "\Psi\circ\Omega"] \arrow[r, "\Pi"'] & \mathscr{P}
\end{tikzcd}$$

Proof. By Proposition 25.15, $\triangle(\Psi\circ\Omega) : \mathcal{N}^1 \to \mathscr{L}^1$. By Proposition 27.2, $\Pi_1 : \mathscr{N}^1 \to \mathscr{P}^1$. By definition, $\Pi : \mathcal{N}^0 \to \mathscr{P}$. By Proposition 25.8, $\Omega : \mathcal{N}^0 \to \mathcal{N}$, and by Proposition 9.6, $\Psi : \mathcal{N} \to \triangle\mathscr{W}$. Hence $\Psi \circ \Omega : \mathcal{N}^0 \to \triangle\mathscr{W}$. Clearly $T : \triangle^2\mathscr{W} \to \mathscr{W}$. By Proposition 25.11, $T : \mathscr{N}^1 \to \mathscr{N}^0$. Clearly $T : \mathscr{P}^1 \to \mathscr{P}^0$.

To show that $\Psi \circ \Omega \circ T = T \circ \triangle(\Psi \circ \Omega)$, suppose that $\alpha \in \mathcal{N}^1$. Then $\alpha \in \triangle\mathcal{N}^0$, so $T(\alpha) = T(a)$ for any $a \in \alpha$; say a has domain n. Then $\Psi(\Omega(T(\alpha))) = \Psi(\Omega(T(a))) = \sum_{i<n} \Psi(\Omega(a_i))$. On the other hand,

$$(T\triangle(\Psi\circ\Omega))(\alpha) = T((\triangle(\Psi\circ\Omega))(\alpha)) = T(\{\Psi\circ\Omega\circ a : a \in \alpha\} = T(\Psi\circ\Omega\circ a) = \sum_{i<n} \Psi(\Omega(a_i)),$$

as desired.

To show that $\Pi \circ T = T \circ \Pi_1$, suppose that $\alpha \in \mathscr{P}^1 = \triangle\mathscr{P}$. Then $T(\alpha) = T(a)$ for any $a \in \alpha$, so $\Pi(T(\alpha)) = \Pi(T(a))$. On the other hand,

$$\begin{aligned} T(\Pi_1(\alpha)) &= T(\{\langle \Pi(a_i) : i < n\rangle : a \in \alpha, \mathrm{dmn}(a) = n\} \\ &= \sum_{i<n} \Pi(a_i) \quad \text{for any } a \in \alpha \\ &= \Pi(T(a)). \end{aligned}$$

□

Let M be an m-monoid and $A \in \triangle^2 M$. Then A satisfies the *new heirarchy property* iff

(N1) For all $m, n \in \omega\backslash\{0\}$, all $\langle \alpha, \beta_1, \ldots, \beta_{m-1}\rangle \in A$ and all $\langle a_0, \ldots, a_{n-1}\rangle \in \alpha$ there exist $\alpha_i \in \triangle M$ for $i < n$ such that $\langle \alpha_0, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{m-1}\rangle \in A$, $\sum_{i<n} \alpha_i = \alpha$, and $\forall i < n[T(\alpha_i) = a_i]$.

$\mathcal{N}^2$ is the set of all $A \in \triangle\mathcal{N}^1$ such that:

(N2) A satisfies the new hierarchy property.

(N3) If $\forall n \in \omega\backslash\{0\}\forall \alpha, \beta[\alpha$ and β have domain n, $\alpha \in A$, and $\forall i < n[\Pi_1(\beta_i) = \Pi_1(\alpha_i)]$, then $\beta \in A$.

(N4) $\forall n \in \omega\forall\langle \alpha, \beta_1 \ldots, \beta_{n-1}\rangle \in A[\Pi_1(\alpha) \neq 0 \rightarrow \exists \gamma_0, \gamma_1[\alpha = \gamma_0 + \gamma_1$ in $\mathcal{N}^1$, $\Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0$, and $\langle \gamma_0, \gamma_1, \beta_1, \ldots, \beta_{n-1}\rangle \in A]]$.

$\mathscr{P}^2$ is the set of all $A \in \triangle\mathscr{P}^1$ which satisfy the new heirarchy property.

Proposition 28.2. *If A and B satisfy the new heirarchy property, then so does $A + B$.*

Proof. Suppose that A and B satisfy the new heirarchy property. Suppose that $m, n \in \omega\backslash\{0\}$, $\langle \alpha, \beta_1, \ldots, \beta_{m-1}\rangle \in A + B$, and $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in \alpha$. Now $A, B \in \triangle^2 M$, so there are $\langle \alpha^0, \beta_1^0, \ldots, \beta_{m-1}^0\rangle \in A$, and $\langle \alpha^1, \beta_1^1, \ldots, \beta_{m-1}^1\rangle \in B$ such that $\langle \alpha, \beta_1, \ldots, \beta_{m-1}\rangle = \langle \alpha^0, \beta_1^0, \ldots, \beta_{m-1}^0\rangle + \langle \alpha^1, \beta_1^1, \ldots, \beta_{m-1}^1\rangle$. Moreover,

$$\langle \alpha^0, \beta_1^0, \ldots, \beta_{m-1}^0\rangle + \langle \alpha^1, \beta_1^1, \ldots, \beta_{m-1}^1\rangle = \langle \alpha^0 + \alpha^1, \beta_1^0 + \beta_1^1, \ldots, \beta_{m-1}^0 + \beta_{m-1}^1\rangle.$$

Hence there are $\langle a_0^0, a_1^0, \ldots, a_{n-1}^0\rangle \in \alpha^0$ and $\langle a_0^1, a_1^1, \ldots, a_{n-1}^1\rangle \in \alpha^1$ such that

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle = \langle a_0^0, a_1^0, \ldots, a_{n-1}^0\rangle + \langle a_0^1, a_1^1, \ldots, a_{n-1}^1\rangle.$$

Since A satisfies the new heirarchy property, there are $\gamma_0^0, \gamma_1^0, \ldots, \gamma_{n-1}^0 \in \triangle M$ such that $\langle \gamma_0^0, \gamma_1^0, \ldots, \gamma_{n-1}^0, \beta_1^0, \ldots, \beta_{m-1}^0\rangle \in A$, $\sum_{i<n} \gamma_i^0 = \alpha^0$, and $\forall i < n[T(\gamma_i^0) = a_i^0]$.

Similarly, there are $\gamma_0^1, \gamma_1^1, \ldots, \gamma_{n-1}^1 \in \triangle M$ such that

$$\langle \gamma_0^1, \gamma_1^1, \ldots, \gamma_{n-1}^1, \beta_1^1, \ldots, \beta_{m-1}^1 \rangle \in B, \ \sum_{i<n} \gamma_i^1 = \alpha^1, \text{ and } \forall i < n[T(\gamma_i^1) = a_i^1].$$

Hence

$$\begin{aligned} &\langle \gamma_0^0 + \gamma_0^1, \ldots, \gamma_{n-1}^0 + \gamma_{n-1}^1, \beta_1, \ldots, \beta_{m-1} \rangle \in A + B, \\ &\sum_{i<n} (\gamma_i^0 + \gamma_i^1) = \alpha, \ \forall i < n[T(\gamma_i^0 + \gamma_i^1) = a_i. \end{aligned}$$

□

Proposition 28.3. *$\mathscr{P}^2$ is a submonoid of $\triangle \mathscr{P}^1$.* □

Proposition 28.4. *Assume that $m, n \in \omega \backslash \{0\}$, α has domain m, β has domain n, $\forall i < m[\alpha_i \in \mathcal{N}^1]$, $\forall j < n[\beta_j \in \mathcal{N}^1]$, γ has domain $m \times n$, $\forall i < m \forall j < n[\gamma_{ij} \in \mathscr{P}^1]$, $\forall i < m[\Pi_1(\alpha_i) = \sum_{j<n} \gamma_{ij}]$, and $\forall j < n[\Pi_1(\beta_j) = \sum_{i<m} \gamma_{ij}]$. Then there is a δ with domain $m \times n$ such that $\forall i < m \forall j < n[\delta_{ij} \in \mathcal{N}^1$ and $\Pi_1(\delta_{ij}) = \gamma_{ij}]$, $\forall i < m[\alpha_i = \sum_{j<n} \delta_{ij}]$, and $\forall j < n[\beta_j = \sum_{i<n} \delta_{ij}]]]$.*

Proof. Assume that $m, n \in \omega \backslash \{0\}$, α has domain m, β has domain n, $\forall i < m[\alpha_i \in \mathcal{N}^1]$. $\forall j < n[\beta_j \in \mathcal{N}^1]$, $\sum_{i<m} \Pi_1(\alpha_i) = \sum_{j<n} \Pi_1(\beta_j)$, γ has domain $m \times n$, $\forall i < m \forall j < n[\gamma_{ij} \in \mathscr{P}^1]$, $\forall i < m[\Pi_1(\alpha_i) = \sum_{j<n} \gamma_{ij}]$, and $\forall j < n[\Pi_1(\beta_j) = \sum_{i<m} \gamma_{ij}]$. Let $\theta_i = T(\alpha_i)$ for all $i < m$, $\psi_j = T(\beta_j)$ for all $j < n$, and $\rho_{ij} = T(\gamma_{ij})$ for all $i < m, j < n$. Then by Proposition 28.1, $\Pi(\theta_i) = \Pi(T(\alpha_i) = T(\Pi_1(\alpha_i)) = \sum_{j<n} T(\gamma_{ij}) = \sum_{j<n} \rho_{ij}$ for all $i < m$, and similarly $\Pi(\psi_j) = \sum_{i<m} \rho_{ij}$ for all $j < n$. Moreover, $\forall i < m[\theta_i \in \mathcal{N}^0]$, $\forall j < n[\psi_j \in \mathcal{N}^0]$, and $\forall i < m \forall j < n[\rho_{ij} \in \mathscr{P}]$. Hence by Proposition 27.1 with m, n, θ, χ, ψ replaced by n, m, θ, ψ, ρ we get $\sigma \in {}^{m \times n}\mathcal{N}^{0^*}$ such that $\forall i < m \forall j < n[\Pi(\sigma_{ij}) = \rho_{ij}]$, $\forall i < m[\theta_i = \sum_{j<n} \sigma_{ij}]$, and $\forall j < n[\psi_j = \sum_{i<m} \sigma_{ij}]$. Then $T(\gamma_{ij}) = \rho_{ij} = \Pi(\sigma_{ij})$ for all $i < m$, $j < n$, so by Proposition 27.3 with α, χ replaced by γ_{ij}, σ_{ij}, there is a δ with domain $m \times n$ such that $\forall i < m \forall j < n[\delta_{ij} \in \mathcal{N}^1$, $\Pi_1(\delta_{ij}) = \gamma_{ij}$ and $T(\delta_{ij}) = \sigma_{ij}]$. Now for any $i < m$, $\Pi_1(\sum\{\delta_{ij} : j < n\}) = \sum\{\gamma_{ij} : j < n\}$ and $T(\sum\{\delta_{ij} : j < n\}) = \sum\{\sigma_{ij} : j < n\}$. Also, $\Pi_1(\alpha_i) = \sum\{\gamma_{ij} : j < n\}$ and $T(\alpha_i) = \theta_i = \sum\{\sigma_{ij} : j < n\}$. Hence by the uniqueness assertion in Proposition 27.3, $\alpha_i = \sum\{\delta_{ij} : j < n\}$. Similarly, $\forall j < n[\beta_j = \sum_{i<m} \delta_{ij}]$.

□

Proposition 28.5. *$\mathcal{N}^2$ is a submonoid of $\triangle \mathcal{N}^1$.*

Proof. Suppose that $A, B \in \mathcal{N}^2$. By Proposition 28.3, $A + B$ satisfies the new hierarchy property. To prove (N2) for $A + B$, suppose that $n \in \omega \backslash \{0\}$, α and β have domain n, $\alpha \in A + B$, and $\forall i < n[\Pi_1(\beta_i) = \Pi_1(\alpha_i)]$. Say $\alpha = \gamma + \delta$ with $\gamma \in A$ and $\delta \in B$. Take any $i < n$. Then $\alpha_i, \gamma_i, \delta_i \in \mathcal{N}^1$ and $\alpha_i = \gamma_i + \delta_i$. Hence $\Pi_1(\alpha_i) = \Pi_1(\gamma_i) + \Pi_1(\delta_i)$. Let $\varepsilon_{f(0,i)} = \gamma_i$ and $\varepsilon_{f(1,i)} = \delta_i$.

For $i < n$ and $k < 2n$ let

$$c_{ik} = \begin{cases} \Pi_1(\varepsilon_k) & \text{if } k = f(u, i) \text{ for some } u \in 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $i < n$,

$$\Pi_1(\beta_i) = \Pi_1(\alpha_i) = \Pi_1(\gamma_i) + \Pi_1(\delta_i) = \Pi_1(\varepsilon_{f(0,i)}) + \Pi_1(\varepsilon_{f(1,i)}) = c_{if(0,i)} + c_{if(1,i)} = \sum_{k<2n} c_{ik}$$

and for all $k < 2n$, with $f(u,j) = k$,

$$\Pi_1(\varepsilon_k) = c_{jk} = \sum_{i<n} c_{ik}.$$

Now we apply Proposition 28.4 with $m, n, \alpha, \beta, \gamma$ replaced by $n, 2n, \beta, \varepsilon, c$; we get d with domain $n \times 2n$ such that $\forall i < n \forall k < 2n[d_{ik} \in \mathcal{N}^1$ and $\Pi_1(d_{ik}) = c_{ik}]$, $\forall i < n[\beta_i = \sum_{k<2n} d_{ik}]$, and $\forall k < 2n[\varepsilon_k = \sum_{i<n} d_{ik}]$. Then

$$\sum_{i<n} \gamma_i = \sum_{i<n} \varepsilon_{f(0,i)} = \sum_{i<n} \sum_{j<n} d_{jf(0,i)} = \sum_{j<n} \sum_{i<n} d_{jf(0,i)} = \sum_{i<n} \sum_{j<n} d_{if(0,j)}.$$

Further,

$$\Pi_1(\gamma_i) = \Pi_1(\varepsilon_{f(0,i)}) = \Pi_1\left(\sum_{j<n} d_{jf(0,i)}\right) = \sum_{j<n} c_{jf(0,i)} = \Pi_1(\varepsilon_{f(0,i)}).$$

Also, $\Pi_1(\sum_{j<n} d_{if(0,j)}) = \sum_{j<n} c_{if(0,j)} = \Pi_1(\varepsilon_{f(0,i)})$. Now we apply (N3) with n, α_i, β_i replaced by $n, \sum_{j<n} d_{if(0,j)}, \varepsilon_{f(0,i)}$; we get $\langle \sum_{j<n} d_{if(0,j)} : i < n\rangle \in A$. Similarly,

$$\left\langle \sum_{j<n} d_{if(1,j)} : i < n \right\rangle \in B.$$

Now for each $i < n$, $\beta_i = \sum_{k<2n} d_{ik} = \sum_{j<n} d_{if(0,j)} + \sum_{j<n} d_{if(1,j)}$. Hence $\beta \in A + B$.

To prove (N4) for $A+B$, suppose that $n \in \omega$, $\langle \alpha, \beta_1, \ldots, \beta_{n-1}\rangle \in A+B$, and $\Pi_1(\alpha) \neq 0$. Say $\langle \alpha, \beta_1, \ldots, \beta_{n-1}\rangle = \langle \alpha^0, \beta_1^0, \ldots, \beta_{n-1}^0\rangle + \langle \alpha^1, \beta_1^1, \ldots, \beta_{n-1}^1\rangle$ with $\langle \alpha^0, \beta_1^0, \ldots, \beta_{n-1}^0\rangle \in A$ and $\langle \alpha^1, \beta_1^1, \ldots, \beta_{n-1}^1\rangle \in B$. Wlog $\Pi_1(\alpha^0) \neq 0$. Then there are γ_0, γ_1 such that $\alpha^0 = \gamma_0 + \gamma_1$, $\Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0$, and $\langle \gamma_0, \gamma_1, \beta_1^0, \ldots, \beta_{n-1}^0\rangle \in A$. Also, $\langle 0, \alpha^1, \beta_1^1, \ldots, \beta_{n-1}^1\rangle \in B$, so $\langle \gamma_0, \gamma_1 + \alpha^1, \beta_1, \ldots, \beta_{n-1}\rangle \in A + B$ and $\Pi_1(\gamma_0), \Pi_1(\gamma_1 + \alpha^1) \neq 0$. □

Now we define, for any $A \in \mathcal{N}^2$,

$$\Pi_2(A) = \{\Pi_1 \circ a : a \in A\}.$$

Proposition 28.6. Π_2 *is a monoid morphism from* $\mathcal{N}^2$ *to* $\mathscr{P}^2$.

Proof. By Proposition 28.1, Π_1 is a morphism from $\mathcal{N}^1$ to $\mathscr{P}^1$. Note that $\Pi_2 = (\triangle \Pi_1)$. Now let $A \in \mathcal{N}^2$. Then by the proof of Proposition 25.12, $\Pi_2(A)$ satisfies C.P. and R.P. For S.P., suppose that $a \in \Pi_2(A)$ and $a_0 \neq o$. Say $a = \Pi_1 \circ b$ with $b \in A$. By (148) there are γ_0, γ_1 such that $b_0 = \gamma_0 + \gamma_1$, $\Pi_1(\gamma_0), \Pi_1(\gamma_1) \neq 0$,

and $\langle \gamma_0, \gamma_1.a_1, \ldots \rangle \in A$. Hence $\Pi_1 \circ \langle \gamma_0, \gamma_1.a_1, \ldots \rangle \in \Pi_2(A)$, as desired. So we have shown that $\Pi_2(A) \in \triangle\mathscr{P}^1$. To check the new heirarchy property, suppose that $m \in \omega\backslash\{0\}$, $\langle \alpha, \beta_1, \ldots \beta_{m-1} \rangle \in \Pi_2(A)$, and $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \alpha$. Say $\langle \alpha, \beta_1, \ldots \beta_{m-1} \rangle = \Pi_1 \circ \langle \alpha', \beta'_1, \ldots \beta'_{m-1} \rangle$ with $\langle \alpha', \beta'_1, \ldots \beta'_{m-1} \rangle \in A$. Then $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \Pi_1(\alpha')$; say $\langle a_0, a_1, \ldots, a_{n-1} \rangle = \langle \Pi(a'_0).\Pi(a'_1), \ldots, \Pi_1(a'_{n-1}) \rangle$ with $\langle a'_0, a'_1, \ldots, a'_{n-1} \rangle \in \alpha'$. Then by the new heirarchy property for A, there are $\alpha'_0, \alpha'_1, \ldots, a'_{n-1}$ such that $\alpha' = \sum_{i<n} \alpha'_i$, $\forall i < n[T(\alpha'_i) = a'_i]$, and $\langle \alpha'_0, \alpha'_1, \ldots, \alpha'_{n-1}, \beta'_1, \ldots, \beta'_{m-1} \rangle \in A$. Then $\alpha = \Pi_1(\alpha') = \sum_{i<n} \Pi_1(\alpha'_i)$. Using Proposition 28.1, for any $i < n$, $T(\Pi_1(\alpha'_i) = \Pi(T(\alpha'_i) = \Pi(a'_i) = a_i$. Finally,

$$\begin{aligned}
&\Pi_1 \circ \langle \alpha'_0, \alpha'_1, \ldots, \alpha'_{n-1}, \beta'_1, \ldots, \beta'_{m-1} \rangle \\
&\quad = \langle \Pi_1(\alpha'_0), \Pi_1(\alpha'_1), \ldots, \Pi_1(\alpha'_{n-1}), \Pi_1(\beta'_1), \ldots \Pi_1(b'_{m-1}) \rangle \\
&\quad = \langle \Pi_1(\alpha'_0), \Pi_1(\alpha'_1), \ldots, \Pi_1(\alpha'_{n-1}), \beta_1, \ldots, \beta_{m-1} \rangle \in \Pi_2(A).
\end{aligned}$$

Thus $\Pi_2(A)$ has the new heirarchy property. So $\Pi_2(A) \in \mathscr{P}^2$.

Now Π_2 preserves $+$:

$$\begin{aligned}
\Pi_2(A+B) &= \{\Pi_1 \circ a : a \in A+B\} = \{\Pi_1 \circ (b+c) : b \in A, c \in B, \mathrm{dmn}(a) = \mathrm{dmn}(b)\} \\
&= \{(\Pi_1 \circ b) + (\Pi_1 \circ c) : b \in A, c \in B, \mathrm{dmn}(a) = \mathrm{dmn}(b)\} = \Pi_2(A) + \Pi_2(B). \square
\end{aligned}$$

Proposition 28.7. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{N}^2 \arrow[r, "\Pi_2"] \arrow[d, "T"'] & \mathscr{P}^2 \arrow[d, "T"] \\
\mathcal{N}^1 \arrow[r, "\Pi_1"'] & \mathscr{P}^1
\end{tikzcd}$$

Proof. Π_2 is a morphism from $\mathcal{N}^2$ to $\mathscr{P}^2$ by Proposition 28.6. T is a morphism from $\mathcal{N}^2$ to $\mathcal{N}^1$ by Proposition 8.11. T is a morphism from $\mathscr{P}^2$ to $\mathscr{P}^1$ by Proposition 8.11. The arrow at the bottom holds by Proposition 28.1. For commutativity, for any $A \in \mathcal{N}^2$ we have

$$\begin{aligned}
T(\Pi_2(A)) &= T(\{\Pi_1 \circ a : a \in A\}) \\
&= \sum_{i<n} \Pi_1(a_i)) \quad \text{for some } a \in A \\
&= \Pi_1\left(\sum_{i<n} a_i\right) \quad \text{for some } a \in A \\
&= \Pi_1(T(A)).
\end{aligned}$$

$\square$

Proposition 28.8. (2.11.5) *If $A \in \mathscr{P}^2$, $\beta \in \mathcal{N}^1$, and $T(A) = \Pi_1(\beta)$, then there is a unique $B \in \mathcal{N}^2$ such that $\Pi_2(B) = A$ and $\Pi_1(T(B)) = \Pi_1(\beta)$.*

Proof. Suppose that $A \in \mathscr{P}^2$, $\beta \in \mathcal{N}^1$, and $T(A) = \Pi_1(\beta)$.

28.8(1) $\forall \alpha \in \mathscr{P}^1[\{\gamma \in \mathcal{N}^1 : \Pi_1(\gamma) = \alpha\}$ is countable].

For, suppose that $\alpha \in \mathscr{P}^1$, and let $X = \{\gamma \in \mathcal{N}^1 : \Pi_1(\gamma) = \alpha\}$. Now $\alpha \in \mathscr{P}^1 = \triangle\mathscr{P}$. Hence $T(\alpha) = T(c) \in \mathscr{P}$. So $T(\alpha)$ is countable. If $\gamma \in X$, then $T(\alpha) = T(\Pi_1(\gamma)) = \Pi(T(\gamma))$ by Proposition 28.1. Thus $T(\gamma) \in T(\alpha)$. So $X = \bigcup_{x \in T(\alpha)}\{\gamma \in X : T(\gamma) = x\}$. We claim that for any $x \in T(\alpha)$ there is at most one $\gamma \in X$ such that $T(\gamma) = x$. For, suppose that $\gamma \in X$ and $T(\gamma) = x$ with $x \in T(\alpha)$. Then $T(\alpha) = \Pi(T(\gamma))$, so by Proposition 27.3 there is a unique $\beta \in \mathcal{N}^1$ such that $\Pi_1(\beta) = \alpha$ and $T(\beta) = T(\gamma)$. So $\beta = \gamma$. This proves 28.8(1).

The following is a consequence of 28.8(1):

28.8(1)$'$ $\forall m \in \omega\backslash\{0\}\forall \alpha \in {}^m\mathscr{P}^1[\{\gamma \in {}^m\mathcal{N}^1 : \Pi_1 \circ \gamma = \alpha\}$ is countable.

Now let

$$B = \{b \in {}^n(\mathcal{N}^1) : n \in \omega, \Pi_1 \circ b \in A \text{ and } \Pi_1\left(\sum_{i<n} b_i\right) = \Pi_1(\beta)\}$$

Now $A \in \mathscr{P}^2 \subseteq \triangle\mathscr{P}^1$, so A is countable. Hence by 28.8(1)$'$, also B is countable. We claim that $B \in \mathcal{N}^2$. First we prove that $B \in \triangle\mathcal{N}^1$. We have $B \in [{}^{<<\omega}\mathcal{N}^1]^\omega$.

C.P.: Suppose that $a \in B$ and $a \prec b$. Say $\mathrm{dmn}(a) = m$, $\mathrm{dmn}(b) = n$, $\lambda : m \to n$, and $\forall j < n[b_j = \sum\{a_i : i < m, \lambda(i) = j\}]$. We have $\Pi_1 \circ a \in A$ and $\forall j < n[\Pi_1(b(j)) = \sum\{\Pi_1(a_i) : i < m, \lambda(i) = j\}]$, so $\Pi_1 \circ a \prec \Pi_1 \circ b$. Hence $\Pi_1 \circ b \in A$. Also, $\sum_{j<n} b_j = \sum_{i<m} a_i$, so $\Pi_1(\sum_{j<n} b_j) = \Pi_1(\sum_{i<n} a_i) = \Pi_1(\beta$. Hence $b \in B$.

R.P.: Suppose that $m, n \in \omega\backslash\{0\}$, $a \in {}^m(\mathcal{N}^1)$, $b \in {}^n(\mathcal{N}^1)$, and $a, b \in B$. By the definition of B, $\Pi_1(\sum_{i<m} a_i) = \Pi_1(\beta) = \Pi_1(\sum_{i<n} b_i)$. Since $a, b \in B$ we have $\Pi_1 \circ a, \Pi_1 \circ b \in A$. Now $A \in \mathcal{P}^2 \subseteq \triangle\mathscr{P}^1$. Hence by R.P. for A there is a c with domain $m \cdot n$ such that $c \in A$, $\forall i < m[\Pi_1(a_i) = \sum_{j<n} c_{f(i,j)}]$ and $\forall j < n[\Pi_1(b_j) = \sum_{i<m} c_{f(i,j)}]$. Define d with domain $m \times n$ by $d_{ij} = c_{f(i,j)}$. Then $\forall i < m[\Pi_1(a_i) = \sum_{j<n} d_{ij}]$ and $\forall j < n[\Pi_1(b_j) = \sum_{i<m} d_{ij}]$. Hence by Proposition 28.4 there is a e with domain $m \times n$ such that $\forall i < m \forall j < n[\Pi_1(e_{ij}) = d_{ij}]$, $\forall i < m[a_i = \sum_{j<n} e_{ij}]$, and $\forall j < n[b_j = \sum_{i<m} e_{ij}]$. Now define g with domain $m \cdot n$ by $g_{f(i,j)} = e_{ij}$. Then $\forall i < m[a_i = \sum_{j<n} g_{f(i,j)}]$, and $\forall j < n[b_j = \sum_{i<m} g_{f(i,j)}]$. Now for any $i < m$ and $j < n$, $\Pi_1(g_{f(i,j)}) = \Pi_1(e_{ij}) = d_{ij} = c_{f(i,j)}$. So $\Pi_1 \circ g = c \in A$. Also,

$$\Pi_1\left(\sum_{k<m\cdot n} g_k\right) = \Pi_1\left(\sum_{i<m}\sum_{j<n} e_{ij}\right) = \Pi_1\left(\sum_{i<m} a_i\right) = \Pi_1(\beta).$$

It follows that $g \in B$. This proves R.P. for B.

S.P.: suppose that $m \in \omega\backslash\{0\}$, $a \in {}^m(\mathcal{N}^1)$, $a \in B$ and $a_0 \neq o$. Thus $\Pi_1 \circ a \in A$ and $\Pi_1(a_0) \neq O$. Hence by S.P. for A, there exist $u, v \in \mathscr{P}^1\backslash\{0\}$ such that $\Pi_1(a_0) = u + v$ and

$\langle u, v, \Pi_1(a_1), \ldots, \Pi_1(a_{m-1})\rangle \in A$. Hence by Proposition 27.3 there exist $u', v' \in \mathcal{N}^1$ such that $\Pi_1(u') = u$ and $\Pi_1(v') = v$. Thus $\Pi_1 \circ \langle u', v', a_1, \ldots, a_{m-1}\rangle \in A$. Also,

$$\begin{aligned}\Pi_1(u' + v' + a_1 + \cdots + a_{m-1}) &= u + v + \Pi_1(a_1 + \cdots + a_{m-1})\\ &= \Pi_1(a_0) + \Pi_1(a_1 + \cdots + a_{m-1})\\ &= \Pi_1(a_0 + \cdots + a_{m-1}) = \Pi_1(\beta).\end{aligned}$$

Hence $\langle u', v', a_1, \ldots, a_{m-1}\rangle \in B$. So S.P. holds.

Thus we have shown that $B \in \triangle\mathcal{N}^1$.

To check the new heirarchy property for B, suppose $m \in \omega\backslash\{0\}$, $\langle \alpha, \gamma_1, \ldots, \gamma_{m-1}\rangle \in B$, and $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in \alpha$. Then

$$\Pi_1\left(\alpha + \sum_{i=1}^{m-1} \gamma_i\right) = \Pi_1(\beta) \text{ and } \langle \Pi_1(\alpha), \Pi_1(\gamma_1), \ldots, \Pi_1(\gamma_{m-1})\rangle \in A,$$

and $\langle \Pi(a_0), \Pi(a_1), \ldots, \Pi(a_{n-1})\rangle \in \Pi_1(\alpha)$. By the new heirarchy property for A, there are $\delta_0, \delta_1, \ldots, \delta_{n-1} \in \mathscr{P}^1$ such that $\Pi_1(\alpha) = \sum_{i<n} \delta_i$, $\forall i < n[T(\delta_i) = \Pi(a_i)]$, and

$$\langle \delta_0, \delta_1, \ldots, \delta_{n-1}, \Pi_1(\gamma_1), \ldots, \Pi_1(\gamma_{m-1})\rangle \in A.$$

By Proposition 27.3, for each $i < n$ there is an ε_i such that $\Pi_1(\varepsilon_i) = \delta_i$ and $T(\varepsilon_i) = a_i$. Now $T(\sum_{i<n} \delta_i) = \Pi(\sum_{i<n} a_i)$, so by Proposition 27.3 there is a unique x such that $\Pi_1(x) = \sum_{i<n} \delta_i$ and $T(x) = \sum_{i<n} a_i$. Now $\Pi_1(\alpha) = \sum_{i<n} \delta_i$ and $T(\alpha) = \sum_{i<n} a_i$. Also $\Pi_1(\sum_{i<n} \varepsilon_i) = \sum_{i<n} \delta_i$ and $T(\sum_{i<n} \varepsilon_i) = \sum_{i<n} a_i$. So by the uniqueness assertion in Proposition 27.3 we have $\alpha = \sum_{i<n} \varepsilon_i$. Now $\Pi_1 \circ \langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}, \gamma_1, \ldots, \gamma_{m-1}\rangle \in A$ and

$$\Pi_1\left(\sum_{i<n} \varepsilon_i + \sum_{i=1}^{m-1} \gamma_i\right) = \Pi_1\left(\alpha + \sum_{i=1}^{m-1} \gamma_i\right) = \Pi_1(\beta).$$

Hence $\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}, \gamma_1, \ldots, \gamma_{m-1}\rangle \in B$. This proves the new heirarchy property for B.

To check (N3) for B, suppose that $n \in \omega\backslash\{0\}$, a and b have domain n, $a \in B$, and $\forall i < n[\Pi_1(a_i) = \Pi_1(b_i)]$. Then $\Pi_1 \circ b = \Pi_1 \circ a \in A$ and $\Pi_1(\sum_{i<n} b_i) = \Pi_1(\sum_{i<n} a_i) = \Pi_1(\beta)$. So $b \in B$.

For (N4) for B, suppose that $n \in \omega$ and $\langle a, b_1, \ldots, b_{n-1}\rangle \in B$, with $\Pi_1(a) \neq 0$. Then $\langle \Pi_1(a), \Pi_1(b_1), \ldots, \Pi_1(b_{n-1})\rangle \in A$, so by S.P. for A, there are nonzero c, d such that $\Pi_1(a) = c + d$ and $\langle c, d, \Pi_1(b_1), \ldots, \Pi_1(b_{n-1})\rangle \in A$. Say $T(c) = [c'']$ and $T(d) = [d'']$. Now $[c'' + d''] = T(c + d) = T(\Pi_1(a)) = \Pi(T(a)) = [T(a)]$. So $T(a) \equiv_{ae} c'' + d''$. Note that $T(a) \in \mathcal{N}^0$. Let $F = \{k \in \omega + 1 : (T(a))(k) \neq c''(k) + d''(k)\}$. So F is finite. Define

$$c'''(k) = \begin{cases} (T(a))(k) & \text{if } k \in F,\\ c''(k) & \text{otherwise;}\end{cases} \qquad d''(k) = \begin{cases} (T(a))(k) & \text{if } k \in F,\\ d''(k) & \text{otherwise.}\end{cases}$$

Thus $T(a) = c''' + d'''$, $[c''] = [c''']$, and $[d''] = [d''']$. Now $T(c) = [c''] = [c'''] = \Pi(c''')$, so by Proposition 27.3 there is a c' such that $\Pi_1(c') = c$ and $T(c') = c'''$. Similarly there

is a d' such that $\Pi_1(d') = d$ and $T(d') = d'''$. Hence $\Pi_1(c' + d') = c + d = \Pi_1(a)$ and $T(c' + d') = c''' + d''' = T(a)$. Now $T(c + d) = T(\Pi_1(a)) = \Pi(T(a))$, so by Proposition 27.3 there is a unique β such that $\Pi_1(\beta) = c + d$ and $T(\beta) = T(a)$. $\beta = a$ works, and so does $c' + d'$. Hence $a = c' + d'$. Also,

$$\begin{aligned}\Pi_1(c' + d' + b_1 + \cdots + b_{m-1}) &= \Pi_1(c' + d') + \Pi_1(b_1 + \cdots + b_{m-1})\\ &= \Pi_1(a) + \Pi_1(b_1 + \cdots + b_{m-1})\\ &= \Pi_1(a + b_1 + \cdots + b_{m-1}) = \Pi_1(\beta).\end{aligned}$$

So $\langle c', d', b_1, \ldots, b_{m-1}\rangle \in B$, as desired.

So, we have shown that $B \in \mathcal{N}^2$.

Clearly $\Pi_2(B) \subseteq A$. Now suppose that $c \in A$. Now $A \in \mathscr{P}^2 \subseteq \triangle\mathscr{P}^1$. Say $c \in {}^m\mathscr{P}^1$. For any $i < m$, $c_i \in \mathscr{P}^1$. Then there is a $d_i \in \mathcal{N}^0$ such that $T(c_i) = \Pi(d_i)$. By Proposition 27.3 there is an $e_i \in \mathcal{N}^1$ such that $\Pi_1(e_i) = c_i$ and $T(e_i) = d_i$. So $\Pi_1 \circ e = c$ and $T \circ e = d$. Then

$$\begin{aligned}\left\{\Pi \circ \theta : \theta \in \sum_{i<m} e_i\right\} &= \Pi_1\left(\sum_{i<n} e_i\right) = \sum_{i<n} \Pi_1(e_i) = \sum_{i<n} c_i\\ &= T(c) = T(A) = \Pi_1(\beta) = \{\Pi \circ \theta : \theta \in \beta\}.\end{aligned}$$

Thus $\Pi_1(\sum_{i<m} e_i) = \Pi_1(\beta)$. Hence $c \in \Pi_2(B)$. So $\Pi_2(B) = A$.

Clearly $\Pi_1(T(B)) = \Pi_1(\beta)$.

Now suppose that also $C \in \mathcal{N}^2$, $\Pi_2(C) = A$, and $\Pi_1(T(C)) = \Pi_1(\beta)$. If $c \in C$, then $\Pi_1 \circ c \in \Pi_2(C) = A = \Pi_2(B)$, so there is a $b \in B$ such that $\Pi_1 \circ c = \Pi_1 \circ b \in A$. Also, $\Pi_1(\sum_{i<m} c_i) = \Pi_1(T(c)) = \Pi_1(T(C))$, so $c \in B$.

If $b \in B$, then $\Pi_1 \circ b \in \Pi_2(B) = A = \Pi_1(C)$, so there is a $c \in C$ such that $\Pi_1 \circ b = \Pi_1 \circ c$. By (147) for C, $b \in C$. Thus $B = C$. □

Let $\mathscr{Q}^2 = \{A \in \mathscr{P}^2 : T(A) \in \mathscr{Q}^1\}$.

Proposition 28.9. *$\mathscr{Q}^2$ is a subsemigroup of $\mathscr{P}^2$.*

Proof. Let $A, B \in \mathscr{Q}^2$. Then $T(A + B) = T(A) + T(B)$, and this sum is in $\mathscr{Q}^1$ by Proposition 27.4. □

Proposition 28.10. *There is a homomorphism $\Gamma_2 : \mathscr{Q}^2 \to \mathcal{N}^2$ such that:*

(i) $\forall A \in \mathscr{Q}^2[\Pi_2(\Gamma_2(A)) = A]$.

(ii) $\forall A \in \mathscr{Q}^2[\Pi_1(T(\Gamma_2(A))) = T(A)]$.

Proof. Suppose that $A \in \mathscr{Q}^2$. Thus $A \in \mathscr{P}^2$. Say $T^2(A) = \Pi(X)$ with $X \in \mathcal{N}^0$. Then by Proposition 27.6 we get $\beta \in \mathcal{N}^1$ such that $\Pi_1(\beta) = T(A)$ and $T(\beta) = X$. By Proposition 28.8 let $\Gamma_2(A)$ be the set B indicated there. Then $\Pi_2(\Gamma_2(A)) = A$ and $\Pi_1(T(\Gamma_2(A))) = \Pi_1(\beta)$. So $\Pi_1(T(\Gamma_2(A))) = T(A)$.

For preservation of $+$, suppose that $A_1, A_2 \in \mathscr{P}^2$. Then $T^2(A_1 + A_2) = T^2(A_1) + T^2(A_2) = \Pi(X_1) + \Pi(X_2) = \Pi(X_1 + X_2)$. Say $T^2(A_1) = \Pi(U_1)$ and $T^2(A_2) = \Pi(U_2)$.

Then there are unique β_1, β_2 such that $\Pi_1(\beta_1) = T(A_1)$, $T(\beta_1) = X_1$, $\Pi_1(\beta_2) = T(A_2)$, and $T(\beta_2) = X_2$. Then we have

$$\Gamma_2(A_1) = \{b \in {}^n(\mathcal{N}^1) : n \in \omega, \Pi_1 \circ b \in A_1 \text{ and } \Pi_1\left(\sum_{i<n} b_i\right) = \Pi_1(\beta_1\};$$

$$\Gamma_2(A_2) = \{b \in {}^n(\mathcal{N}^1) : n \in \omega, \Pi_1 \circ b \in A_2 \text{ and } \Pi_1\left(\sum_{i<n} b_i\right) = \Pi_1(\beta_2)\}.$$

Hence

$$\Gamma_2(A_1 + A_2) = \{b \in {}^n(\mathcal{N}^1) : n \in \omega, \Pi_1 \circ b \in A_1 + A_2 \text{ and } \Pi_1\left(\sum_{i<n} b_i\right) = \Pi_1(\beta_1 + \beta_2)\}.$$

Hence Γ_2 preserves $+$. □

Proposition 28.11. *$\triangle^2(\Psi \circ \Omega) \restriction \mathcal{N}^2$ is an injective morphism from $\mathcal{N}^2$ into $\triangle^3\mathscr{W}$.*

Proof. By Proposition 25.15, $(\triangle(\Psi \circ \Omega)) \restriction \mathcal{N}^1$ is an injective morphism from $\mathcal{N}^1$ into $\mathscr{L}^1$. Hence by Proposition 25.123, $(\triangle^2(\Psi \circ \Omega)) \restriction \triangle\mathcal{N}^1$ is an injective morphism from $\triangle\mathcal{N}^1$ into $\triangle\mathscr{L}^1$. By Proposition 28.5, $\mathcal{N}^2$ is a submonoid of $\triangle\mathcal{N}^1$. So $(\triangle^2(\Psi \circ \Omega)) \restriction \mathcal{N}^2$ is an injective morphism of $\mathcal{N}^2$ into $\triangle\mathscr{L}^1$. By definition $\triangle\mathscr{L}^1 \subseteq \triangle^3\mathscr{W}$. □

29. First reduction step

Proposition 29.1. *Suppose that N is a countable submonoid of $\mathscr{P}^2$ such that:*

(i) N has the refinement property.

(ii) $\forall A \in N \forall(\alpha_0, \alpha_1) \in A \exists A_0, A_1 \in N[A = A_0 + A_1$ and $T(A_0) = \alpha_0$ and $T(A_1) = \alpha_1]$.

Then $N \cap \mathscr{Q}^2$ is isomorphic to a subsemigroup of **NBA**.

Proof. Let $N_1 = \{B \in \mathcal{N}^2 : \Pi_2(B) \in N\}$.

29.1(1) N_1 is a submonoid of $\mathcal{N}^2$.

For, suppose that $B, C \in N_1$. Then $B + C \in \mathcal{N}^2$, and $\Pi_2(B + C) = \Pi_2(B) + \Pi_2(C) \in N$. So $B + C \in N_1$, proving 29.1(1).

29.1(2) $N \cap \mathscr{Q}^2$ is a submonoid of $\mathscr{P}^2$, and Γ_2 is an injective homomorphism of $N \cap \mathscr{Q}^2$ into N_1.

In fact, N is a submonoid of $\mathscr{P}^2$ by definition, and $\mathscr{Q}^2$ is a submonoid of $\mathscr{P}^2$ by Proposition 28.9. So $N \cap \mathscr{Q}^2$ is a submonoid of $\mathscr{P}^2$. For any $A \in N \cap \mathscr{Q}^2$ we have $\Pi_2(\Gamma_2(A)) = A \in N$ by Proposition 28.10, so $\Gamma_2(A) \in N_1$. If $\Gamma_2(A) = \Gamma_2(A')$, then $A = \Pi_2(\Gamma_2(A)) = \Pi_2(\Gamma_2(A')) = A'$. Thus Γ_2 is injective.

29.1(3) N_1 is countable.

For, $N_1 = \bigcup\{\{B \in \mathcal{N}^2 : \Pi_2(B) = X\} : X \in N\}$ so, since N is countable, it suffices to show that for any $X \in N$ the set $\{B \in \mathcal{N}^2 : \Pi_2(B) = X\}$ is countable.

29.1(4) $\forall X \in \mathscr{P}[\{\theta \in \mathcal{N}^0 : \Pi(\theta) = X\}$ is countable].

For, suppose that $X \in \mathscr{P}$. Say $X = [\psi]_{ae}$. Then $\{\theta \in \mathcal{N}^0 : \Pi(\theta) = X\} = [\psi]_{ae}$ is countable by Proposition 26.2.

29.1(5) $\forall X \in \mathscr{P}^1[\{b \in \mathcal{N}^1 : \Pi_1(b) = X\}$ is countable].

For, assume that $X \in \mathscr{P}^1$. Let $Y = \{b \in \mathcal{N}^1 : \Pi_1(b) = X\}$. Let $Z = \{x \in \mathcal{N}^0 : \Pi(x) = T(X)\}$. So Z is countable by 29.1(4). If $b \in Y$, then $T(X) = T(\Pi_1(b)) = \Pi(T(b))$, so that $T(b) \in Z$. Suppose that $b, b' \in Y$ and $T(b) = T(b')$. Then $T(\Pi_1(b)) = \Pi(T(b))$ and so by Proposition 27.3 there is a unique β such that $\Pi_1(\beta) = \Pi_1(b)$ and $T(\beta) = T(b)$. So $\beta = b$. But also $\Pi_1(b') = \Pi_1(b)$ and $T(b') = T(b)$, so $b = b'$. Hence $T \restriction Y$ is a one-one function mapping Y into Z; so Y is countable.

Now we can prove 29.1(3). Suppose that $X \in N$, and let $Y = \{B \in \mathcal{N}^2 : \Pi_2(B) = X\}$. Let $Z = \{x \in \mathcal{N}^1 : \Pi_1(x) = T(X)\}$. So Z is countable by (154). Suppose that $b, b' \in Y$ and $T(b) = T(b')$. Then $T(\Pi_2(b)) = \Pi_1(T(b))$ and so by Proposition 28.6 there is a unique β such that $\Pi_2(\beta) = \Pi_2(b)$ and $\Pi_1(T(\beta)) = \Pi_1(T(b))$. So $\beta = b$. But also $\Pi_2(b') = \Pi_2(b)$ and $\Pi_1(T(b')) = \Pi_1(T(b))$, so $b = b'$. Hence $T \restriction Y$ is a one-one function mapping Y into Z; so Y is countable. This proves 29.1(3).

Now by Proposition 28.11, $\mathcal{N}^2$ is isomorphic to a submonoid of $\triangle^3\mathscr{W}$. Hence it suffices now to check conditions (i) and (ii) of Proposition 24.5 for N_1.

29.1(6) N_1 has the refinement property.

For, we use Proposition 19.7. Suppose that $A, B \in {}^2N_1$ and $A_0 + A_1 = B_0 + B_1$. Then $\Pi_2(A_0) + \Pi_2(A_1) = \Pi_2(B_0) + \Pi_2(B_1)$ and $T(A_0) + T(A_1) = T(B_0) + T(B_1)$. Now $\Pi_2(A_0), \ldots, \Pi_2(B_1) \in N$, so by (i) there is a $C \in {}^{2\times 2}N$ such that $\forall i < 2[\Pi_2(A_i) = \sum_{j<2} C_{ij}]$ and $\forall j < 2[\Pi_2(B_j) = \sum_{i<2} C_{ij}]$. For all $i, j < 2$ let $\gamma_{ij} = T(C_{ij})$. Then $\forall i < 2[\Pi_1(T(A_i)) = T(\Pi_2(A_i)) = \sum_{j<2} \gamma_{ij}]$, and similarly $\forall j < 2[\Pi_1(T(B_j)) = \sum_{i<2} \gamma_{ij}]$. By Proposition 28.4 there is a $\rho \in {}^{2\times 2}\mathcal{N}^1$ such that $\forall i, j < 2[\Pi_1(\rho_{ij}) = \gamma_{ij}]$, $\forall i < 2[T(A_i) = \sum_{j<2} \rho_{ij}]$, and $\forall j < 2[T(B_j) = \sum_{i<2} \rho_{ij}]$. Thus $\forall i, j < 2[T(C_{ij}) = \gamma_{ij} = \Pi_1(\rho_{ij})]$, so by Proposition 28.8 there is a $D \in {}^{2\times 2}\mathcal{N}^2$ such that $\forall i, j < 2[\Pi_2(D_{ij}) = C_{ij}$ and $T(D_{ij}) = \rho_{ij}]$. It follows that $\forall i < 2[T(A_i) = \sum_{j<2} \rho_{ij} = \sum_{j<2} T(D_{ij}) = T(\sum_{j<2} D_{ij})$ and $\Pi_2(A_i) = \sum_{j<2} C_{ij} = \sum_{j<2} \Pi_2(D_{ij}) = \Pi_2(\sum_{i<2} D_{ij}]$. Now $\forall i < 2[\Pi_1(T(A_i)) = T(\Pi_2(A_i))]$, so by Proposition 28.8, for each $i < 2$ there is a unique X such that $\Pi_2(X) = \Pi_2(A_i)$ and $T(X) = T(A_i)$. It follows that $\forall i < 2[A_i = \sum_{j<2} D_{ij}]$. Similarly, $\forall j < 2[B_j = \sum_{i<2} D_{ij}]$. This proves 29.1(6).

29.1(7) $\forall n \in \omega\backslash\{0\}\forall A \in N\forall \alpha \in A[\mathrm{dmn}(\alpha) = n \rightarrow \exists B \in {}^nN[A = \sum_{i<n} B_i$ and $\forall i < n[T(B_i) = \alpha_i]]$.

We prove 29.1(7) by induction on n. It is trivial for $n = 1$, and (ii) gives $n = 2$. Now assume it for n, and suppose that $\alpha \in A$ with $\mathrm{dmn}(\alpha) = n+1$. Then $\langle \sum_{i<n} \alpha_i, \alpha_n) \in A$, so by (ii) there are $C_0, C_1 \in N$ such that $A = C_0 + C_1$, $T(C_0) = \sum_{i<n} \alpha_i$, and $T(C_1) = \alpha_n$. Then by the inductive hypothesis there is a $B \in {}^nN$ such that $C_0 = \sum_{i<n} B_i$, and $\forall i < n[T(B_i) = \alpha_i]$. This gives 29.1(7) for $n + 1$.

To prove Proposition 24.5(ii), suppose that $A \in N_1$, $n \in \omega$, $\alpha \in A$, and $\mathrm{dmn}(\alpha) = n$. So $T(A) = \sum_{i<n} \alpha_i$ and $\Pi_2(A) \in N$. Now $\Pi_1 \circ \alpha \in \Pi_2(A)$. By 29.1(7) there is $B \in N$ with

domain n such that $\Pi_2(A) = \sum_{i<n} B_i$ and $\forall i < n[T(B_i) = \Pi_1(\alpha_i)]$. By Proposition 28.8 there is $C \in \mathcal{N}^2$ with domain n such that $\forall i < n[\Pi_2(C_i) = B_i$ and $T(C_i) = \alpha_i]$. Hence $\forall i < n[C_i \in N_1]$. Now $\Pi_2(\sum_{i<n} C_i) = \sum_{i<n} B_i = \Pi_2(A)$ and $T(\sum_{i<n} C_i) = \sum_{i<n} \alpha_i = T(A)$. Hence by Proposition 28.8, $A = \sum_{i<n} C_i$. □

30. Distinguished elements

An element e of an m-monoid M is *distinguished* iff $\forall f \in M[e+f = e \to f = 0]$. Note that 0 is distinguished. M_s is the set of all distinguished elements of M. M is *distinguished* iff every element of M is a sum of distinguished elements.

Proposition 30.1. *$\forall \theta \in \mathcal{N}^0[[\theta]_{ae}$ is distinguished iff $[\theta]_{ae} \neq [\omega^c]_{ae}]$.*

Proof. $\Rightarrow$: $[\omega^c]_{ae} + [\omega]_{ae} = [\omega]_{ae}$ and $[\omega]_{ae} \neq [0^c]_{ae}$, so $[\omega^c]_{ae}$ is not distinguished.

$\Leftarrow$: Obviously $[0^c]$ is distinguished. Now suppose that $\theta \in \mathcal{N}^0$ satisfies (135), $\psi \in \mathcal{N}^0$, and $[\theta]_{ae} + [\psi]_{ae} = [\theta]_{ae}$. Then $F \stackrel{\text{def}}{=} \{k \in \omega + 1 : \theta(k) + \psi(k) \neq \theta(k)\}$ is finite. Clearly $\forall k \in (\omega + 1)\backslash F[\psi(k) = 0]$, so $[\psi]_{ae} = [0^c]_{ae}$. □

Proposition 30.2. *In $\mathscr{P}$, $[\omega^c]_{ae}$ is not a sum of distinguished elements. Hence $\mathscr{P}$ is not distinguished.*

Proof. Suppose that $[\omega^c]_{ae} = \sum_{i<m} a_i$, each $a_i \in \mathscr{P}$. Then some a_i is equal to $[\omega^c]_{ae}$, and so by Proposition 30.1 is not distinguished. □

Proposition 30.3. *If e is a distinguished element of M, then e has the Schröder-Bernstein property.*

Proof. Assume that e is a distinguished element of M and $e \leq b \leq e$. Thus there are $c, d \in M$ such that $b = e + c$ and $e = b + d$. Thus $e = e + c + d$, so $c + d = 0$, hence $c = d = 0$, hence $b = e$. □

Proposition 30.4. *In $\mathscr{P}$, $[\omega^c]_{ae}$ has the Schröder-Bernstein property, but $[\omega^c]_{ae}$ is not distinguished.*

Proof. By Propositions 30.1, $[\omega^c]_{ae}$ is not distinguished. Now suppose that $[\omega^c]_{ae} \leq a \leq [\omega^c]_{ae}$. Choose $b, c \in \mathscr{P}$ such that $a = [\omega^c]_{ae} + b$ and $[\omega^c]_{ae} = a + c$. Clearly then $a = [\omega^c]_{ae}$. □

For m-monoids M, N, we say that N is a *distinguished extension* of M if N is an extension of M and $M_s \subseteq N_s$.

Proposition 30.5. *If P is a distinguished extension of N and N is a distinguished extension of M, then P is a distinguished extension of M.* □

Proposition 30.6. *If $\langle M_k : k \in \omega\rangle$ is a system of m-monoids such that $\forall k \in \omega[M_{k+1}$ is a distinguished extension of $M_k]$, then $\forall k \in \omega[\bigcup_{i\in\omega} M_i$ is a distinguished extension of $M_k]$.* □

Proposition 30.7. *If e is a nonzero element of an m-monoid M, then there exist a distinguished extension N of M and $e_0, e_1 \in N$ such that*

(i) $e = e_0 + e_1$ and e_0 and e_1 are distinguished and nonzero.

(ii) N is generated as a monoid by $M \cup \{e_0, e_1\}$.

(iii) $\forall m, n \in \omega \forall f, f' \in M[f' = f + me_0 + ne_1 \to m = n$ and $f' = f + me]$.

(iv) If $\Lambda : M \to M'$ is a morphism of m-monoids, then $\forall f_0, f_1 \in M'[\Lambda(e) = f_0 + f_1 \to \exists \Lambda'[\Lambda \subseteq \Lambda'$ and $\Lambda' : N \to M'$ is a morphism and $\forall i < 2[\Lambda'(e_i) = f_i]]]$.

Proof. Let $(\omega, +)$ be the natural monoid. Define $P = M \times \omega \times \omega$. Define

$$\begin{aligned}(f, m, n) \sim (f', m', n') \quad \text{iff} \quad & (f, m, n), (f', m', n') \in P,\ m - n = m' - n', \\ & f + me = f' + m'e, \text{ and } f + ne = f' + n'e.\end{aligned}$$

30.7(1) $\sim$ is a congruence relation on P.

In fact, it is clearly an equivalence relation on P. Now suppose that $(f, m, n) \sim (f', m', n')$ and $(f'', m'', n'') \sim (f''', m''', n''')$. Then

$$\begin{aligned}(f, m, n) + (f'', m'', n'') &= (f + f'', m + m'', n + n''); \\ (f', m', n') + (f''', m''', n''') &= (f' + f''', m' + m''', n' + n'''); \\ m + m'' - (n + n'') &= m - n + m'' - n'' \\ &= m' - n' + m''' - n''' \\ &= m' + m''' - (n' + n'''); \\ f + f'' + (m + m'')e &= f + me + f'' + m''e \\ &= f' + m'e + f''' + m'''e \\ &= f' + f''' + (m' + m''')e \\ f + f'' + (n + n'')e &= f + ne + f'' + n''e \\ &= f' + n'e + f''' + n'''e \\ &= f' + f''' + (n' + n''')e.\end{aligned}$$

This proves 30.7(1).

Let $N = P/\sim$ and let Π be the natural mapping from P to N. For any $f \in M$ let $\Psi(f) = [(f, 0, 0)]_\sim$.

30.7(2) Ψ is an injective morphism from M to N.

In fact, clearly Ψ is a morphism from M to N. Now suppose that $\Psi(f) = \Psi(f')$. Thus $(f, 0, 0) \sim (f', 0, 0)$. Clearly $f = f'$. This proves 30.7(2).

30.7(3) If f is distinguished in M, then $\Psi(f)$ is distinguished in N.

For, suppose that f is distinguished in M, and $\Psi(f) + [(f', m, n)]_\sim = \Psi(f)$. Thus $(f, 0, 0) + (f', m, n) \sim (f, 0, 0)$, i.e., $(f + f', m, n) \sim (f, 0, 0)$. Hence $f + f' + me = f$; since f is distinguished, this implies that $f' + me = 0$, hence $f' = 0 = m$. Also, $m - n = 0$, so $m = n$. Hence $[(f', m, n)] = 0$, proving 30.7(3).

Now let $e_0 = [(0,1,0)]_\sim$ and $e_1 = [(0,0,1)]_\sim$.

30.7(4) e_0 is distinguished.

For, suppose that $e_0 + [(f,m,n)]_\sim = e_0$. Thus $(f, m+1, n) \sim (0,1,0)$, so $m+1-n = 1$, $f + (m+1)e = e$, and $f + ne = 0$. Hence $f = n = 0$ and $m = 0$. So $[(f,m,n0]_\sim = 0$. This proves 30.7(4). Similarly,

30.7(5) e_1 is distinguished.

30.7(6) $\Psi(e) = e_0 + e_1$.

In fact, $\Psi(e) = [(e,0,0)]_\sim$, and $e_0 + e_1 = [(0,1,1)]_\sim$. We have $0-0 = 1-1$, $e+0e = 0+1e$, and $e + 0e = 0 + 1e$. So $\Psi(e) = [(e,0,0)]_\sim = [(0,1,1)]_\sim = e_0 + e_1$.

30.7(7) $e_0 \neq 0$.

In fact, suppose that $[(0,1,0)]_\sim = [(0,0,0)]_\sim$. then $1-0 = 0-0$, contradiction. Similarly,

30.7(8) $e_1 \neq 0$.

30.7(9) N is generated by $\Psi[M] \cup \{e_0, e_1\}$.

In fact, let $[(f,m,n)]_\sim \in N$. Then $\Psi(f) + me_0 + ne_1 = [(f,m,n)]_\sim$, as desired.

30.7(10) $\forall m,n \in \omega \forall f, f' \in M[\Psi(f') = \Psi(f) + me_0 + ne_1 \to m = n$ and $f' = f + me]$.

For, suppose that $m, n \in \omega$, $f, f' \in M$, and $\Psi(f') = \Psi(f) + me_0 + ne_1$. Thus $(f', 0, 0) \sim (f,m,n)$, so $0 - 0 = m - n$ and $f' = f + me$. So $m = n$, proving 30.7(4).

Now suppose that $\Lambda : M \to M'$ is a morphism of m-monoids, $f_0, f_1 \in M'$, and $\Lambda(e) = f_0 + f_1$. Define $\Lambda' : N \to M'$ by $\Lambda'([(f,m,n)]_\sim) = \Lambda(f) + mf_0 + nf_1$. Then Λ' is well-defined: Suppose that $[(f,m,n)]_\sim = [(f',m',n.)]_\sim$. Thus $(f,m,n) \sim (f',m',n')$, so $m - n = m' - n'$, $f + me = f' + m'e$, and $f + ne = f' + n'e$. Say $m \geq n$. Then also $m' \geq n'$, and

$$\begin{aligned}
\Lambda(f) + mf_0 + nf_1 &= \Lambda(f) + n(f_0 + f_1) + (m-n)f_0 = \Lambda(f) + n\Lambda(e) + (m-n)f_0 \\
&= \Lambda(f + ne) + (m' - n')f_0 = \Lambda(f' + n'e) + (m' - n')f_0 \\
&= \Lambda(f') + n'\Lambda(e) + (m' - n')f_0 = \Lambda(f') + n'(f_0 + f_1) + (m' - n')f_0 \\
&= \Lambda(f') + m'f_0 + n'f_1.
\end{aligned}$$

So Λ' is well-defined.

Λ' preserves the operation:

$$\begin{aligned}
\Lambda'([(f,m,n)]_\sim + [(f',m',n')]_\sim) &= \Lambda'([(f+f', m+m', n+n')]_\sim) \\
&= \Lambda(f+f') + (m+m')f_0 + (n+n')f_1 \\
&= \Lambda(f) + \Lambda(f') + mf_0 + m'f_0 + nf_1 + n'f_1 \\
&= \Lambda'([(f,m,n)]_\sim) + \Lambda'([(f',m',n')]_\sim)).
\end{aligned}$$

Clearly $\Lambda'([f,0,0)]_\sim) = \Lambda(f)$, and $\Lambda'(e_i) = f_i$ for $i < 2$. $\square$

Proposition 30.8. *If M is distinguished and N is as in Proposition 30.7, then N is distinguished.* □

Proposition 30.9. (2.13.4) *If M is a countable m-monoid, then there is a distinguished extension N of M such that N is distinguished, countable and atomless.*

Proof. Assume that M is a countable m-monoid. Let $\langle e_k; k < \omega\rangle$ enumerate $M\backslash\{0\}$. We define $N_0 = M$, and N_{k+1} is obtained from N_k by applying Proposition 30.7 to N_k and e_k. Thus we have $e_k = e_{k0} + e_{k1}$ for some distinguished nonzero elements of N_{k+1}. Let $P = \bigcup_{k\in\omega} N_k$. Thus

30.9(1) P is a countable distinguished extension of M such that for each nonzero $x \in M$ there are nonzero distinguished $e, e' \in P$ such that $x = e + e'$.

Now let $Q^0 = M$ and let Q^{k+1} be obtained from Q^k by the proof of 30.9(1). Let $N = \bigcup_{k\in\omega} Q^k$. Then N is a distinguished extension of M such that for every nonzero $x \in N$ there are nonzero distinguished elements $e, e' \in N$ such that $x = e + e'$. So N is atomless. □

31. Constructing refinements

$M_2(\omega)$ is the monoid of 2×2 matrices with entries in ω under matrix addition. For $i, j < 2$, ε_{ij} is the 2×2 matrix with 1 in the (i, j)-position and 0 elsewhere. Define $\alpha_i = \sum_{j<2} \varepsilon_{ij}$ for $i < 2$ and $\beta_j = \sum_{i<2} \varepsilon_{ij}$ for $j < 2$. Define $\varphi : M_2(\omega) \to \mathbb{Z}$ by $\varphi(\sum_{i,j<2} r_{ij}\varepsilon_{ij}) = r_{00} + r_{11} - r_{01} - r_{10}$.

Proposition 31.1. *φ is a semigroup homomorphism from $M_2(\omega)$ to $\mathbb{Z}$.*

Proof.

$$\begin{aligned}
\varphi\left(\sum_{i,j<2} r_{ij}\varepsilon_{ij} + \sum_{i,j<2} r'_{ij}\varepsilon_{ij}\right) &= \varphi\left(\sum_{i,j<2} (r_{ij} + r'_{ij})\varepsilon_{ij}\right) \\
&= r_{00} + r'_{00} + r_{11} + r'_{11} - r_{01} - r'_{01} - r_{10} - r'_{10} \\
&= r_{00} + r_{11} - r_{01} - r_{10} + r'_{00} + r'_{11} - r'_{01} - r'_{10} \\
&= \varphi\left(\sum_{i,j<2} r_{ij}\varepsilon_{ij}\right) + \varphi\left(\sum_{i,j<2} r'_{ij}\varepsilon_{ij}\right).
\end{aligned}$$

□

Proposition 31.2. $\forall i < 2[\varphi(\alpha_i) = \varphi(\beta_i) = 0]$. □

Proposition 31.3.

$$\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

□

A 2×2 matrix $\gamma \in M_2(\omega)$ is *counter-diagonal* iff $\gamma_{00} = \gamma_{11} = 0$.

Proposition 31.4. *Every $\gamma \in M_2(\omega)$ can be represented in the form*

$$\gamma = \delta + \sum_{i<2} m_i\alpha_i + \sum_{j<2} n_j\beta_j \tag{31.4(1)}$$

where δ is diagonal or counter-diagonal, $\forall i < 2[m_i, n_i \in \omega]$, and $n_0n_1 = 0$.

Proof. Suppose that $\gamma \in M_2(\omega)$. We prove the existence of a decomposition 31.4(1) by induction on $p = \sum_{i,j<2} \gamma_{ij}$. Note that if γ is diagonal or counter-diagonal then we can take each m_i and n_i equal to 0. If $p = 0$, then γ is diagonal. Assume the result for values $\leq p$, and now assume that $\sum_{i,j<2} \gamma_{ij} = p + 1$. We may assume that γ is not diagonal or counter-diagonal. Thus

$$(\gamma_{01} \neq 0 \text{ or } \gamma_{10} \neq 0) \text{ and } (\gamma_{00} \neq 0 \text{ or } \gamma_{11} \neq 0).$$

This gives four cases.

Case 1. $\gamma_{01} \neq 0 \neq \gamma_{00}$.

Subcase 1.1. $\gamma_{01} \leq \gamma_{00}$. Let $\gamma' = \gamma - \gamma_{01}\alpha_0$. Then $\sum_{i,j<2} \gamma'_{ij} \leq p - 1$, so γ' can be written in the form 31.4(1), and hence so can $\gamma = \gamma' + \gamma_{01}\alpha_0$.

Subcase 1.2. $\gamma_{00} < \gamma_{01}$. Let $\gamma' = \gamma - \gamma_{00}\alpha_0$; 31.4(1) follows.

Case 2. $\gamma_{01} \neq 0 \neq \gamma_{11}$.

Subcase 2.1. $\gamma_{01} \leq \gamma_{11}$. Let $\gamma' = \gamma = \gamma_{01}\beta_1$; 31.4(1) follows.

Subcase 2.2. $\gamma_{11} < \gamma_{01}$. Similarly.

Case 3. $\gamma_{10} \neq 0 \neq \gamma_{00}$.

Subcase 3.1. $\gamma_{10} \leq \gamma_{00}$. Let $\gamma' = \gamma - \gamma_{10}\beta_0$; 31.4(1) follows.

Subcase 3.2. $\gamma_{00} < \gamma_{10}$. Similarly.

Case 4. $\gamma_{10} \neq 0 \neq \gamma_{11}$.

Subcase 4.1. $\gamma_{10} \leq \gamma_{11}$. Let $\gamma' = \gamma - \gamma_{10}\alpha_1$; 31.4(1) follows.

Subcase 4.2. $\gamma_{11} < \gamma_{10}$. Similarly.

Thus we have the representation 31.4(1). To see that we can add the condition $n_0n_1 = 0$, we take two cases. Recall that $\alpha_0 + \alpha_1 = \beta_0 + \beta_1$.

Case 1. $n_0 \leq n_1$. Then

$$\begin{aligned}
\delta + \sum_{i<2} m_i\alpha_i + \sum_{j<2} n_j\beta_j &= \delta + \sum_{i<2} m_i\alpha_i + n_0(\beta_0 + \beta_1) + (n_1 - n_0)\beta_1 \\
&= \delta + \sum_{i<2} m_i\alpha_i + n_0(\alpha_0 + \alpha_1) + (n_1 - n_0)\beta_1 \\
&= \delta + (m_0 + n_0)\alpha_0 + (m_1 + n_0)\alpha_1 + (n_1 - n_0)\beta_1.
\end{aligned}$$

Case 2. $n_1 < n_0$. Similarly. □

A representation of γ as in Proposition 31.4 is called a *reduced representation* of γ.

Proposition 31.5. *In a reduced representation of a matrix $\gamma \in M_2(\omega)$, $\varphi(\gamma) = \varphi(\delta)$.*

Proof. In fact, for each $i < 2$ we have

$$m_i\alpha_i = \sum_{j<2} m_i\varepsilon_{ij} = \begin{pmatrix} m_i & m_i \\ 0 & 0 \end{pmatrix},$$

so $\varphi(m_i\alpha_i) = 0$. Similarly $\varphi(m_i\beta_i) = 0$. Hence the proposition holds. □

Proposition 31.6. *In a reduced representation of a matrix $\gamma \in M_2(\omega)$, $\delta = 0$ iff $\varphi(\gamma) = 0$.*

Proof. Let $\gamma \in M_2(\omega)$, and consider a reduced representation of γ as in Proposition 31.3. If $\delta = 0$, then $\varphi(\gamma) = \varphi(\delta) = 0$ by Proposition 31.4. Assume that $\varphi(\gamma) = 0$. Then $\varphi(\delta) = \varphi(\delta) = 0$ by Proposition 31.4. If δ is diagonal, then $\delta_{00} + \delta_{11} = 0$, hence $\delta = 0$. If δ is counter-diagonal, then $\delta_{01} + \delta_{10} = 0$, hence $\delta = 0$. □

Proposition 31.7. *If γ has the reduced representation 34.1(1) with $\delta = 0$, then in any reduced representation*

$$\gamma = \delta' + \sum_{i<2} m'_i\alpha_i + \sum_{j<2} n'_j\beta_j$$

we must have $\delta' = 0$, $\forall i < 2[m_i = m'_i]$, and $\forall j < 2[n_j = n'_j]$.

Proof. By Proposition 31.5 we have $\varphi(\gamma) = \varphi(\delta') = \varphi(\delta) = 0$. Clearly $\varphi(A) \not\sim 0$ if $A \neq 0$ is diagonal or counter-diagonal. So $\delta' = 0$. So assume that we have reduced representations

$$\gamma = \sum_{i<2} m_i\alpha_i + \sum_{j<2} n_j\beta_j = \sum_{i<2} m'_i\alpha_i + \sum_{j<2} n'_j\beta_j.$$

Then

$$m_0\alpha_0 = \begin{pmatrix} m_0 & m_0 \\ 0 & 0 \end{pmatrix}; \quad m_1\alpha_1 = \begin{pmatrix} 0 & 0 \\ m_1 & m_1 \end{pmatrix};$$
$$n_0\beta_0 = \begin{pmatrix} n_0 & 0 \\ n_0 & 0 \end{pmatrix}; \quad n_1\beta_1 = \begin{pmatrix} 0 & n_1 \\ 0 & n_1 \end{pmatrix}.$$

Say wlog $n_0 = 0$. Then

$$\sum_{i<2} m_i\alpha_i + \sum_{j<2} n_j\beta_j = \begin{pmatrix} m_0 & m_0 + n_1 \\ m_1 & m_1 + n_1 \end{pmatrix}.$$

Case 1. $n'_0 = 0$. Then also

$$\sum_{i<2} m'_i\alpha_i + \sum_{j<2} n'_j\beta_j = \begin{pmatrix} m'_0 & m'_0 + n'_1 \\ m'_1 & m'_1 + n'_1 \end{pmatrix}.$$

It follows that $m_0 = m'_0$, $m_1 = m'_1$, $n_0 = n'_0$, and $n_1 = n'_1$.

Case 2. $n'_1 = 0$. Then

$$\sum_{i<2} m'_i\alpha_i + \sum_{j<2} n'_j\beta_j = \begin{pmatrix} m'_0 + n'_0 & m'_0 \\ m'_1 + n'_0 & m'_1 \end{pmatrix}$$

This gives the equations

$$\begin{aligned} m_0 &= m'_0 + n'_0 \\ m_1 &= m'_1 + n'_0 \\ m_0 + n_1 &= m'_0 \\ m_1 + n_1 &= m'_1 \end{aligned}$$

It follows that $n'_0 = n_1 = 0$, $m_0 = m'_0$, and $m_1 = m'_1$. So $n_0 = n'_0$ and $n_1 = n'_1$. □

Proposition 31.8. *Suppose that M is an m-monoid, $e_0, e_1, f_0, f_1 \in M\backslash\{0\}$, and $e_0+e_1 = f_0+f_1$. Then there is a distinguished extension N of M such that N contains distinguished elements g_{ij} for $i, j < 2$ such that:*

(i) $\forall i < 2[e_i = \sum_{j<2} g_{ij}]$.

(ii) $\forall j < 2[f_j = \sum_{i<2} g_{ij}]$.

(iii) N is generated by $M \cup \{g_{ij} : i, j < 2\}$.

(iv) $\forall g, g' \in M \forall r \in {}^{2\times 2}\omega[g' = g + \sum_{i,j<2} r_{ij}g_{ij} \to r_{00} + r_{11} - r_{01} - r_{10} = 0$ *and* $\exists m_i, n_i \in \omega$ *for* $i < 2[\forall i, j < 2[r_{ij} = m_i + n_j]$ *and* $g' = g + \sum_{i<2} m_i e_i + \sum_{j<2} n_j f_j]]]$.

(v) If $\Lambda : M \to M'$ is a morphism, $\forall i, j < 2[h_{ij} \in M']$, $\forall i < 2[\Lambda(e_i) = \sum_{j<2} h_{ij}]$, and $\forall j < 2[\Lambda(f_j) = \sum_{i<2} h_{ij}]$, then there is an extension Λ' of Λ to a morphism of N to M' such that $\forall i, j < 2[\Lambda'(g_{ij}) = h_{ij}]$.

Proof. Let $P = M \times M_2(\omega)$. We define $(g, \gamma) \sim (g', \gamma')$ iff there exist $\delta \in M_2(\omega)$ and $m_i, n_i, m'_i, n'_i \in \omega$ for $i < 2$ such that the following conditions hold:

31.8(1) $$\gamma = \delta + \sum_{i<2} m_i\alpha_i + \sum_{j<2} n_j\beta_j, \quad \gamma' = \delta + \sum_{i<2} m'_i\alpha_i + \sum_{j<2} n'_j\beta_j, \quad \text{and}$$

31.8(2) $$g + \sum_{i<2} m_i e_i + \sum_{j<2} n_i f_i = g' + \sum_{i<2} m'_i e_i + \sum_{j<2} n'_i f_i$$

We may assume that the representations in 31.8(1) are reduced. In fact, let

$$\delta = \delta' + \sum_{i<2} m''_i\alpha''_i + \sum_{j<2} n''_j\beta''_j$$

be a reduced representation of δ. Then the first equation of 31.8(1) becomes

$$\gamma = \delta' + \sum_{i<2}(m_i + m''_i)\alpha_i + \sum_{j<2}(n_j + n''_j)\beta_j,$$

and the argument at the end of the proof of Proposition 31.3 can be used to put this in reduced form, with δ' retained. Namely, if $n_0 + n_0'' \le n_1 + n_1''$, then

$$\begin{aligned}
&\sum_{i<2}(m_i + m_i'')\alpha_i + \sum_{j<2}(n_j + n_j'')\beta_j \\
&= \sum_{i<2}(m_i + m_i'')\alpha_i + \sum_{j<2}(n_j + n_j'')\beta_j + (n_0 + n_0'')\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - (n_0 + n_0'')\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \sum_{i<2}(m_i + m_i'')\alpha_i + \sum_{j<2}(n_j + n_j'')\beta_j + (n_0 + n_0'')(\alpha_0 + \alpha_1) - (n_0 + n_0'')(\beta_0 + \beta_1) \\
&= (m_0 + m_0'' + n_0 + n_0'')\alpha_0 + (m_1 + m_1'' + n_0 + n_0'')\alpha_1 + (n_1 + n_1'' - n_0 - n_0'')\beta_1,
\end{aligned}$$

while if $n_1 + n_1'' < n_0 + n_0''$ then

$$\begin{aligned}
&\sum_{i<2}(m_i + m_i'')\alpha_i + \sum_{j<2}(n_j + n_j'')\beta_j \\
&= (m_0 + m_0'' + n_0 + n_0'')\alpha_0 + (m_1 + m_1'' + n_1 + n_1'')\alpha_1 + (n_0 + n_0'' - n_1 - n_1'')\beta_0.
\end{aligned}$$

The second equation in 31.8(1) is treated similarly.

The condition 31.8(2) can be treated similarly. Namely, if $n_0 + n_0'' \le n_1 + n_1''$, then

$$\begin{aligned}
&\sum_{i<2}(m_i + m_i'')e_i + \sum_{j<2}(n_j + n_j'')f_j \\
&= \sum_{i<2}(m_i + m_i'')e_i + (n_0 + n_0'')(f_0 + f_1) + (n_1 + n_1'' - n_0 - n_0'')f_1 \\
&= \sum_{i<2}(m_i + m_i'')e_i + (n_0 + n_0'')(e_0 + e_1) + (n_1 + n_1'' - n_0 - n_0'')f_1 \\
&= (m_0 + m_0'' + n_0 + n_0'')e_0 + (m_1 + m_1'' + n_0 + n_0'')e_1 + (n_1 + n_1'' - n_0 - n_0'')f_1,
\end{aligned}$$

while if $n_1 + n_1'' < n_0 + n_0''$ then

$$\begin{aligned}
&\sum_{i<2}(m_i + m_i'')e_i + \sum_{j<2}(n_j + n_j'')f_j \\
&= (m_0 + m_0'' + n_0 + n_0'')e_0 + (m_1 + m_1'' + n_1 + n_1'')e_1 + (n_0 + n_0'' - n_1 - n_1'')f_0.
\end{aligned}$$

Now

$$\begin{aligned}
&g + \sum_{i<2}(m_i + m_i'')e_i + \sum_{j<2}(n_j + n_j'')f_j \\
&= g' + \sum_{i<2}(m_i' + m_i'')e_i + \sum_{j<2}(n_j' + n_j'')f_j.
\end{aligned}$$

So 38.1(2) can be changed corresponding to the change of 38.1(1).

31.8(3) If $(g,\gamma) \sim (g',\gamma')$, then $\varphi(\gamma) = \varphi(\gamma')$.

This follows from Proposition 31.5.

Now for any $(g,\gamma) \in M \times M_2(\omega)$ it is clear that $(g,\gamma) \sim (g,\gamma)$. So $\sim$ is reflexive on $M \times M_2(\omega)$. Clearly $\sim$ is symmetric. Next, let $(g,\gamma) \sim (g',\gamma')$ and $(g'',\gamma'') \sim (g''',\gamma''')$. Say

$$\gamma = \delta + \sum_{i<2} m_i\alpha_i + \sum_{j<2} n_j\beta_j, \quad \gamma' = \delta + \sum_{i<2} m'_i\alpha_i + \sum_{j<2} n'_j\beta_j,$$

$$g + \sum_{i<2} m_i e_i + \sum_{j<2} n_i f_i = g' + \sum_{i<2} m'_i e_i + \sum_{j<2} n'_i f_i,$$

$$\gamma'' = \delta'' + \sum_{i<2} m''_i\alpha_i + \sum_{j<2} n''_j\beta_j, \quad \gamma''' = \delta'' + \sum_{i<2} m'''_i\alpha_i + \sum_{j<2} n'''_j\beta_j,$$

$$g'' + \sum_{i<2} m''_i e_i + \sum_{j<2} n''_i f_i = g''' + \sum_{i<2} m'''_i e_i + \sum_{j<2} n'''_i f_i.$$

Then

$$\gamma + \gamma'' = \delta + \delta'' + \sum_{i<2}(m_i + m''_i)\alpha_i + \sum_{j<2}(n_j + n''_j)\beta_j;$$
$$\gamma' + \gamma''' = \delta + \delta'' + \sum_{i<2}(m'_i + m'''_i)\alpha_i + \sum_{j<2}(n'_j + n'''_j)\beta_j;$$

$$\begin{aligned} g + g'' &+ \sum_{i<2}(m_i + m''_i)e_i + \sum_{j<2}(n_i + n''_i)f_i \\ &= g' + g''' + \sum_{i<2}(m'_i + m'''_i)e_i + \sum_{j<2}(n'_i + n'''_i)f_i \end{aligned}$$

Thus $(g + g'', \gamma + \gamma'') \sim (g' + g''', \gamma' + \gamma''')$.

It follows that the transitive closure $\approx$ of $\sim$ is a congruence relation on P.

Next we claim

31.8(4) If $(g,0) \approx (g',\theta)$, then θ has the form $\sum_{i<2} p_i\alpha_i + \sum_{j<2} q_i\beta_i$, and $g = g' + \sum_{i<2} p_i e_i + \sum_{j<2} q_i f_i$.

For, say $(g,0) = (g_0,\gamma_0) \sim (g_1,\gamma_1) \sim \cdots \sim (g_s,\gamma_s) = (g',\theta)$. By (171) we have $\varphi(\gamma_i) = 0$ for all i. Hence in the reduced representations of the γ_i, all the δ-s are 0 by Proposition 31.6. So we get

$$\gamma_k = \sum_{i<2} m_i^k\alpha_i + \sum_{j<2} n_j^k\beta_j \quad \text{and}$$

$$g_k + \sum_{i<2} m_i^k e_i + \sum_{j<2} n_j^k f_j = g_{k+1} + \sum_{i<2} m_i^{k+1} e_i + \sum_{j<2} n_j^{k+1} f_j.$$

By uniqueness (Proposition 31.7), $m_i^0 = 0 = n_i^0$ for all $i < 2$. Hence the desired conclusion holds.

We let $N = P/\approx$, and let $\Omega : P \to N$ be the natural map. Define $\Psi(g) = \Omega(g, 0)$ for any $g \in M$. Clearly Ψ is a morphism from M to N. We claim that Ψ is one-one. For, suppose that $\Psi(g) = \Psi(g')$. Thus $(g, 0) \approx (g', 0)$, so by 31.8(4) we get $g = g'$. Thus Ψ is one-one.

31.8(5) If g is distinguished in M, then $\Psi(g)$ is distinguished in N.

For suppose that g is distinguished in M, and $\Psi(g) + [(h, \gamma)]_\approx = \Psi(g)$. Then $(g, 0) \approx (g + h, \gamma) \approx$. Hence by 31.8(4), γ has the form $\sum_{i<2} m_i\alpha_i + \sum_{j<2} n_i\beta_i$, and $g = g + h + \sum_{i<2} m_i e_i + \sum_{j<2} n_i f_i$. Since g is distinguished, it follows that $h = 0$ and $\sum_{i<2} m_i e_i + \sum_{j<2} n_i f_i = 0$. So $(h, \gamma) \sim (0, 0)$, as desired.

31.8(6) $\forall i < 2[(e_i, 0) \sim (0, \alpha_i)$ and $(f_i, 0) \sim (0, \beta_i)]$.

In fact, for $(e_0, 0) \sim (0, \alpha_0)$ we have

$$0 = 0 + \sum_{i<2} 0\alpha_i + \sum_{i<2} 0\beta_i,\ \ \alpha_0 = 0 + \alpha_0 + 0\alpha_1 + \sum_{j<2} 0\beta_j \quad \text{and}$$

$$e_0 + \sum_{i<2} 0\alpha_i + \sum_{j<2} 0\beta_j = 0 + e_0 + 0e_1 + \sum_{j<2} 0\beta_j.$$

Similarly for the other three cases.

Now we define for $i, j < 2$, $g_{ij} = [(0, \varepsilon_{ij})]_\approx$. Then $\forall i < 2[\sum_{j<2} g_{ij} = [(0, \alpha_i)]_\approx = [(e_i, 0)]_\approx = \Psi(e_i)]$, using 31.8(6). Similarly, $\forall j < 2[\sum_{i<2} g_{ij} = \Psi(f_j)]$. This gives (i) and (ii) of the proposition.

31.8(7) N is generated by $\Psi[M] \cup \{g_{ij} : i, j < 2\}$.

In fact, if $(g, \gamma) \in P$, then $[(g, \gamma)]_\approx = \Psi(g) + \sum_{i,j<2} \gamma_{ij} g_{ij}$.

31.8(8) $\forall i, j < 2[g_{ij}$ is distinguished$]$.

For, suppose that $i, j < 2$ and $g_{ij} + [(h, \gamma)]_\approx = g_{ij}$. Thus $(h, \gamma + \varepsilon_{ij}) \approx (0, \varepsilon_{ij})$; say

$$(0, \varepsilon_{ij}) = (g_0, \gamma_0) \sim \cdots \sim (g_s, \gamma_s) = (h, \gamma + \varepsilon_{ij}).$$

Thus

$$\varepsilon_{ij} = \varepsilon_{ij} + \sum_{i<2} 0\alpha_i + \sum_{j<2} 0\beta_j, \quad \gamma_1 = \varepsilon_{ij} + \sum_{i<2} m_i^1\alpha_i + \sum_{j<2} n_i^1\beta_j,$$
$$0 = g^1 + \sum_{i<2} m_i^1 e_i + \sum_{j<2} n_i^1 f_j.$$

It follows that $g^1 = 0$ and $m_i^1 = n_i^1$ for all $i < s$, and $h = \gamma = 0$. So 31.8(8) holds.

Now we prove (iv). Suppose that $g, g' \in M$, $r \in {}^{2\times 2}\omega$, and $\Psi(g') = \Psi(g) + \sum_{i,j<2} r_{ij} g_{ij}$. Thus $(g', 0) \sim (g_1, \gamma_1) \sim \cdots \sim (g, \sum_{i,j<2} r_{ij}\varepsilon_{ij})$. By 31.8(4) we have $0 = \varphi(\gamma_1) = \cdots = \varphi(r)$. So $r_{00} + r_{11} - r_{01} - r_{10} = 0$.

Case 1. $r_{00} \leq r_{01}$. Let $n_0 = 0$, $n_1 = r_{01} - r_{00}$, $m_0 = r_{00}$, $m_1 = r_{10}$

Case 2. $r_{01} < r_{00}$. Let $n_0 = r_{00} - r_{01}$. $n_1 = 0$, $m_0 = r_{01}$, $m_1 = r_{11}$.

Then in either case we have $\forall i, j < 2[r_{ij} = m_i + n_j]$. Now

$$\begin{aligned}\sum_{i<2} m_i \Psi(e_i) + \sum_{j<2} n_j \Psi(f_j) &= m_0 \Psi(e_0) + m_1 \Psi(e_1) + n_0 \Psi(f_0) + n_1 \Psi(f_1) \\ &= m_0 g_{00} + m_0 g_{01} + m_1 g_{10} + m_1 g_{11} \\ &\quad + n_0 g_{00} + n_0 g_{10} + n_1 g_{01} + n_1 g_{11} \\ &= r_{00} g_{00} + r_{01} g_{01} + r_{10} g_{10} + r_{11} g_{11}\end{aligned}$$

This proves (iv).

Now for (v), suppose that $\Lambda : M \to M'$ is a morphism, $\forall i, j < 2[h_{ij} \in M']$, $\forall i < 2[\Lambda(e_i) = \sum_{j<2} h_{ij}]$, and $\forall j < 2[\Lambda(f_j) = \sum_{i<2} h_{ij}]$. Define $\Gamma : P \to M'$ by

$$\Gamma(g, r) = \Lambda(g) + \sum_{i,j<2} r_{ij} h_{ij}.$$

Clearly Γ preserves $+$. Now

$$\begin{aligned}\Gamma(0, \alpha_0) &= \Gamma\left(0, \sum_{j<2} \varepsilon_{0j}\right) = \Gamma\left(0, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = h_{00} + h_{01} = \Lambda(e_0); \\ \Gamma(0, \alpha_1) &= \Gamma\left(0, \sum_{j<2} \varepsilon_{1j}\right) = \Gamma\left(0, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right) = h_{10} + h_{11} = \Lambda(e_1); \\ \Gamma(0, \beta_0) &= \Gamma\left(0, \sum_{i<2} \varepsilon_{i0}\right) = \Gamma\left(0, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) = h_{00} + h_{10} = \Lambda(f_0); \\ \Gamma(0, \beta_1) &= \Gamma\left(0, \sum_{i<2} \varepsilon_{i1}\right) = \Gamma\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) = h_{01} + h_{11} = \Lambda(f_1).\end{aligned}$$

Suppose now that $(g, \gamma) \sim (g', \gamma')$ as in (169), (170). Then

$$\begin{aligned}\Gamma(g, \gamma) &= \Gamma(g, 0) + \gamma(0, \gamma) \\ &= \Lambda(g) + \Gamma(0, \delta) + \sum_{i<2} m_i \Gamma(0, \alpha_i) + \sum_{j<2} n_j \Gamma(0, \beta_j) \\ &= \Lambda(g) + \Gamma(0, \delta) + \sum_{i<2} m_i \Lambda(e_i) + \sum_{j<2} n_j \Lambda(f_j) \\ &= \Gamma(0, \delta) + \Lambda\left(g + \sum_{i<2} m_i e_i + \sum_{j<2} n_j f_j\right)\end{aligned}$$

$$
\begin{aligned}
&= \Gamma(0,\delta) + \Lambda\left(g' + \sum_{i<2} m'_i e_i + \sum_{j<2} n'_j f_j\right) \\
&= \Lambda(g') + \Gamma(0,\delta) + \sum_{i<2} m'_i \Lambda(e_i) + \sum_{j<2} n_j \Lambda(f_j) \\
&= \Lambda(g') + \Gamma(0,\delta) + \sum_{i<2} m'_i \Gamma(0,\alpha_i) + \sum_{j<2} n'_j \Gamma(0,\beta_j) \\
&= \Gamma(g',\gamma').
\end{aligned}
$$

Thus Γ induces Γ' defined by $\Gamma'([(g,\gamma)]_\approx) = \Gamma(g,\gamma)$. We have $\Gamma'(g_{ij}) = \Gamma'([(0.\varepsilon_{ij})]_\approx) = \Gamma(0,\varepsilon_{ij}) = h_{ij}$. □

32. Models

$\mathscr{R}$ is the set of all countable nonempty subsets α of ${}^{<<\omega}\mathscr{P}$ such that α has C.P. and $\forall m,n \in \omega \forall a,b \in \alpha[\mathrm{dmn}(a) = m \text{ and } \mathrm{dmn}(b) = n \to \sum_{i<m} a_i = \sum_{j<n} b_j]$.

Proposition 32.1. *$\mathscr{R}$ is an m-monoid which contains $\triangle\mathscr{P}$ as a submonoid.*

Proof. The sum of elements of $\mathscr{R}$ is defined as in the definition of $\triangle M$. If $\alpha,\beta \in \mathscr{R}$, $a,b \in \alpha+\beta$, $\mathrm{dmn}(a) = m$, $\mathrm{dmn}(b) = n$, then there exist $c,d \in \alpha$ and $e,f \in \beta$ such that $\mathrm{dmn}(c) = \mathrm{dmn}(e) = m$ and $a = c+e$, and $\mathrm{dmn}(d) = \mathrm{dmn}(f) = n$ and $b = d+f$. Then $\sum_{i<m} c_i = \sum_{i<n} d_i$ and $\sum_{i<m} e_i = \sum_{i<n} f_i$, so $\sum_{i<m} a_i = \sum_{i<m}(c_i+e_i) = \sum_{i<n}(d_i+f_i) = \sum_{i<n} b_i$. By the proof of Proposition 8.7, $\alpha+\beta$ has C.P. Hence $\alpha+\beta \in \mathscr{R}$. Clearly $\mathscr{R}$ contains $\triangle\mathscr{P}$ as a submonoid. □

Proposition 32.2. *If $\alpha,\beta \in \mathscr{R}$ and*

$$\forall m,n \in \omega\backslash\{0\}\forall a \in \alpha\forall b \in \beta[\mathrm{dmn}(a) = m \textit{ and } \mathrm{dmn}(b) = n \to \sum_{i<m} a_i = \sum_{j<n} b_j,$$

then $\alpha \cup \beta \in \mathscr{R}$. □

For any $a \in \mathscr{P}$ let

$$\hat{a} = \{b \in {}^{<\omega}\mathscr{P} : \exists l \in \mathrm{dmn}(b)[b_l = a \text{ and } \forall j \neq l[b_l = 0]]\}.$$

Proposition 32.3. *If $a \in \mathscr{P}$, then $\hat{a} \in \mathscr{R}$.*

Proof. For the collection property, suppose that $b \in \hat{a}$ and $b \prec c$. Say $b_l = a$ and $\forall s \neq l[b_s = 0]$. Also say b has domain m, c has domain n. $\lambda : m \to n$, and $\forall j < n[c_j = \sum\{b_i : \lambda(i) = j\}$. Say $\lambda(l) = k$. Then $c_k = a$ and $c_s = 0$ for all $s \neq k$. So $c \in \hat{a}$. Thus $\hat{a}$ has C.P. Clearly the other condition for $\hat{a} \in \mathscr{R}$ holds. □

Proposition 32.4. *$T(\hat{a}) = a$.* □

Proposition 32.5. *If $\alpha \in \mathscr{R}$, $b \in \alpha$, and $\mathrm{dmn}(b) = m$, then $\sum_{i<m} \hat{b_i} \subseteq \alpha$.*

Proof. Suppose that $c \in \sum_{i<m} \hat{b_i}$; say c has domain n. Then there are $d_i \in \hat{b_i}$ for each $i < m$ such that the d_i have common domain n and $\forall j < n[c_j = \sum_{i<m} d_i(j)]$. For each $i < m$ let $\lambda_i < n$ be such that $d_i(\lambda_i) = b_i$ and $d_i(s) = 0$ for all $s \neq \lambda_i$. Then for all $j < n$, $c_j = \sum\{b_i : \lambda_i = j\}$. So $b \prec c$, hence $c \in \alpha$. □

Let M be a countable m-monoid. A *submodel* of M is a mapping $\Psi : M \to \mathscr{R}$ such that

(M1) $\forall e \in M[\Psi(e) = 0 \text{ iff } e = 0]$.

(M2) $\forall e, f \in M[\Psi(e) + \Psi(f) \subseteq \Psi(e + f)]$.

A *model* of M is an m-monoid morphism Ψ of M to $\mathscr{P}^1$ such that

(M3) $\forall e \in M_s \forall f \in M[e \neq f \to \Psi(e) \neq \Psi(f)]$.

Proposition 32.6. *If Ψ is a model of M, then $\forall e \in M[\Psi(e) = 0$ iff $e = 0]$.*

Proof. Suppose that Ψ is a model of M and $e \in M$. Clearly $e = 0$ implies that $\Psi(e) = 0$. If $e \neq 0$ then, since 0 is distinguished, $\forall f \in M[0 \neq f \to 0 = \Psi(0) \neq \Psi(f)]$. □

Proposition 32.7. *Let $\Psi : M \to \mathscr{R}$ be a submodel of an m-monoid M. For any $e \in M$ let $\Psi'(e) = T(\Psi(e))$. Then Ψ' is a morphism from M to $\mathscr{P}$.*

Proof. If $e = f = 0$, clearly $\Psi'(e + f) = \Psi'(e) + \Psi'(f)$. If $e \neq 0$ or $f \neq 0$, then by (M1) and (M2), $\Psi(e) + \Psi(f)$ is a nonempty subset of $\Psi(e + f)$. Hence $T(\Psi(e + f)) = T(\Psi(e) + \Psi(f)) = T(\Psi(e)) + T(\Psi(f))$. □

Proposition 32.8. *Let E be a generating set for a countable m-monoid M. Suppose that $\Psi : E \to \mathscr{R}$ is a mapping such that*

(i) $\forall e \in E[\Psi(e) = 0$ iff $e = 0]$.

(ii) $\forall m, n \in \omega \forall e \in {}^mE \forall f \in {}^nE[\sum_{i<m} e_i = \sum_{j<n} f_j \to \sum_{i<m} T(\Psi(e_i)) = \sum_{j<n} T(\Psi(f_j))]$.

For each $g \in M$ let

$$\Psi^+(g) = \bigcup \left\{ \sum_{i<m} \Psi(e_i) : m \in \omega, e \in {}^mE, \sum_{i<m} e_i = g \right\}.$$

Then Ψ^+ is a submodel of M, and $\forall e \in M[\Psi(e) \subseteq \Psi^+(e)]$.

Proof. Assume the hypotheses. Suppose that $g \in M$; we want to show that $\Psi^+(g) \in \mathscr{R}$. Clearly $\Psi^+(g)$ is countable and nonempty. Clearly also $\sum_{i<m} \Psi(e_i) \in \mathscr{R}$ for any $m \in \omega$ and $e \in {}^mE$, so $\Psi^+(g) \subseteq {}^{<<\omega}\mathscr{P}$. C.P. is closed under unions, so $\Psi^+(g)$ has C.P. Now suppose that $m, n \in \omega$, $a, b \in \Psi^+(g)$, $\mathrm{dmn}(a) = m$, and $\mathrm{dmn}(b) = n$. Say $a \in \sum_{i<p} \Psi(e_i)$ with $e \in {}^pE$ and $\sum_{i<p} e_i = g$; and $b \in \sum_{i<q} \Psi(e'_i)$ with $e' \in {}^qE$ and $\sum_{i<q} e'_i = g$. By (ii), $\sum_{i<p} T(\Psi(e_i)) = \sum_{i<q} T(\Psi(e'_i))$. By Proposition 32.1, $\sum_{i<p} \Psi(e_i), \sum_{i<q} \Psi(e'_i) \in \mathscr{R}$, and by Proposition 32.2, $\sum_{i<p} \Psi(e_i) \cup \sum_{i<q} \Psi(e'_i) \in \mathscr{R}$. Hence $\sum_{i<m} a_i = \sum_{i<n} b_i$. So $\Psi^+(g) \in \mathscr{R}$.

Clearly $\Psi^+(0) = 0$. If $g \neq 0$, choose $m \in \omega$ and $e \in {}^mE$ so that $\sum_{i<m} e_i = g$. Say $i < m$ and $e_i \neq 0$. Then by (i), $\Psi(e_i) \neq 0$. So $\sum_{j<m} \Psi(e_j) \neq 0$. Thus $\Psi^+(g) \neq 0$. Thus (M1) holds for Ψ^+.

Now suppose that $g, h \in M$ and $x \in \Psi^+(g) + \Psi^+(h)$. Then there exist y, z with equal domains such that $x = y + z$, $y \in \Psi^+(g)$, and $z \in \Psi^+(h)$. Say $y \in \sum_{i<m} \Psi(e_i)$ with $m \in \omega$, $e \in {}^mE$, and $\sum_{i<m} e_i = g$; and $z \in \sum_{i<n} \Psi(e'_i)$ with $n \in \omega$, $e' \in {}^nE$, and $\sum_{i<n} e'_i = h$. Clearly then $x \in \Psi^+(g + h)$.

Clearly $\forall e \in M[\Psi(e) \subseteq \Psi^+(e)]$. □

The mapping Ψ^+ of Proposition 32.8 is called the *closure* of Ψ.

Proposition 32.9. *Suppose that Ψ is a submodel of M, and $\Theta : M \to \mathscr{R}$ satisfies $\forall e \in M[\Psi(e) \subseteq \Theta(e)]$. Then Θ satisfies (i) and (ii) of Proposition 32.8.*

Proof. (i) is clear. (ii) holds since for any $e \in M$ $\Psi(e) \subseteq \Theta(e)$ and so $T(\Psi(e)) = T(\Theta(e))$.

33. Expansions of submodels

For any $\alpha \subseteq {}^{<<\omega}\mathscr{P}$ let

$$\mathrm{Dom}(\alpha) = \{a \in \mathscr{P} : a \text{ is an entry in some } x \in \alpha\}.$$

If M is a collection of subsets of ${}^{<<\omega}\mathscr{P}$, then

$$\mathrm{Dom}(M) = \bigcup_{\alpha \in M} \mathrm{Dom}(\alpha).$$

Proposition 33.1. *If M is a submonoid of $\mathscr{R}$, then* $\mathrm{Dom}(M)$ *is a submonoid of $\mathscr{P}$.*

Proof. Suppose that M is a submonoid of $\mathscr{R}$ and $a, b \in \mathrm{Dom}(M)$. Say $a \in \mathrm{Dom}(\alpha)$ and $b \in \mathrm{Dom}(\beta)$ with $\alpha, \beta \in M$. Say $a = x_i$ with $x \in \alpha$ and $b = y_j$ with $y \in \beta$. Say x has domain m. Define $\lambda : m \to 2$ by $\lambda(k) = 0$ if $k \neq i$ and $\lambda(i) = 1$. Let $c_0 = \sum\{x_k : k < m, k \neq i\}$ and $c_1 = i$. Then $\forall j < 2[c_j = \sum\{x_k : \lambda(k) = j\}$. So $\langle \sum\{x_k : k < m, k \neq i\}, x_i\rangle \in \alpha$. Similarly, $\langle \sum\{y_k : k < m, k \neq j\}, y_j\rangle \in \beta$. It follows that $x_i + y_j$ is an entry in a member of $\alpha + \beta$. □

Let M and N be m-monoids, with M a submonoid of N. If Ψ is a submodel of M and Θ is a submodel of N, then Θ is an *expansion* of Ψ iff

(E) $\forall e \in M[\Theta(e) \cap {}^{<<\omega}(\mathrm{Dom}(\Psi[M]) = \Psi(e)]$.

Proposition 33.2. *If Θ is an expansion of Ψ, then $\forall e \in M[\Psi(e) \subseteq \Theta(e)$ and $T(\Psi(e)) = T(\Theta(e))]$.*

Proof. Assume that Θ is an expansion of Ψ. Then $\forall e \in M[\Psi(e) \subseteq \Theta(e) \subset \mathscr{R}]$, so the proposition follows from the definition of $\mathscr{R}$. □

Proposition 33.3. *If Θ is an expansion of Ψ, then* $\mathrm{Dom}(\Psi[M]) \subseteq \mathrm{Dom}(\Theta[M])$.

Proof.

$$\begin{aligned}\mathrm{Dom}(\Psi[M]) &= \bigcup_{\alpha\in\Psi[M]} \mathrm{Dom}(\alpha) = \bigcup_{e\in M} \mathrm{Dom}(\Psi(e)) \\ &\subseteq \bigcup_{e\in M} \mathrm{Dom}(\Theta(e)) = \bigcup_{\alpha\in\Theta[M]} \mathrm{Dom}(\alpha) = \mathrm{Dom}(\Theta[M]).\end{aligned}$$ □

Proposition 33.4. *If Θ is an expansion of Ψ, then $\forall e, f \in M[\Psi(e) \neq \Psi(f) \to \Theta(e) \neq \Theta(f)]$.*

Proof. Suppose that Θ is an expansion of Ψ, $e, f \in M$, and $\Theta(e) = \Theta(f)$. Then $\Psi(e) = \Theta(e) \cap {<<^\omega}\Psi[M] = \Theta(f) \cap {<<^\omega}\Psi[M] = \Psi(f)$. □

Proposition 33.5. *Suppose that:*
(i) M_0, M_1, M_2 are m-monoids.
(ii) M_0 is a submonoid of M_1 and M_1 is a submonoid of M_2.
(iii) $\Theta_0, \Theta_1, \Theta_2$ are submodels of M_0, M_1, M_2 respectively.
(iv) Θ_1 is an expansion of Θ_0 and Θ_2 is an expansion of Θ_1.
Then M_0 is a submonoid of M_2, and Θ_2 is an expansion of Θ_0.

Proof. Clearly M_0 is a submonoid of M_2. For any $e \in M_0$ we have

$$\Theta_2(e) \cap {<<^\omega}(\mathrm{Dom}(\Theta_0[M_0]) \subseteq \Theta_2(e) \cap {<<^\omega}(\mathrm{Dom}(\Theta_1[M_1]) = \Theta_1(e).$$

Hence

$$\Theta_2(e) \cap {<<^\omega}(\mathrm{Dom}(\Theta_0[M_0]) \subseteq \Theta_1(e) \cap {<<^\omega}(\mathrm{Dom}(\Theta_0[M_0]) = \Theta_0(e).$$

On the other hand, clearly $\Theta_0(e) \subseteq \Theta_2(e) \cap {<<^\omega}(\mathrm{Dom}(\Theta_0[M_0])$. So Θ_2 is an expansion of Θ_0. □

Proposition 33.6. *Let $M_0 \subseteq M_1 \subseteq \cdots$ be an ω-chain of submonoids of an m-monoid M, with $\bigcup_{k\in\omega} M_k = M$. Suppose that each M_{k+1} is a distinguished extension of M_k. For each $k \in \omega$ let Ψ_k be a submodel of M_k, and suppose that $\forall k \in \omega[\Psi_{k+1}$ is an expansion of $\Psi_k]$. For each $k \in \omega$ and $e \in M_k$ let $\Theta_k(e) = \bigcup_{n\geq k} \Psi_n(e)$. Then the following hold:*
(i) $\forall k \in \omega[\Theta_k$ is a submodel of $M_k]$.
(ii) $\forall k \in \omega[\Theta_k$ is an expansion of $\Psi_k]$.
(iii) $\forall k, n \in \omega[k \leq n \to \Theta_k = \Theta_n \restriction M_k]$.
(iv) $\forall k \in \omega[\forall n \geq k[\Psi_n$ is a model$] \to \Theta_k$ is a model$]$.

Proof. Assume the hypotheses. (i): (M1) is clear. For (M2), suppose that $e, f \in M_k$ and $x \in \Theta_k(e) + \Theta_k(f)$. So there exist $y \in \Theta_k(e)$ and $z \in \Theta_k(f)$ such that $x = y + z$. Say $k \leq n$ and $y \in \Psi_n(e)$ and $z \in \Psi_n(f)$. So $x \in \Psi_n(e) + \Psi_n(f) \subseteq \Psi_n(e + f) \subseteq \Theta_k(e + f)$. This proves (M2).

For (ii), assume that $k \in \omega$. Suppose that $e \in M_k$. Then

$$\Theta_k(e) \cap {<<^\omega}(\mathrm{Dom}(\Psi_k[M_k]) = \bigcup_{n\geq k} (\Psi_n(e) \cap {<<^\omega}(\mathrm{Dom}(\Psi_k[M_k]))) = \Psi_k(e).$$

For (iii), suppose that $n, k \in \omega$ and $k \leq n$. Then for any $e \in M_k$, if $k \leq m \leq n$ then $\Psi_m(e) \subseteq \Psi_n(e)$, and so $\Theta_n(e) = \bigcup_{m \geq n} \Psi_m(e) = \bigcup_{m \geq k} \Psi_m(e) = \Theta_k(e)$. For (iv), assume that $k \in \omega$ and $\forall n \geq k[\Psi_n$ is a model$]$. Suppose that $e \in M_{ks}$ and $f \in M_k$, with $e \neq f$. Say $\Theta_k(e) = \Psi_m(e)$ and $\Theta_k(f) = \Psi_m(f)$. Then $e \in M_{ms}$, so $\Psi_m(e) \neq \Psi_m(f)$. □

Proposition 33.7. *Suppose that the hypotheses of Proposition 33.6 hold. For each $e \in M$ let $\Theta_\omega(e) = \Theta_k(e)$, where k is minimum such that $e \in M_k$. Then*

(i) Θ_ω is a submodel of M.

(ii) $\forall k < \omega[\Theta_\omega$ is an expansion of $\Psi_k]$.

Proof. Assume the hypotheses. For (i), clearly $\Theta_\omega(e) = 0$ iff $e = 0$. Now take $e, f \in M$. Say k minimum such that $e \in M_k$, and l is minimum such that $f \in M_l$. Say $k \leq l$. Take any $u \in \Theta_\omega(e) + \Theta_\omega(f)$. Then $\Theta_\omega(e) = \Theta_k(e)$ and $\Theta_\omega(f) = \Theta_l(f)$. Say $u = x + y$ with $x \in \Theta_k(e)$ and $y \in \Theta_l(f)$. Then $x \in \Theta_l(e)$, so $u \in \Theta_l(e) + \Theta_l(f) \subseteq \Theta_l(e+f)$. Say s is minimum such that $e+f \in M_s$. So $s \leq l$. $\Theta_s(e+f) = \bigcup_{m \geq s} \Psi_m(e+f)$; if $s \leq m \leq l$ then $\Psi_m(e+f) \leq \Psi_l(e+f)$. so $\Theta_s(e+f) = \Theta_l(e+f)$. So $\Theta_\omega(e+f) = \Theta_l(e+f)$. Hence (i) holds.

For (ii), suppose that $k \in \omega$ and $e \in M_k$. Let s be minimum such that $e \in M_s$. Then

$$\begin{aligned}\Theta_\omega(e) \cap {}^{<<\omega}(\mathrm{Dom}(\Psi_k[M_k])) &= \Theta_s(e) \cap {}^{<<\omega}(\mathrm{Dom}(\Psi_k[M_k])) \\ &= \Theta_k(e) \cap {}^{<<\omega}(\mathrm{Dom}(\Psi_k[M_k])) \\ &= \Psi_k(e).\end{aligned}$$ □

The submodel Θ_ω is called the *limit* of $\langle \Psi_k : k \in \omega \rangle$.

Proposition 33.8. *Under the notation of Proposition 33.7, if all Ψ_k are models, then Θ_ω is a model.*

Proof. Suppose that $e \in M_s$ and $f \in M$. Say $e \in M_k$ and $f \in M_k$. Clearly $e \in M_{ks}$. Hence $\Psi_k(e) \neq \Psi_k(f)$. By Proposition 33.7(ii), clearly $\Theta_\omega(e) \neq \Theta_\omega(f)$. □

34. Second reduction step

Proposition 34.1. *Let $M_0 \subseteq M_1 \subseteq \cdots$ be a chain of countable submonoids of an m-monoid M, and suppose that for each $k \in \omega$, Ψ_k is a model of M_k, with Ψ_{k+1} an expansion of Ψ_k. Also assume:*

(i) Each M_k is distinguished, and M_{k+1} is a distinguished extension of M_k.

(ii) $\forall k \in \omega \forall e_0, e_1, f_0, f_1 \in M_k[e_0 + e_1 = f_0 + f_1 \rightarrow \exists g \in {}^{2\times 2}M_{k+1}[\forall i < 2[e_i = \sum_{j<2} g_{ij}]$ and $\forall j < 2[f_j = \sum_{i<2} g_{ij}]]]$.

(iii) $\forall k \in \omega \forall e \in M_k \forall n \in \omega \forall \langle a_0, \ldots, a_{n-1} \rangle \in \Psi_k(e) \exists \langle e'_0, \ldots, e'_{n-1} \rangle \in M_{k+1}[e = \sum_{i<n} e'_i$ and $\forall i < n[T(\Psi_{k+1}(e'_i)) = a_i]]$.

(iv) $\Psi_0(M_0^*) \subseteq \mathscr{Q}^1$.

Then M_0^* is isomorphic to a subsemigroup of **NBA**.

Proof. We may assume that $M = \bigcup_{k \in \omega} M_k$. Hence M is a countable m-monoid. By Proposition 19.7 and (ii), M is an r-monoid.

We claim that M is atomless. For, suppose that $e \in M^*$. Say $e \in M_k$. Then $\Psi_k(e) \neq 0$. Now $\Psi_k(e) \in \mathscr{P}^1 = \triangle\mathscr{P}$, so by the splitting property there is an $(a_0, a_1) \in \Psi_k(e)$ with $a_0 \neq 0 \neq a_1$. Then (iii) shows that e is not an atom.

By Propositions 33.7 and 33.8, Θ_ω is a model of M and is an expansion of each Ψ_k. For each $e \in M$ let

$$\Lambda(e) = \left\{ \Theta_\omega \circ f : \sum_{i<n} f_i = e \right\}.$$

We are going to apply Proposition 29.1 to $\Lambda[M_0^*]$.

Let $N = \{\Lambda(e) : e \in M\}$. Note that $\Lambda = (\triangle\Theta_\omega) \circ \delta_M$. In fact, if $e \in M$, then

$$(\triangle\Theta_\omega)(\delta_M(e)) = (\triangle\Theta_\omega)\left(\left\{ b : \sum_{i<n} b_j = e \right\}\right) = \left\{ \Theta_\omega \circ f : \sum_{i<n} f_i = e \right\}.$$

Now by Proposition 19.12, δ_M is a morphism from M to $\triangle M$. Θ_ω is a morphism from M to $\mathscr{P}^1$. Since M is distinguished, the proof of Proposition 25.9 gives: $\triangle\Theta_\omega$ is a morphism from $\triangle M$ to $\triangle\mathscr{P}^1$. Hence Λ is a morphism from M to N, and N is a submonoid of $\triangle\mathscr{P}^1$. Since Λ maps M onto N, it follows that N has the refinement property. Now suppose that $A \in N$ and $(\alpha_0, \alpha_1) \in A$. Then there exist $g \in M$ such that $A = \Lambda(g)$, and $f_0, f_1 \in M$ with $f_0 + f_1 = g$, $\alpha_0 = \Omega_\omega(f_0)$, and $\alpha_1 = \Theta_\omega(f_1)$. Say $A_0 = \Lambda(f_0)$ and $A_1 = \Lambda(f_1)$. So $A = A_0 + A_1$. Moreover,

$$T(A_0) = T(\Lambda(f_0)) = \sum_{i<m} \Omega_\omega(g_i) = \Omega_\omega\left(\sum_{i<m} g_i\right) = \Omega_\omega(f_0) = \alpha_0,$$

and similarly $T(A_1) = \alpha_1$. This checks (ii) of Proposition 29.1.

34.1(1) $N \subseteq \mathscr{P}^2$.

In fact, if $e \in M$, then by the above, $\Lambda(e) \in \triangle\mathscr{P}^1$, so we just need to show that $\Lambda(e)$ has the new heirarchy property. Suppose that $m, n \in \omega\backslash\{0\}$, $\langle \alpha, \beta_1, \ldots, \beta_{m-1}\rangle \in \Lambda(e)$, and $\langle a_0, \ldots, a_{n-1}\rangle \in \alpha$. Then there are $f, g_1, \ldots, g_{m-1}$ such that $f + \sum_{i<m} g_i = e$, $\alpha = \Theta_\omega(f)$, and $\forall i < m[\beta_i = \Theta_\omega(g_i) = \beta_i]$. Since $\langle a_0, \ldots, a_{n-1}\rangle \in \alpha$, there is a $k < \omega$ such that $f \in M_k$ and $\langle a_0, \ldots, a_{n-1}\rangle \in \Psi_k(f)$. By (iii), there are $e_0, \ldots, e_{n-1} \in M_{k+1}$ such that $\sum_{i<n} e_i = f$ and $\forall i < n[T(\Psi_{k+1}(e_i)) = a_i]$. Hence $\forall i < n[T(\Omega_\omega(e_i)) = a_i]$. Now $\sum_{i<n} e_i + \sum_{i<m} g_i = e$, so $\langle \Theta_\omega(e_0), \ldots, \Theta_\omega(e_{n-1}), \Theta_\omega(g_0), \ldots, \Theta_\omega(g_{m-1})\rangle \in \Lambda(e)$. This checks the new heirarchy property; so 34.1(1) holds.

34.1(2) $\Lambda[M_0^*] \subseteq \mathscr{Q}^2$.

In fact, let $e \in M_0^*$. By (iv), $\Psi_0(e) \in \mathscr{Q}^1$, so $T(\Psi_0(e)) = [\omega^c]_{\mathrm{ae}}$. Also, $T(\Lambda(e)) = \Omega_\omega(e) \supseteq \Psi_0(e)$. Hencee $T^2(\Lambda(e)) = [\omega^c]_{ae}$. This proves (182).

Now by Proposition 29.1, $\Lambda[M_0^*]$ is isomorphic to a subsemigroup of **NBA**. Hence it suffices now to show that Λ is injective. Suppose that $e, f \in M$ and $\Lambda(e) = \Lambda(f)$. Since M is distinguished, we can write $e = \sum_{i<m} e_i$ with each $e_i \in M_s$. Then $\Theta_\omega \circ e \in \Lambda(e) = \Lambda(f)$, so there exist g_i so that $\sum_{i<m} g_i = f$ and $\Theta_\omega \circ e = \Theta_\omega \circ g$. Since Ω_ω is a model, $\forall i < m[g_i = e_i]$, so $e = f$. $\square$

35. Two expansions

Proposition 35.1. *Suppose that M is a countable m-monoid, Θ is a submodel of M, $e \in M^*$, $\langle a, b_0, \dots, b_{m-1}\rangle \in \Theta(e)$, $a \neq 0$, and $\exists i < m[b_i \neq 0]$. Then there exist a distinguished extension N of M and an expansion Θ' of Θ to a submodel of N such that:*

(i) $\exists e_0, e_1[e_0, e_1$ are distinguished in N, $e = e_0 + e_1$, and N is generated by $M \cup \{e_0, e_1\}]$.

(ii) $\langle a\rangle \in \Theta'(e_0)$ and $\langle b_0, \dots, b_{m-1}\rangle \in \Theta'(e_1)$.

(iii) $\Theta' \restriction M = \Theta$.

Proof. By Proposition 30.7 let N be a distinguished extension of M with $e_0, e_1 \in N$ such that e_0, e_1 are distinguished and nonzero, $e = e_0 + e_1$, N is generated by $M \cup \{e_0, e_1\}$, and Proposition 30.7(iii),(iv) hold. Define $\Psi : M \cup \{e_0, e_1\} \to \mathscr{R}$ by

$$\begin{aligned}
\Psi(f) &= \Theta(f) \quad \text{if} \quad f \in M;\\
\Psi(e_0) &= \hat{a};\\
\Psi(e_1) &= \sum_{i<m} \hat{b}_i.
\end{aligned}$$

35.1(1) The hypotheses of Proposition 32.8 hold for Ψ.

Proposition 32.8(i) is clear. Now by Proposition 32.7, $T \circ \Theta$ is a morphism from M to $\mathscr{P}$. Moreover,

35.1(2) There is a morphism $\Lambda' : N \to \mathscr{P}$ such that $T \circ \Theta \subseteq \Lambda'$, $\Lambda'(e_0) = T(\hat{a}) = a$, and $\Lambda'(e_1) = T(\sum_{i<m} \hat{b_i}) = \sum_{i<m} b_i$.

This holds by Proposition 30.7(iv), since $T(\Theta(e)) = a + \sum_{i<p} b_i$.

For Proposition 32.8(ii), suppose $p, n \in \omega$, $u \in {}^p(M \cup \{e_0, e_1\})$, $v \in {}^n(M \cup \{e_0, e_1\})$, and $\sum_{i<p} u_i = \sum_{j<n} v_j$. Say $\sum_{i<p} u_i = g + pe_0 + qe_1$ and $\sum_{j<n} v_j = g' + p'e_0 + q'e_1$ with $g, g' \in M$. Then

$$\begin{aligned}
\sum_{i<p} T(\Psi(u_i)) &= T(\Psi(g')) + pT(\Psi(e_0)) + qT(\Psi(e_1))\\
&= T(\Theta(g)) + pa + q\sum_{i<m} b_i\\
&= \Lambda'(g) + p\Lambda'(e_0) + q\Lambda'(e_1)\\
&= \Lambda'(g + pe_0 + qe_1) = \Lambda'\left(\sum_{i<p} u_i\right) = \Lambda'\left(\sum_{j<n} v_j\right)\\
&= \Lambda'(g') + p'\Lambda'(e_0) + q'\Lambda'(e_1)\\
&= T(\Psi(g')) + p'T(\Psi(e_0)) + q'T(\Psi'(e_1))\\
&= \sum_{j<n} T(\Psi(v_j)).
\end{aligned}$$

Thus Proposition 32.8(ii) holds. So (183) holds. We thus get a submodel Ψ^+ of N.

To see that $\Psi^+ \restriction M = \Theta$, suppose that $g \in M$. Take e' such that $\sum_{i<m} e'_i = g$. Then we can write $\sum_{i<m} e'_i = f + me_0 + ne_1$ with $f \in M$. By Proposition 30.7(iii) we have $m = n$ and $\sum_{i<m} e'_i = f + me$. Hence $\sum_{i<m} \Psi(e'_i) = \Psi\left(\sum_{i<m} e'_i\right) = \Psi(f + me) = \Psi(g) = \Theta(g)$. So $\Psi^+ \restriction M = \Theta$. In particular, Ψ^+ is an expansion of Θ.

Now $\langle a \rangle \in \hat{a} = \Psi(e_0) \subseteq \Psi^+(e_0)$. Also, $\langle b_0, \ldots, b_{m-1} \rangle \in \sum_{i<m} \hat{b}_i = \Psi(e_1) \subseteq \Psi^+(e_1)$. $\square$

Proposition 35.2. *Suppose that M is a countable m-monoid, Θ is a submodel of M, and $e_0 + e_1 = f_0 + f_1$ in M^*.*

Then there exist a distinguished extension N of M and elements $g_{00}, g_{01}, g_{10}, g_{11}$ of N such that N is generated by $M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$, the g_{ij} are distinguished elements, $\forall i < 2[e_i = \sum_{j<2} g_{ij}]$, and $\forall j < 2[f_j = \sum_{i<2} g_{ij}]$. Also, there is a submodel Λ of N such that $\forall e \in M[\Theta(e) \subseteq \Lambda(e)]$.

Proof. We take N and g_{ij} as in Proposition 31.8. So it remains only to find Λ. For all $i < 2$ let $a_i = T(\Theta(e_i))$ and $b_i = T(\Theta(f_i))$. Then $a_i, b_i \in \mathscr{P}$ for each $i < 2$. By Proposition 26.6 there is a $c : 2 \times 2 \to \mathscr{P}$ such that $\forall i < 2[a_i = \sum_{j<2} c_{ij}]$ and $\forall j < 2[b_j = \sum_{i<2} c_{ij}]$. Now define $\Psi : M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\} \to \mathscr{R}$ by:

$$\Psi(x) = \begin{cases} \Theta(x) & \text{if } x \in M, \\ \hat{c}_{ij} & \text{if } x = g_{ij}. \end{cases}$$

35.2(1) The hypotheses of Proposition 32.8 hold for Ψ.

In fact, Proposition 32.8(i) is clear. For Proposition 32.8(ii), assume that $m, n \in \omega$, $h \in {}^mE$, $f \in {}^nE$, and $\sum_{i<m} h_i = \sum_{j<n} f_j$, where $E = M \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$. Now $T \circ \Theta$ is a morphism from M into $\mathscr{P}$, $\forall i < 2[T(\Theta(e_i)) = a_i = \sum_{j<2} c_{ij}]$, and $\forall j < 2[T(\Theta(f_j)) = b_j = \sum_{i<2} c_{ij}]$. Hence by Proposition 31.8(v) there is an extension Λ of $T \circ \Theta$ to a morphism of N into $\mathscr{P}$ such that $\forall i, j < 2[\Lambda(g_{ij}) = c_{ij}]$. Note that for any $i, j < 2$, $T(\Psi(g_{ij})) = c_{ij} = \Lambda(g_{ij})$. So Λ is an extension of $T \circ \Psi$. Hence

$$\sum_{i<m} T(\Psi(h_i)) = \sum_{i<m} \Lambda(h_i) = \Lambda\left(\sum_{i<m} h_i\right) = \Lambda\left(\sum_{j<n} f_j\right) = \sum_{j<n} \Lambda(f_j) = \sum_{j<n} T(\Psi(f_j)).$$

Thus 35.2(1) holds.

By Proposition 32.8, Ψ^+ is a submodel of N. Clearly $\forall e \in M[\Theta(e) \subseteq \Psi^+(e)]$. $\square$

36. The basic lemma

Proposition 36.1. *Suppose that M is a countable m-monoid, Ψ is a submodel of M, and $e, f \in M$. Suppose that $e \not\leq f$ and $a = T(\Psi(e))$.*

Then there exist $a_0, a_1 \in \mathscr{P}$ such that $a = a_0 + a_1$ and there is an expansion Θ of Ψ such that $(a_0, a_1) \in \Theta(e) \backslash \Theta(f)$.

Proof. $\mathrm{Dom}(\Psi[M])$ is countable, so by Proposition 26.7 there exist a_0, a_1 such that $a = a_0 + a_1$ and a_0 is linearly independent from $\mathrm{Dom}(\Psi[M])$, with $a_0 \neq [\omega^c]_{ae}$. Define $\Lambda : M \to \mathscr{R}$ by:

$$\Lambda(g) = \begin{cases} \Psi(g) & \text{if } g \neq e, \\ \Psi(e) \cup (\hat{a}_0 + \hat{a}_1) & \text{if } g = e. \end{cases}$$

Clearly $\Lambda : M \to \mathscr{R}$. Suppose that $m, n \in \omega$, $h \in {}^m M$, $f' \in {}^n M$, and $\sum_{i<m} h_i = \sum_{j<n} f'_j$. Say $\sum_{i<m} h_i = x + sa$ with $x \in M \backslash \{a\}$, and $\sum_{j<n} f'_j = y + ta$ with $y \in M \backslash \{a\}$. Then

$$\begin{aligned} \sum_{i<m} T(\Lambda(h_i)) &= T(\Lambda(x)) + sT(\Lambda(e)) = T(\Psi(x)) + sT(\Psi(e) \cup (\hat{a}_0 + \hat{a}_1)) \\ &= T(\Lambda(x)) + sa = T(\Lambda(x)) + sT(\lambda(e)) \\ &= T(\Lambda(x + se)) = T(\Lambda(y + te) \\ &= \sum_{j<n} T(\Lambda(f'_j)). \end{aligned}$$

By Proposition 32.8, let Θ be the closure of Λ; so Θ is a submodel of M. Moreover, $(a_0, a_1) \in \hat{a}_0 + \hat{a}_1 \subseteq \Lambda(e) \subseteq \Theta(e)$. If $\sum_{i<m} u_i = f$, then each $u_i \neq e$, and so $\Psi(f) = \sum_{i<m} \Psi(u_i) = \sum_{i<m} \Lambda(u_i)$; hence $\Psi(f) = \Theta(f)$. Since $a_0 \notin \mathrm{Dom}(\Psi[M])$, it follows that $(a_0, a_1) \notin \Theta(f)$.

It remains only to show that Θ is an expansion of Ψ. Suppose that $v \in M$. Obviously $\Psi(v) \subseteq \Theta(v) \cap {}^{<<\omega}(\mathrm{Dom}(\Psi[M]))$. Now suppose that $b \in \Theta(v)$, with $\mathrm{dmn}(b) = m$ and $\forall i < m[b_i \in \mathrm{Dom}(\Psi[M])]$. By the definition of closure, there is an $m \in \omega$ and $w \in {}^m M$ such that $\sum_{i<m} w_i = v$ and $b \in \sum_{i<m} \Lambda(w_i)$.

Case 1. $\forall i < m[w_i \neq e]$. Then $\sum_{i<m} \Lambda(w_i) = \sum_{i<m} \Psi(w_i) = \Psi(\sum_{i<m} w_i) = \Psi(v)$. So $b \in \Psi(v)$.

Case 2. $\exists i < m[w_i = e]$. Then there exist $y \in M \backslash \{e\}$ and $p \in \omega$ such that $\sum_{i<m} w_i = y + pe$. Then $b \in \Lambda(y) + p\Lambda(e) = \Psi(y) + p(\Psi(e) \cup (\hat{a}_0 + \hat{a}_1))$. Thus we may assume that $b = b' + b_1 + \cdots + b_q + b_{q+1} + \cdots + b_p$, with $b' \in \Psi(y)$, $b_1, \ldots, b_q \in \Psi(e)$, and $b_{q+1}, \ldots, b_p \in \hat{a}_0 + \hat{a}_1$. For each $r \in \{q+1, \ldots, p\}$ choose c_0^r, c_1^r such that $b_r = c_0^r + c_1^r$, $c_0^r \in \hat{a}_0$, $c_1^r \in \hat{a}_1$. Moreover, there are $d', d_1, \ldots, d_p \in M$ and $\gamma' \in \Psi(d')$, $\gamma_1 \in \Psi(d_1)$, $\ldots$ $\gamma_p \in \Psi(d_p)$ such that b' is an entry in γ' and $\forall r \in \{1, \ldots, p\}[b_r$ is an entry in $\gamma_r]$. Thus for all $r = q+1. \ldots, p$, $c_0^r + c_1^r$ is an entry in $\gamma_r \subseteq \mathrm{Dom}(\Psi[M])$. It follows that a_0 and a_1 appear at the same place in c_0^r and c_1^r. Thus for all $r = q+1, \ldots, p$, $b_r \in \hat{a}$. Now clearly $\hat{a} \subseteq \Psi(e)$, so for all $r = q+1, \ldots, p$, $b_r \in \Psi(e)$. Hence $b \in \Psi(y) + p\Psi(e) = \Psi(y + pe) = \Psi(\sum_{i<m} w_i) = \Psi(v)$. $\square$

Proposition 36.2. *Suppose that M is a countable m-monoid, Ψ is a submodel of M, $e_0, e_1 \in M$, and $a \in \Psi(e_0 + e_1)$. Then there is a submodel Θ of M such that $\forall e \in M[\Psi(e) \subseteq \Theta(e)]$ and $a \in \Theta(e_0) + \Theta(e_1)$.*

Proof. We proceed by induction on the length n of a. For $n = 1$, say $a = \langle x \rangle$. Then $x = T(\Psi(e_0 + e_1)) = T(\Psi(e_0)) + T(\Psi(e_1)) = T(\Psi(e_0) + \Psi(e_1))$, so $a \in \Psi(e_0) + \Psi(e_1)$.

Now suppose inductively that $n > 1$. Write $a = \langle a_i : i < n \rangle$. We may assume that $a_0 \neq 0$ and $\exists i[0 < i < n$ and $a_i \neq 0]$. Then by Proposition 35.1 there exist a distinguished extension N of M and an expansion Θ' of Θ to a submodel of N such that:

36.2(1) $\exists f_0, f_1[f_0, f_1$ are distinguished in N, $e_0 + e_1 = f_0 + f_1$, and N is generated by $M \cup \{f_0, f_1\}]$.

36.2(2) $\langle a_0 \rangle \in \Theta'(f_0)$ and $\langle a_1, \ldots, a_{n-1} \rangle \in \Theta'(f_1)$.

36.2(3) $\Theta' \restriction M = \Theta$.

Now by Proposition 35.2 there exist a distinguished extension N' of N and elements $g_{00}, g_{01}, g_{10}, g_{11}$ of N' such that N' is generated by $N \cup \{g_{00}, g_{01}, g_{10}, g_{11}\}$, the g_{ij} are distinguished elements, $\forall i < 2[e_i = \sum_{j<2} g_{ij}]$, and $\forall j < 2[f_j = \sum_{i<2} g_{ij}]$. Also, there is a submodel Λ of N' such that $\forall e \in N[\Theta'(e) \subseteq \Lambda(e)]$. Hence $\langle a_1, \ldots, a_{n-1} \rangle \in \Theta'(f_1) \subseteq \Lambda(f_1) = \Lambda(g_{01} + g_{11})$. By the inductive hypothesis there is a submodel Θ'' of N' such that $\forall e \in N'[\Lambda(e) \subseteq \Theta''(e)]$ and $\langle a_1, \ldots, a_{n-1} \rangle \in \Theta''(g_{01}) + \Theta''(g_{11})$. Say $\langle a_1, \ldots, a_{n-1} \rangle = b_0 + b_1$ with $b_0 \in \Theta''(g_{01})$ and $b_1 = \Theta''(g_{11})$. Also, $a_0 = T(\Theta'(f_0)) = T(\Lambda(f_0)) = T(\Theta''(f_0)) = T(\Theta''(g_{00})) + T(\Theta''(g_{10}))$. Let $a'_0 = T(\Theta''(g_{00}))$ and $a'_1 = T(\Theta''(g_{10}))$. Then for any $i < 2$, $\langle a'_i \rangle^\frown b_i = (\langle a'_i \rangle^\frown 0) + (0^\frown b_i) \in \Theta''(g_{i0}) + \Theta''(g_{i1}) \subseteq \Theta''(g_{i0} + g_{i1}) = \Theta''(e_i)$. Now let $\Theta''' = \Theta'' \restriction M$. Then Θ''' is a submodel of M, $\forall e \in M[\Theta(e) \subseteq \Theta'''(e)]$, and $a \in \Theta'''(e_0) + \Theta'''(e_1)$. □

Proposition 36.3. *Suppose that M is a countable m-monoid, Ψ is a submodel of M, $e \in M$, $m, n \in \omega$, $a \in {}^m(\Psi(e))$, and $b \in {}^n(\Psi(e))$. Then there is a submodel Θ of M such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$ and there is a $c \in {}^{m \times n}(\Theta(e))$ such that $\forall i < m[a_i = \sum_{j<n} c_{ij}]$ and $\forall j < n[b_j = \sum_{i<m} c_{ij}]$.*

Proof. By Proposition 35.1 and induction, there exist an extension N of M, a submodel Θ of M and elements $e' \in {}^m N$ such that $\forall i < m[T(\Theta(e'_i)) = a_i]$, $\sum_{i<m} e'_i = e$, and $\Theta \restriction M = \Psi$. Now $b \in {}^n(\Psi(e)) \subseteq {}^n(\Theta(e))$, so by Proposition 36.2 there is a submodel Λ of M such that $\forall x \in M[\Theta(x) \subseteq \Lambda(x)]$ and $b \in \sum_{i<m} \Lambda(e'_i)$. Say $b = \sum_{i<m} c_i$ with each $c_i \in {}^n(\Lambda(e_i))$. Thus $\forall j < n[b_j = \sum_{i<m} c_{ij}]$. Then $\forall i < m[\sum_{j<n} c_{ij} = T(\Lambda(e'_i)) = T(\Theta(e'_i)) = a_i]$. Now $c \in {}^{m \times n}\mathscr{R}$, and

$$\begin{aligned} c &= c_0^\frown \cdots^\frown c_{m-1} \\ &= (c_0^\frown 0 \cdots^\frown 0) + (0^\frown c_1^\frown 0 \cdots^\frown 0) + \cdots + (0^\frown 0^\frown \cdots^\frown c_{m-1}) \\ &\in \sum_{i<m} \Lambda(e_i) \subseteq \Lambda\left(\sum_{i<m} e_i\right) = \Lambda(e). \end{aligned}$$

Thus Λ is as desired. □

Proposition 36.4. *If Ψ is a submodel of M, $(a, b) \in \Psi(e)$ with $a \neq 0$, $\exists i < \mathrm{dmn}(b)[b_i \neq 0]$, then there exist a_0, a_1 and Θ such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$, $a_0 \neq 0 \neq a_1$, $a = a_0 + a_1$, and $(a_0, a_1, b) \in \Theta(e)$.*

Proof. By Proposition 35.1 there exist e_0, e_1, an distinguished extension N of M, and an expansion Ψ' of Ψ to a submodel of N, such that $e = e_0 + e_1$, $a = T(\Psi'(e_0))$, $b \in \Psi'(e_1)$, and $\Psi' \restriction M = \Psi$. Then by Proposition 36.1, taking $f = 0$, there exist $a_0, a_1 \in \mathscr{P}$ and an expansion Θ of Ψ' such that $a = a_0 + a_1$ and $(a_0, a_1) \in \Theta(e_0)$. Hence $(a_0, a_1, b) \in \Theta(e_0) + \Theta(e_1) \subseteq \Theta(e)$. □

Proposition 36.5. *If M is a countable m-monoid and Ψ is a submodel of M, then there is a submodel Θ of M such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$ and for all $e, f \in M$ with e distinguished and $e \neq f$ we have $\Theta(e) \neq \Theta(f)$.*

Proof. Let $\langle (e_i, f_i) : i \in \omega \rangle$ list all pairs (e, f) such that $e, f \in M$, e is distinguished, and $e \neq f$. Define $\Theta^0 = \Psi$. Suppose that Θ^i has been defined so it is a submodel of M and $\forall j < i \forall x \in M[\Theta^j(x) \subseteq \Theta^i(x)]$. By Proposition 30.3 we have $e_i \not\leq f_i$ or $f_i \not\leq e_i$. Hence by Proposition 36.1 there is a submodel Θ^{i+1} of M such that $\forall x \in M[\Theta^i(x) \subseteq \Theta^{i+1}(x)]$ and $\Theta^{i+1}(e_i) \neq \Theta^{i+1}(f_i)$.

Now define $\Lambda(x) = \bigcup_{i \in \omega} \Theta^i(x)$ for all $x \in M$. Clearly Λ is as desired. □

Proposition 36.6. *If M is a countable distinguished m-monoid and Ψ is a submodel of M, then there is a submodel Θ of M such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$ and $\forall e_0, e_1 \in M[\Psi(e_0 + e_1) \subseteq \Theta(e_0 + e_1)$.*

Proof. This follows by an easy construction using Proposition 36.2. □

Proposition 36.7. *If M is a countable distinguished m-monoid and Ψ is a submodel of M, then there is a model Θ of M such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$.*

Proof. This follows by an inductive construction using Propositions 36.5 and 36.6. □

37. Proof of Ketonen's theorem

Proposition 37.1. *Let M be a countable distinguished m-monoid, and suppose that Ψ is a model of M. Suppose that $e \in M$ and $(a_0, a_1) \in \Psi(e)$. Then there exist a distinguished extension of M to a countable distinguished monoid N, a model Θ of N such that $\forall x \in M[\Psi(x) \subseteq \Theta(x)]$, and elements e_0, e_1 of N, such that $e = e_0 + e_1$ and $\forall i < 2[T(\Theta(e_i)) = a_i]$.*

Proof. By Propositions 35.1 and 36.7. □

Proposition 37.2. *Let M be a countable distinguished m-monoid, and suppose that Ψ is a model of M. If $e_0 + e_1 = f_0 + f_1$ in M^*, then there exist a distinguished extension N of M and elements g_{ij} of N for $i, j < 2$ such that N is countable and distinguished, $\forall i < 2[e_i = \sum_{j<2} g_{ij}]$, and $\forall j < 2[f_j = \sum_{i<2} g_{ij}]$.*

Proof. By Propositions 36.3 and 36.7. □

Theorem 37.3. (Ketonen) *Every countable monoid can be isomorphically embedded into* **NBA**.

Proof. Let M be a countable monoid. Let 0 be a new element, define $M' = M \cup \{0\}$, and define for any $x, y \in M'$,

$$x +_{M'} y = \begin{cases} x +_M y & \text{if } x, y \in M, \\ x & \text{if } y = 0, \\ y & \text{if } x = 0 \neq y. \end{cases}$$

Thus M' is a countable m-monoid. By Proposition 30.9 it can be embedded in a countable atomless distinguished m-monoid N. For each $x \in N$ let $\Psi(x) = ([\omega^c]_{ae})\hat{}$. Obviously Ψ is a submodel of N. By Proposition 36.7 there is a model Ψ_0 of N such that $\forall x \in N[\Psi(x) \subseteq \Psi_0(x)]$. Note that $\forall x \in N[T(\Psi(x)) = T(\Psi_0(x)) = [\omega^c]_{ae}]$, so $\Psi_0[P] \subseteq \Psi_0[N^*] \subseteq \mathscr{Q}^1$.

Let $M_0 = N$ and $\Psi_0 = \Psi$. Suppose that we have defined countable distinguished m-monoids $M_0, \ldots, M_m$ and models Ψ_k of M_k for $k \leq m$ such that if $k < m$ then M_{k+1} is a distinguished extension of M_k, $\forall x \in M_k[\Psi_k(x) \subseteq \Psi_{k+1}(x)]$, and the following conditions hold:

37.3(1) $\forall e_0, e_1, f_0, f_1 \in M_k[e_0 + e_1 = f_0 + f_1 \to \exists g \in {}^{2\times 2}M_{k+1}[\forall i < 2[e_i = \sum_{j<2} g_{ij}]$ and $\forall j < 2[f_j = \sum_{i<2} g_{ij}]]]$.

37.3(2) $\forall e \in M_k \forall (a_0, a_1) \in \Psi_k(e) \exists e_0, e_1 \in M_{k+1}[e = e_0 + e_1$ and $\forall i < 2[T(\Psi_{k+1}(e_i)) = a_i]]$.

Using Propositions 37.1 and 37.2 we define M_{k+1} and Ψ_{k+1} satisfying the above conditions.

Now define $R = \bigcup_{k\in\omega} M_k$ and $\Theta(x) = \bigcup\{\Psi_k(x) : x \in M_k\}$. By Proposition 34.1, M is a submonoid of **NBA**. □

References

Ershov, Ju. [79] *Distributive lattices with relative complements.* (Russian) Alg. i Log. 18 (1979), 680-722, 755. English translation: Alg. and Log. 18 (1979), 431-459.

Koppelberg, S. **General theory of Boolean algebras.** 312pp.

Pierce, R. *Countable Boolean algebras.* Handbook of Boolean algebras, v. 3, 775–876.

Pinus, A. **Constructions of Boolean algebras.** Novosibirsk 1994, 208pp.

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