CU Boulder

Math 2130

Test 2

Section 002 (Instructor Farid Aliniaeifard)

Friday, Oct 6, 2017, 10:00 - 10:40 am

NAME (print):____

(Family) (Given)

SIGNATURE:____

STUDENT NUMBER:____________________________________

Instructions:

1. Time allowed: 40 minutes.
2. NO CALCULATORS OR OTHER AIDS
3. There are 5 questions on 5 pages. Last page is blank.
4. Questions can be solved in more than one way.
5. You are expected to write clearly and carefully.

Question	**Points**	**Marks**
1	10	
2	10	
3	10	
4	10	
5	5	
Total	45	

1. (10 points) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Is A diagonalizable? If so write $A = PDP^{-1}$, where P is invertible and D is diagonal.

Solution. We first need to compute the eigenvalues.

$$det(A - \lambda I) = det(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix}) = (2-\lambda)^2(1-\lambda).$$

Therefore, $\lambda = 2$ and $\lambda = 1$.

The eigenspace corresponding to $\lambda = 1$ is the solution set of $(A - I)x = 0$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, x_3 is free and we have $x_1 = 0$, $x_2 = -x_3$. Let $x_3 = t$. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}.$$

So, a basis for the eigenspace corresponding to $\lambda = 1$ is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

The eigenspace corresponding to $\lambda = 2$ is the solution set of $(A - 2I)x = 0$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced echelon form of the augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}.$$

Therefore, x_2 and x_3 are free and we have $x_1 = -x_3$. Let $x_2 = t$ and $x_3 = s$. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix}.$$

So, a basis for the eigenspace corresponding to $\lambda = 1$ is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Therefore,

$$P = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

2. (10 points)

   (a) Write a formula for the determinant of an $n \times n$ matrix A.

   (b) Let $\{v_1, \ldots, v_n\}$ is a subset of a vector space W. What does it mean $W = sapn\{v_1, \ldots, v_n\}$? What does it mean $\{v_1, \ldots, v_n\}$ is linearly independent.

   (c) A subspace of a vector space W is ....

   (d) Let A be an $n \times n$ matrix. The characteristic equation of A is ...

Solution.

(a) Any cofactor expression is a formula for A, for instance the cofactor expression down to jth column is

$$det(A) = (-1)^{1+j} a_{1j} det(A_{1j}) + (-1)^{2+j} a_{2j} det(A_{2j}) + \ldots + (-1)^{n+j} a_{nj} det(A_{nj}),$$

where A_{ij} denote the submatrix formed by deleting the ith row and jth columns of A.

(b) $W = sapn\{v_1, \ldots, v_n\}$ means that every element $w \in W$ is a linear combination of $v_1, \ldots, v_n$, i.e., there are scalars $c_1, \ldots, c_n$ such that $w = c_1 v_1 + \ldots + c_n v_n$. Also $\{v_1, \ldots, v_n\}$ is linearly independent if we have $c_1 v_1 + \ldots + c_n v_n = 0$ for any scalars $c_1, \ldots, c_n$, implies that $c_1 = \ldots = c_n = 0$.

(c) A subspace of a vector space W is a non-empty subset of H of W such that

   i. $0 \in H$.

   ii. $u + v \in H$ for every $u, v \in H$.

   iii. $cv \in H$ for every scalar c and any vector $v \in H$.

(d) The characteristic equation of A is $det(A - \lambda I) = 0$.

3. (10 points) Let W be a vector space with a basis $\{v_1, v_2, v_3\}$. Define two other bases for W,

$$\mathcal{B} = \{v_1 - v_2, -v_3, v_1 + v_3\} \text{ and } \mathcal{C} = \{v_1 + v_2, v_2 + v_3, v_1 + v_2 + v_3\}.$$

(a) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

(b) Let $w = 2v_1 + 4v_2 + 3v_3$. Write $[w]_{\mathcal{B}}$.

Solution.

(a) Let $\mathcal{E} = \{v_1, v_2, v_3\}$. Since W has dimension 3 there is an isomorphism from W to $\mathbb{R}^3$ where $v \mapsto [v]_{\mathcal{E}}$. Note that

$$[v_1 - v_2]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad [-v_3]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad [v_1 + v_3]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$[v_1 + v_2]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [v_2 + v_3]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad [v_1 + v_2 + v_3]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We only need to find the matrix of change of bases form

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

We have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 & -2 & 2 \end{array}\right]$$

Therefore, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & -1 \\ 2 & -1 & 2 \end{bmatrix}$.

(b) Note that $w = 2v_1 + 4v_2 + 3v_3 = -4(v_1 - v_2) + 3(-v_3) + 6(v_1 + v_2)$. Therefore,

$$[w]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix}.$$

4. (10 points) Let V be a vector space with a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ and W a vector space with a basis $\mathcal{C} = \{w_1, w_2, w_3\}$. Let T be a function from V to W such that $T(v_1) = w_1 + w_2$, $T(v_2) = w_1 + w_2 - w_3$, and $T(v_3) = 2w_1 + 2w_2 - w_3$.

   (a) Is T a linear transformation? Justify your answer.

   (b) If T a linear transformation, write matrix T relative to the basis $\mathcal{B}$ and $\mathcal{C}$.

Solution.

(a) No. Assume that T maps every vector to zero except v_1, v_2, and v_3. Then if T is a linear transformation, we have

$$T(2v_1) = 2w_1 + 2w_2 \neq 0.$$

(b) The matrix T relative to the basis $\mathcal{B}$ and $\mathcal{C}$ is

$$[[T(v_1)]_{\mathcal{C}} \quad [T(v_2)]_{\mathcal{C}} \quad [T(v_3)]_{\mathcal{C}}] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

5. (5 points) Mark each statement True or False.

   (a) If $A = PDP^{-1}$ for an invertible matrix P and a diagonal matrix D, then P and D are unique.

   (b) If v_1 and v_2 are linearly independent eigenvectors, then they correspond to different eigenvalues.

   (c) Let T be a linear transformation. Then T is one-to-one if and only if $ker\ T = \{0\}$.

   (d) If $T : V \to W$ is an isomorphism and H is a subspace of V of dimension n, then image of H is also of dimension n.

   (e) If $W = span\{v_1, v_2, v_3, v_4, v_5\}$ and $v_1 \in span\{v_2, v_3, v_4, v_5\}$, then $dimW = 4$.

   Solution.

   (a) False.

   (b) False.

   (c) True.

   (d) True.

   (e) False.

The end. Have a great weekend