CU Boulder

Math 2130

Test 1

Section 002 (Instructor Farid Aliniaeifard)

Friday, Oct 6, 2017, 10:00 - 10:40 am

NAME (print): ______________________________
(Family) (Given)

SIGNATURE: ______________________________

STUDENT NUMBER: ______________________________

Instructions:

1. Time allowed: 40 minutes.
2. NO CALCULATORS OR OTHER AIDS
3. There are 5 questions on 5 pages. Last page is blank.
4. Questions can be solved in more than one way.
5. You are expected to write clearly and carefully.

Question	**Points**	**Marks**
1	5	
2	5	
3	5	
4	5	
5	5	
Total	25	

1. (5 points) Let

$$\begin{array}{lllll} & -2x_2 & +x_3 & = 1 \\ x_1 & -2x_2 & & = 1 \\ 2x_1 & -4x_2 & & = 2 \end{array}$$

Is the system consistent? if so write the solution set.
Solution. The augmented matrix of of the system is

$$\begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 2 & -4 & 0 & 2 \end{bmatrix}$$

An echelon form is

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the echelon form does not have a row of the form $[0\ 0\ \ldots\ 0\ b]$, $b \neq 0$, we can say that the system is consistent.

To find the solution set we need to find the reduced echelon form. The reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that x_3 is a free variable and x_1 and x_2 are basic variables. Let $x_3 = t$. Then

$$x_1 = x_3 = t \qquad x_2 = -1/2 + 1/2x_3 = -1/2 + 1/2t$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -1/2 + 1/2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

Therefore, the solution set is

$$\left\{ \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

2. (5 points) For each of the following give the definition.

   (a) Linear independent set of vectors.

   (b) Span of a set of vectors

   (c) Basis for a subspace.

Solution.

(a) A set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly independent if

$$c_1 v_1 + \ldots + c_p v_p = 0$$

for some $c_1, \ldots, c_p \in \mathbb{R}$, then $c_1 = \ldots = c_p = 0$.

(b) The span of a set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is the set of all linear combinations of $\{v_1, \ldots, v_p\}$, i.e., $Span\{v_1, \ldots, v_p\} = \{c_1 v_1 + \ldots + c_p v_p : c_1, \ldots, c_p \in \mathbb{R}\}$.

(c) Let H be a subspace. A set of vectors $\{v_1, \ldots, v_p\}$ in H is a basis for H if $\{v_1, \ldots, v_p\}$ is linearly independent and spans H.

3. (5 points)

   (a) Show that
   $$T(x_1, x_2, x_3) = (3x_2 - x_1, 2x_1 + x_3)$$
   is a linear transformation.

   (b) Find the standard matrix for T.

   (c) Is T onto?

   (d) Is T one-to-one?

Solution.

(a) Note that
$$T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = (3x_2 - x_1, 2x_1 + x_3) + (3y_2 - y_1, 2y_1 + y_3) =$$
$$(3x_2 + 3y_2 - x_1 - y_1, 2x_1 + 2y_2 + x_3 + y_3) =$$
$$(3(x_2 + y_2) - (x_1 + y_1), 2(x_1 + y_2) + (x_3 + y_3)) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3).$$
Therefore, T is linear.

(b) The standard matrix is $[T(e_1)|T(e_2)|T(e_3)]$ which is the same as
$$\begin{bmatrix} -1 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

(c) An echelon form of the standard matrix is
$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$
Since it does have pivot positions in each row we conclude that T is onto.

(d) But it is not one-to-one because the echelon form has a free variable, and so $T(x) = Ax = 0$ has a non-trivial solution.

4. (5 points) Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

(a) Is $\{v_1, v_2, v_3\}$ linearly independent?

(b) Find a basis β for $Span\{v_1, v_2, v_3\}$.

(c) Is $b = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ in V? if so write $[b]_\beta$.

Solution.

(a) If the equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a trivial solution then the vectors are linearly independent. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix}$$

An echelon form is

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it has a free variable, therefore, the system has infinitely many solution and so the vectors are not linearly independent.

(b) Since the echelon form has pivot positions in the first and second row, a basis is

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(c) If

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

has a solution, then b is in the subspace. If you solve the equation, you will see that $x_1 = 2$ and $x_2 = 1$ is a solution, and so

$$[b]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

5. (5 points) Mark each statement True or False. Justify only one of them.

   (a) The dimension of the null space of A is the same as the number of free variables in equation $Ax = 0$.

   (b) The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ab = cd$.

   (c) A linear transformation is onto if the standard matrix of T has pivot position in each column.

Solution.

(a) True. Let n be the number of columns of A. Note that $RankA + dim\ NulA = n$. Since the $rankA$ is the same as the number of basic variables, and the number of basic variables plus the number of free variables is n, we conclude that the $dim\ NulA$ is the same as the number of free variables.

(b) False. A counterexample is $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$.

(c) False. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ be the standard matrix of a linear transformation T. Then if you check you see A has pivot positions in each column but T is not onto.

The end. Have a great weekend