CU Boulder

Math 2130

Sample-Test 2

Section 002 (Instructor Farid Aliniaeifard)

NAME (print):____

(Family) (Given)

SIGNATURE:____

STUDENT NUMBER:____

Instructions:

1. Time allowed: 50 minutes.
2. NO CALCULATORS OR OTHER AIDS
3. There are 5 questions on 5 pages. Last page is blank.
4. Questions can be solved in more than one way.
5. You are expected to write clearly and carefully. You will be graded for both content and presentation.

Question	**Points**	**Marks**
1	5	
2	5	
3	5	
4	5	
5	5	
Total	25	

1. (5 points) Diagonalize the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -5 \\ 8 & 0 & 9 \end{bmatrix}.$$

Solution. First we need to find the eigenvalues. The eigenvalues are the roots of $det(A - \lambda I)$.

$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 0 & 0 \\ -8 & 4-\lambda & -5 \\ 8 & 0 & 9-\lambda \end{bmatrix} = (\lambda - 1)(\lambda - 4)(\lambda - 9).$$

Therefore, the eigenvalues are $\lambda = 1, \lambda = 4$, and $\lambda = 9$.

The eigenspace corresponding to $\lambda = 1$ is the set of solutions of $(A - I)x = 0$, which is

$$\left\{ t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

The eigenspace corresponding to $\lambda = 4$ is the set of solutions of $(A - 4I)x = 0$, which is

$$\left\{ t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

The eigenspace corresponding to $\lambda = 9$ is the set of solutions of $(A - 9I)x = 0$, which is

$$\left\{ t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Therefore,

$$P = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

and we have $A = PDP^{-1}$.

2. (5 points) Let $\mathcal{B} = \{1+t, 1+t^2, 1+t+t^2\}$ and $\mathcal{C} = \{2-t, -t^2, 1+t^2\}$ be bases for $\mathbb{P}_2$.

   (a) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$.

   (b) Let $f = 2+4t+3t^2$. Write $[f]_{\mathcal{C}}$.

Solution. We know that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = [[1+t]_{\mathcal{C}} \quad [1+t^2]_{\mathcal{C}} \quad [1+t+t^2]_{\mathcal{C}}].$$

There is an isomorphism from $\mathbb{P}_2 \to \mathbb{R}^3$ defined by $f \mapsto [f]_{\mathcal{E}}$ where $\mathcal{E} = \{1, t, t^2\}$. So we can define two bases

$$\mathcal{B} = [[1+t]_{\mathcal{E}} \quad [1+t^2]_{\mathcal{E}} \quad [1+t+t^2]_{\mathcal{E}}]$$

and

$$\mathcal{C} = [[2-t]_{\mathcal{E}} \quad [-t^2]_{\mathcal{E}} \quad [1+t^2]_{\mathcal{E}}]$$

for $\mathbb{R}^3$. Therefore,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

If we reduce the following matrix

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{array}\right]$$

to

$$[I|\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}]$$

we will find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$.

(b) Do it as an exercise.

3. (5 points) This question is about definitions.

4. (5 points) Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of a matrix A corresponding to the eigenvalue 3 and that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue -2. Compute $A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Solution. We will find real numbers x_1 and x_2 such that

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This is a system with augmented matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

Which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore, $x_1 = 2$ and $x_1 = 1$. We have that

$$A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = A^2(2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}) = 2A^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2(A(A \begin{bmatrix} 1 \\ 1 \end{bmatrix})) + 3(A(A \begin{bmatrix} 2 \\ 1 \end{bmatrix})) =$$

$$2(3(A \begin{bmatrix} 1 \\ 1 \end{bmatrix})) + 3(-2(A \begin{bmatrix} 2 \\ 1 \end{bmatrix})) = 18(\begin{bmatrix} 1 \\ 1 \end{bmatrix})) + 12(\begin{bmatrix} 2 \\ 1 \end{bmatrix})).$$

5. (5 points) The last question will be True or False question.

The end. Have a great weekend