Introduction to Probability and Statistics (3510)
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Solutions to the Final

April, 2010

1. Small n and normal means that we use the t-distribution. We have $\overline{x} = 8.02$, $\overline{S}_n^2 = \frac{1}{5}\sum_1^6 (x_i - \overline{x})^2 = 0.03$ and $\overline{S}_n = 0.17$. Here the error probability is $\alpha = 0.1$ and the degree of freedom is five, so we look up $t_{5,0.95} = 2.015$. Need to check if margin of error is not more than 0.1. Is
$$\frac{\overline{S}_n}{\sqrt{6}} \leq 0.1?$$
This would mean $1.7 \cdot 2.015 = 3.43 \leq \sqrt{6}$, which is false. **Conclusion**: no, we cannot be 90% certain.

2. Instead of saying that the error is not more than 0.1, we can equivalently require that the margin of error in a confidence interval does not exceed 0.1. Then the question is: what is the confidence level for that interval? Here, although we do not know σ, we can still upper estimate it:
$$\sigma = \sqrt{p(1-p)} \leq 1/2,$$
and we have to find a $0 < q < 1$ such that z_q satisfies
$$\frac{\sigma}{\sqrt{50}} \cdot z_q \leq 0.1.$$
It is enough to have
$$\frac{1/2}{\sqrt{50}} \cdot z_q \leq 0.1,$$
giving $z_q = 1.41$ and $q = 1 - \frac{\alpha}{2} = 0.92$. So $\alpha = 0.16$. **Conclusion:** we can be 84% confident.

3. Here the info that $\lambda < 5$ is in fact superfluous! Since the (unknown) mean is $\lambda/2$, therefore we need that

$$\alpha = P\left(rX < \lambda/2\right) = P\left(X < \lambda/2r\right).$$

Since $X \sim U\left([0,\lambda]\right)$, therefore this is the same as

$$\alpha = 1/2r,$$

and so the **final answer** is $r = \frac{1}{2\alpha}$.

4. **No, we cannot believe this.** The reason is that, using the CLT,

$$P\left(\sum \leq 450\right) \simeq P\left(Z \leq \frac{-50}{0.5 \cdot \sqrt{1000}}\right) = P\left(Z < -\sqrt{10}\right) < P\left(Z < -3\right),$$

where Z is standard normal. So, with a fair coin, getting less than 450 heads has around 0.001 probability. In fact, we tend to believe that the coin was biased and the heads probability was less than 0.5 (or she is not telling the truth about the number of heads she got).

5. We use Poisson model with $\lambda = Var = 4.5$ and so the **sought probability is** $e^{-4.5} = 0.01$, **that is 1%.**

6. If A is the event of having exactly one black, then

$$P(A) = P(A \mid H)P(H) + P(A \mid T)P(T) = P(A \mid H) \cdot 2/3 + P(A \mid T) \cdot 1/3.$$

Now,

$$P(A \mid H) = 3 \cdot (2/5) \cdot (3/5)^2$$

and

$$P(A \mid T) = \frac{2 \cdot \binom{3}{2}}{\binom{5}{3}} = 3/5.$$

(Alternatively, one can calculate the probabilities of BWW, WBW, WWB and each one of them turns out to be 1/5.) So the **final answer** is $61/125 = 0.488$. (Remark: the two conditional probabilities are in fact binomial and hypergeometric, respectively, but one does not need to know the formulas for these simple computations.)