Math 2400 Spring 2020, Exam 1

February 10, 2020

PRINT YOUR NAME: Solution

PRINT INSTRUCTOR'S NAME: ______________________

Mark your section/instructor:

☐	Section 001	Braden Balentine	8:00 - 8:50
☐	Section 002	Xingzhou Yang	8:00 - 8:50
☐	Section 003	Sebastian Bozlee	9:00 - 9:50
☐	Section 004	Xingzhou Yang	9:00 - 9:50
☐	Section 005	Mark Pullins	10:00 - 10:50
☐	Section 006	Athena Sparks	10:00 - 10:50
☐	Section 007	Trevor Jack	10:00 - 10:50
☐	Section 008	Michael Wheeler	12:00 - 12:50
☐	Section 009	Elizabeth Scott-Janda	1:00 - 1:50
☐	Section 010	Hanson Smith	2:00 - 2:50
☐	Section 011	Corey Lyons	2:00 - 2:50
☐	Section 012	Sean O'Rourke	3:00 - 3:50
☐	Section 014	Sean O'Rourke	4:00 - 4:50

Question	Points	Score
1	4	
2	5	
3	12	
4	8	
5	4	
6	4	
7	8	
8	12	
9	8	
10	13	
11	11	
12	11	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- For cylindrical coordinates use (r, θ, z), and for spherical coordinates use (ρ, θ, ϕ).
- When done, give your exam to your proctor, who will mark your name off on a photo roster.

1. (4 points) **Circle ONE.** The volume of the parallelepiped determined by the vectors $\langle 1, 1, 1\rangle$, $\langle -1, 4, 2\rangle$, and $\langle 0, 3, 0\rangle$ is:

   (A) 5

   (B) 3

   (C) -3

   (D) $\frac{\sqrt{22}}{3}$

   (E) 9

2. (5 points) **Circle ONE.** The normal vector to the plane that contains both the point $(2, 5, 2)$ and the line $\vec{r}(t) = \langle 1,\ 3,\ 1\rangle + t\langle 0,\ 1,\ 1\rangle$ is:

   (A) $\langle 3, -2, 2\rangle$

   (B) $\langle 1, -1, 1\rangle$

   (C) $\langle 3, 2, -2\rangle$

   (D) $\langle 2, -1, 1\rangle$

   (E) $\langle -1, 0, 1\rangle$

3. **Note: No partial credit for this problem.**

Suppose $\vec{a}, \vec{b}, \vec{c}$ are vectors in $\mathbb{R}^3$ such that

$$\vec{a} \times \vec{b} = \langle 4, \ -3, 1 \rangle \qquad \vec{b} \times \vec{c} = \langle 3, \ 2, \ 0 \rangle \qquad \vec{a} \cdot \vec{b} = 3.$$

For (a) – (e), write ONE number, or ONE vector for each answer.

(a) (2 points) $\vec{b} \cdot \vec{a} =$ 3

(b) (2 points) $\vec{c} \times \vec{b} =$ $\langle -3, -2, 0 \rangle$

(c) (2 points) $\vec{c} \cdot \langle 3, \ 2, \ 0 \rangle =$ 0

(d) (2 points) $|\vec{a} \times \vec{b}| =$ $\sqrt{26}$

(e) (2 points) $2\left(\vec{b} \times \vec{c}\right) - \left(\vec{a} \times \vec{b}\right) =$ $\langle 2, 7, -1 \rangle$

(f) (2 points) **Circle ONE.** The scalar projection (component), $\text{comp}_{\vec{b}}\vec{a}$, of $\vec{a}$ onto $\vec{b}$ is

(A) Positive (B) Negative (C) Zero (D) Not enough information to decide

4. (a) (4 points) **Circle ONE.** The best description of the solid given by the inequalities in cylindrical coordinates:

$$0 \leq r \leq 2, \quad -\pi/2 \leq \theta \leq \pi/2, \quad 0 \leq z \leq 2.$$

(A) One quarter of a sphere with a radius of 2 in the region where $x \geq 0$ and $y \geq 0$

(B) One quarter of a sphere with a radius of 2 in the region where $x \geq 0$ and $z \geq 0$

(C) One quarter of a sphere with a radius of 2 in the region where $y \geq 0$ and $z \geq 0$

(D) Half of cylinder with a radius of 2 and a height of 2 lying in the region where $x \geq 0$ and $y \geq 0$

(E) Half of cylinder with a radius of 2 and a height of 2 lying in the region where $x \geq 0$ and $z \geq 0$

(F) Half of cylinder with a radius of 2 and a height of 2 lying in the region where $y \geq 0$ and $z \geq 0$

(b) (4 points) **Circle ONE.** The best description of the solid given by the inequalities in spherical coordinates:

$$0 \leq \rho \leq 2, \quad 0 \leq \theta \leq \pi/2, \quad 0 \leq \phi \leq \pi$$

(A) One quarter of a solid sphere with a radius of 2 in the region where $x \geq 0$ and $y \geq 0$

(B) One quarter of a solid sphere with a radius of 2 in the region where $x \geq 0$ and $z \geq 0$

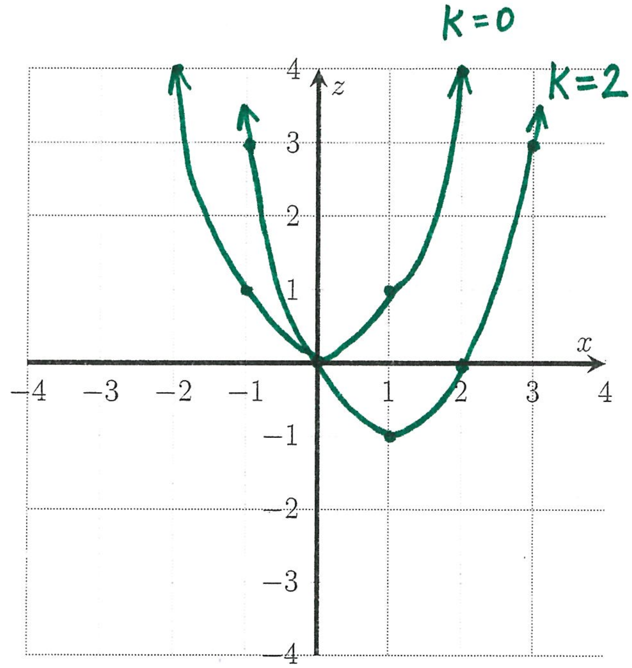
(C) One quarter of a solid sphere with a radius of 2 in the region where $y \geq 0$ and $z \geq 0$

(D) Half of a solid cylinder with a radius of 2 and a height of 2 lying in the region where $x \geq 0$ and $y \geq 0$

(E) Half of a solid cylinder with a radius of 2 and a height of 2 lying in the region where $x \geq 0$ and $z \geq 0$

(F) Half of a solid cylinder with a radius of 2 and a height of 2 lying in the region where $y \geq 0$ and $z \geq 0$

5. (4 points) **Sketch and label** the traces in $y = k$ of $z = x(x - y)$ below for $k = 0$ and $k = 2$.



6. (4 points) **Circle ONE.** Compute the length of the helix

$$\vec{r}(t) = \langle \sin(t),\ t,\ \cos(t) \rangle$$

for t between 0 and 2π.

(A) 2π

(B) $\arcsin(2)$

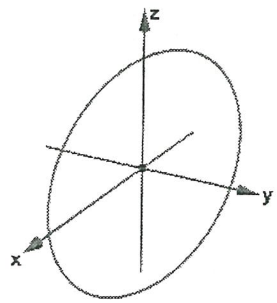
(C) $\ln(2)$

(D) $2\pi\sqrt{2}$

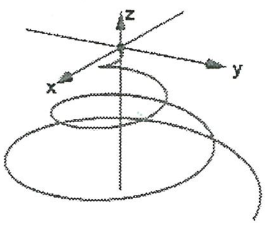
(E) $\frac{\sqrt{3}}{2}$

7. (8 points) Match each curve with one of the equations on the right side. Not all equations will be matched.

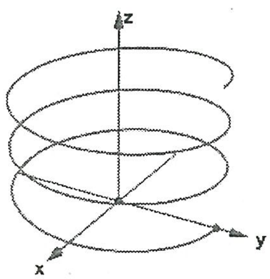
(i) B



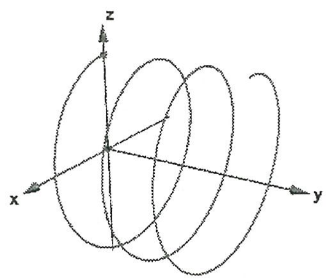
(ii) E



(iii) A



(iv) C



(A) $\vec{r}(t) = \langle \sin t,\ \cos t,\ t \rangle$,
$0 \le t \le 6\pi$

(B) $\vec{r}(t) = \langle -\cos t,\ -\sin t,\ \cos t \rangle$,
$0 \le t \le 6\pi$

(C) $\vec{r}(t) = \langle \sin t,\ t,\ \cos t \rangle$,
$0 \le t \le 6\pi$

(D) $\vec{r}(t) = \langle t \sin t,\ t,\ t \cos t \rangle$,
$0 \le t \le 6\pi$

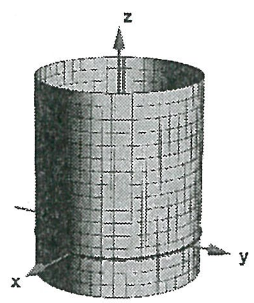
(E) $\vec{r}(t) = \langle t \sin t,\ t \cos t,\ -t \rangle$,
$0 \le t \le 6\pi$

(F) $\vec{r}(t) = \langle t \sin t,\ t \cos t,\ t \rangle$,
$0 \le t \le 6\pi$

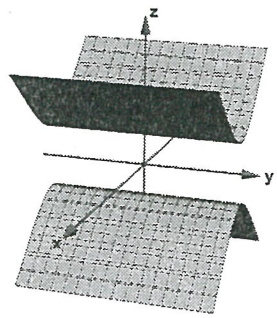
(G) $\vec{r}(t) = \langle t,\ \sin t,\ \cos t \rangle$,
$0 \le t \le 6\pi$

8. (12 points) Match each 3D surface with one of the equations on the right side. Not all equations will be matched.

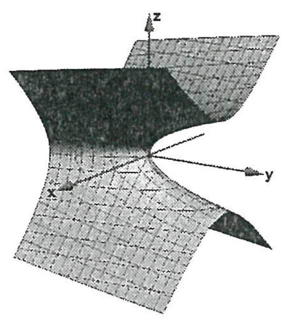
(i) B



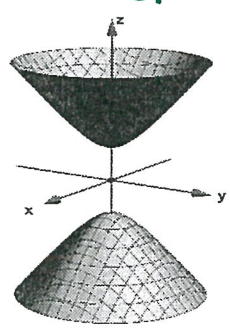
(ii) A



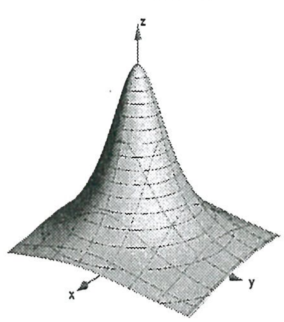
(iii) D



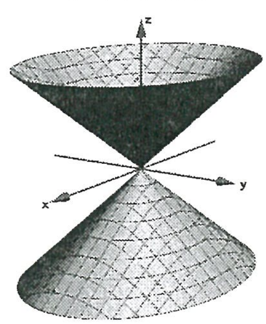
(iv) G



(v) H



(vi) C



(A) $x^2 - z^2 + 4 = 0$

(B) $x^2 + y^2 - 4 = 0$

(C) $x^2 + y^2 - z^2 = 0$

(D) $x^2 + y - z^2 = 0$

(E) $x^2 - y^2 + z = 0$

(F) $x^2 + y^2 - z^2 - 1 = 0$

(G) $x^2 + y^2 - z^2 + 1 = 0$

(H) $z - \frac{1}{1+x^2+y^2} = 0$

9. (a) (4 points) **Circle ONE.** At how many points does the curve with vector equation

$$\vec{r}(t) = \langle t\sin(t),\ t\cos(t),\ t^2\rangle, \quad \text{for } -\infty < t < \infty,$$

intersect the cylinder $x^2 + y^2 = 4$?

(A) 0 points (i.e., the curve and cylinder do not intersect)

(B) 1 point

(C) 2 points

(D) 3 points

(E) infinitely many points

(b) (4 points) **Circle ONE.** At how many points does the curve with vector equation

$$\vec{r}(t) = \langle t\sin(t),\ t\cos(t),\ t^2\rangle, \quad \text{for } t \geq 0,$$

intersect the elliptic paraboloid $z = x^2 + y^2$?

(A) 0 points (i.e., the curve and elliptic paraboloid do not intersect)

(B) 1 point

(C) 2 points

(D) 3 points

(E) infinitely many points

10. Consider

$$\text{Plane A:} \quad 3(x+2)+3y-21z=51, \quad \text{and} \quad \text{Plane B:} \quad x+y=7z.$$

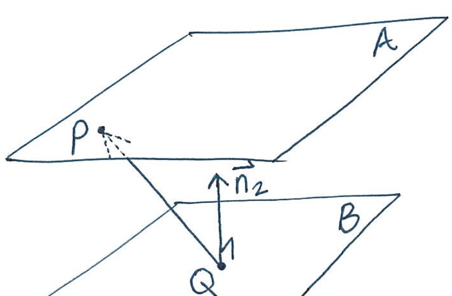
(a) (4 points) Show that the two planes are parallel.

Plane A: $3(x+2)+3y-21z=51$ has normal vector $\vec{n}_1=\langle 3,3,-21\rangle$

Plane B: $x+y=7z \Leftrightarrow x+y-7z=0$ has normal vector $\vec{n}_2=\langle 1,1,-7\rangle$

Since $3\langle 1,1,-7\rangle=\langle 3,3,-21\rangle$, that is $3\vec{n}_2=\vec{n}_1$, $\vec{n}_1$ and $\vec{n}_2$ are in the same direction, so plane A and plane B are parallel.

(b) (9 points) Find the distance between the two planes.



The projection of $\overrightarrow{QP}$ onto $\vec{n}_2$ has length the distance between the two planes.

$P(15,0,0)$ is on plane A, since $3(15+2)+3\cdot 0-21\cdot 0=51$

$Q(0,0,0)$ is on plane B, since $0+0=7\cdot 0$.

So $\overrightarrow{QP}=\langle 15,0,0\rangle$

$$\text{proj}_{\vec{n}_2}\overrightarrow{QP}=\frac{\langle 1,1,-7\rangle\cdot\langle 15,0,0\rangle}{\left(\sqrt{1^2+1^2+(-7)^2}\right)^2}\langle 1,1,-7\rangle=\frac{15}{51}\langle 1,1,-7\rangle$$

$$\left|\text{proj}_{\vec{n}_2}\overrightarrow{QP}\right|=\left|\frac{15}{51}\langle 1,1,-7\rangle\right|=\frac{15}{51}\sqrt{1^2+1^2+(-7)^2}=\boxed{\frac{15}{\sqrt{51}}}$$

11. Consider the space curves

$$\vec{r}_1(t) = \langle \sin(t),\ 1-\cos(t),\ t^3 \rangle \qquad \text{and} \qquad \vec{r}_2(s) = \langle 0,\ 2s,\ 4s \rangle.$$

(a) (4 points) Find the derivative of $\vec{r}_1(t)$.

$\vec{r}_1\,'(t) = \langle \cos(t),\ \sin(t),\ 3t^2 \rangle$

(b) (7 points) Find the angle of intersection (angle between tangent lines) of $\vec{r}_1(t)$ and $\vec{r}_2(s)$ at the point $(0,0,0)$.

When $t=0$, $\vec{r}_1(0) = \langle \sin(0), 1-\cos(0), 0^3 \rangle = \langle 0,0,0 \rangle$

$s=0$, $\vec{r}_2(0) = \langle 0, 2\cdot 0, 4\cdot 0 \rangle = \langle 0,0,0 \rangle$,

So $\vec{r}_1(t)$ and $\vec{r}_2(t)$ intersect when $s=t=0$.

When $s=0$, $\vec{r}_2\,'(s) = \langle 0,2,4 \rangle$, so $\vec{r}_2\,'(0) = \langle 0,2,4 \rangle$

$t=0$, $\vec{r}_1(t) = \langle \cos(t), \sin(t), 3t^2 \rangle$

so $\vec{r}_1(0) = \langle 1,0,0 \rangle$

Since $\vec{r}_1\,'(0) \cdot \vec{r}_2\,'(0) = \langle 1,0,0 \rangle \cdot \langle 0,2,4 \rangle = 0$,

$\vec{r}_1\,'(0)$ and $\vec{r}_2\,'(0)$ are orthogonal, that is

the angle of intersection is 90° (or $\frac{\pi}{2}$).

12. (11 points) Find a parametric representation for the part of the plane $z - x = 3$ that lies inside the cylinder $x^2 + y^2 = 1$. You must specify the bounds for the parameter(s).

$$\begin{cases} x = x \\ y = y \\ z = 3 + x \end{cases} \quad \text{with} \quad x^2 + y^2 \leq 1$$

OR

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 3 + r\cos\theta \end{cases} \quad \text{with} \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{array}$$