Math 2400, Final Exam

May 7, 2019

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50 AM
☐	Section 002	Harrison Stalvey	8:00–8:50 AM
☐	Section 003	Daniel Martin	9:00–9:50 AM
☐	Section 004	Albert Bronstein	9:00–9:50 AM
☐	Section 005	Xingzhou Yang	10:00–10:50 AM
☐	Section 006	Mark Pullins	10:00–10:50 AM
☐	Section 007	János Englander	10:00–10:50 AM
☐	Section 008	John Willis	12:00–12:50 PM
☐	Section 009	Taylor Klotz	1:00–1:50 PM
☐	Section 010	János Englander	2:00–2:50 PM
☐	Section 011	Harrison Stalvey	2:00–2:50 PM
☐	Section 012	Xingzhou Yang	3:00–3:50 PM
☐	Section 013	Trevor Jack	4:00–4:50 PM

Question	Points	Score
1	8	
2	10	
3	3	
4	10	
5	10	
6	10	
7	10	
8	10	
9	3	
10	10	
11	10	
12	6	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 150 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like $\mathbf{100/7}$ or expressions like $\mathbf{\ln(3)/2}$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (8 points) **Note: No partial credit for this problem.**

Let $\vec{\boldsymbol{a}} = \langle \mathbf{2},\ \mathbf{-2},\ \mathbf{1} \rangle$, $\vec{\boldsymbol{b}} = \langle \mathbf{1},\ \mathbf{-2},\ \mathbf{3} \rangle$. Compute

(a) $|\vec{\boldsymbol{a}}| =$ $\underline{\quad 3 \quad}$

$\sqrt{4+4+1}$

(b) $\mathbf{2}\vec{\boldsymbol{a}} - \mathbf{3}\vec{\boldsymbol{b}} =$ $\underline{\quad \langle 1, 2, -7 \rangle \quad}$

$\langle 4, -4, 2 \rangle + \langle -3, 6, -9 \rangle$

(c) $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} =$ $\underline{\quad 9 \quad}$

$2 + 4 + 3$

(d) $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} =$ $\underline{\quad \langle -4, -5, -2 \rangle \quad}$

$$\begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle -6+2,\ 1-6,\ -4+2 \rangle$$

2. (10 points) Determine if the following function is **continuous** at the origin. Explain your reasoning.

$$f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

To be continuous, the limit of f(x,y) must equal 1 for every path of inputs (x,y) that approach (0,0)

Consider the path (0,y) with y approaching 0 : $\lim\limits_{(0,y)\to(0,0)} \dfrac{0-y}{0+y} = -1 \neq 1 = f(0,0)$

Therefore, f is not continuous at (0,0)

3. (3 points) Let $\boldsymbol{D}$ be a region in the $\boldsymbol{xy}$-plane with area $\boldsymbol{A}$ and boundary curve $\boldsymbol{C}$.

Which of the following vector fields will make $\int_C \vec{F} \cdot d\vec{r} = A$? Circle your answer.

Let F = < P , Q >

By Green's Theorem: $\int_C \vec{F} \cdot \vec{dr} = \iint_A (Q_x - P_y)\, dA$

So, we want vector fields such that: $Q_x - P_y = 1$

(A) $\vec{F} = \langle x + y,\ x \rangle$

$Q_x - P_y = 1 - 1 = 0$ No

(B) $\vec{F} = \frac{1}{2} \langle x,\ x - y \rangle$

$Q_x - P_y = \frac{1}{2} - 0 = \frac{1}{2}$ No

(C) $\vec{F} = \langle y^2 - y + \cos x^2,\ 2xy \rangle$

$Q_x - P_y = 2y - (2y - 1) = 1$ Yes

(D) $\vec{F} = \langle -y,\ 2x \rangle$

$Q_x - P_y = 2 + 1 = 3$ No

4. (10 points) Given three points $\boldsymbol{A}(\mathbf{1}, -\mathbf{1}, \mathbf{1})$, $\boldsymbol{B}(\mathbf{1}, \mathbf{2}, \mathbf{1})$, $\boldsymbol{C}(\mathbf{2}, \mathbf{1}, \mathbf{2})$,

(a) Determine if the points are on the same line.

We can test if they're on the same line if any two vectors between the points are parallel, which we can test by checking if their cross product is the zero vector.

$$\vec{AB} = \langle 0, 3, 0 \rangle \qquad \begin{vmatrix} i & j & k \\ 0 & 3 & 0 \\ 1 & 2 & 1 \end{vmatrix} = \langle 3 \; \ldots$$

$$\vec{AC} = \langle 1, 2, 1 \rangle$$

we can stop right there...

Not collinear

(b) Find the equation of a plane containing all three points.

The equation of a plane is ax+by+cz=d, where <a,b,c> is a normal vector to the plane. So... we should finish that cross product from part (a)...

$$\langle a, b, c \rangle = \langle 3, 0, -3 \rangle \rightarrow 3x - 3z = d$$

Now pick d so that each of the three points makes the plane equation true.

$$3(1) - 3(1) = 0$$

Check that 3x-3z=0 works for the other two points

$$\boxed{3x - 3z = 0}$$

5. (10 points) Let $\vec{\boldsymbol{F}} = \langle -2xy,\ y,\ z\rangle$. Evaluate the line integral

$$\int_C \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}}$$

where C is given by $\vec{r}(t) = \langle t,\ \cos(2t),\ t + \sin(2t)\rangle$ for $0 \le t \le \pi$.

$$\int_0^{\pi} \langle -2t\cos(2t), \cos(2t), t+\sin(2t)\rangle \cdot \langle 1, -2\sin(2t), 1+2\cos(2t)\rangle\, dt$$

$$= \int_0^{\pi} (-2t\cos(2t) - 2\cos(2t)\sin(2t) + (t+\sin(2t))(1+2\cos(2t)))\, dt$$

$$= \int_0^{\pi} (\cancel{-2t\cos(2t)} - \cancel{2\cos(2t)\sin(2t)} + t + \cancel{2t\cos(2t)} + \sin(2t) + \cancel{2\cos(2t)\sin(2t)})\,dt$$

$$= \int_0^{\pi} (t + \sin(2t))\, dt = \left[\frac{t^2}{2} - \frac{\cos(2t)}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$$

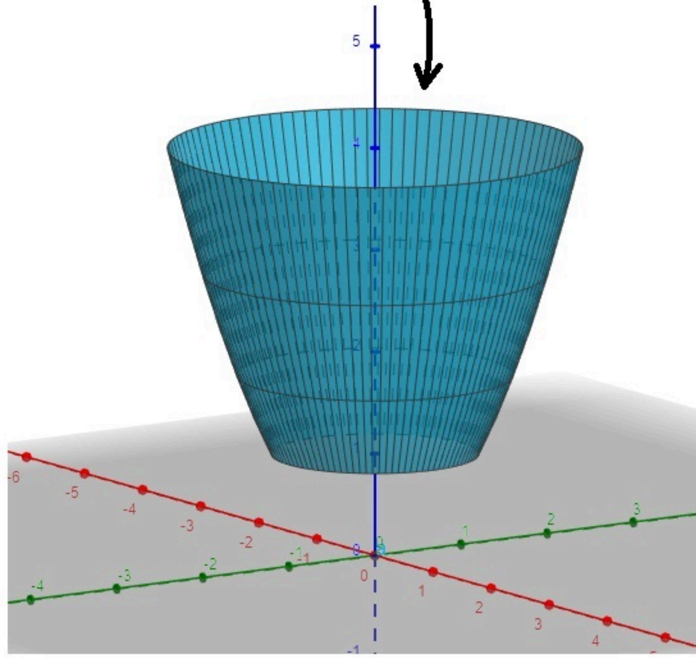
6. (10 points) $\boldsymbol{S}$ is the part of the paraboloid $\boldsymbol{z = x^2 + y^2}$ that lies between the planes $\boldsymbol{z = 1}$ and $\boldsymbol{z = 4}$.

(a) Parametrize the surface $\boldsymbol{S}$. Include the bounds for the parameter(s).

$x = r\cos(t) \qquad 0 \le t \le 2\pi$

$y = r\sin(t) \qquad 1 \le r \le 2$

$z = r^2$



(b) Use your parameterization from part (a) to rewrite the integral with appropriate bounds. You need not evaluate this rewritten integral.

$$\iint_{\boldsymbol{S}} z\,\mathrm{d}\boldsymbol{S}.$$

$$\vec{dS} = \begin{vmatrix} i & j & k \\ \cos(t) & \sin(t) & 2r \\ -r\sin(t) & r\cos(t) & 0 \end{vmatrix}$$

$$= \langle -2r^2\cos(t), -2r^2\sin(t), r \rangle$$

$$dS = \|\vec{dS}\| = \sqrt{4r^4 + r^2}$$

$$\int_0^{2\pi}\int_1^2 r^2\sqrt{4r^4+r^2}\,dr\,dt$$

7. (10 points) Use the **Method of Lagrange Multipliers** to find the maximum and minimum values of the function

$$f(x, y) = x + y$$

under the constraint $g(x, y) = 2x^2 + 2xy + y^2 = 4$.

$f_x = 1 = f_y \quad g_x = 4x + 2y \quad g_y = 2x + 2y$

$\nabla g = \lambda \nabla f \to \begin{array}{l} 4x + 2y = \lambda \\ 2x + 2y = \lambda \end{array} \to \begin{array}{l} x = 0 \\ y = \frac{\lambda}{2} \end{array}$

$g(0, y) = y^2 = 4 \longrightarrow y = \pm 2$

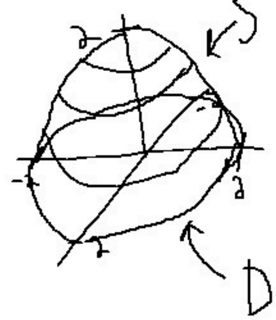
$f(0,2) = 2$ max

$f(0,-2) = -2$ min

8. (10 points) Let $\boldsymbol{S}$ be the part of the hemisphere $\boldsymbol{x^2+y^2+z^2=4}$ above the $\boldsymbol{xy}$-plane, with positive (outward) orientation and let

$$\vec{F} = \langle xz\sin(yz) + x^3,\ \cos(yz),\ 3zy^2 + z^3 - 7\rangle .$$

Find $\iint_S \vec{F}\cdot d\vec{S}$ using the **Divergence Theorem**. **Hint:** $\boldsymbol{S}$ is **not** a closed surface.



Warning: No credit if the Divergence Theorem is not used!

Divergence Theorem: $\iint_S \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{D} = \iiint_V \operatorname{div}(\vec{F})\,dV$

Note: We need D to make the left side of this equation an integral over a closed surface

$D: x = r\cos\theta \quad 0 \le r \le 2$

$y = r\sin\theta \quad 0 \le \theta \le 2\pi$

$z = 0$

$$d\vec{D} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r\rangle$$

correct orientation $\langle 0, 0, -r\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}(\vec{F})\,dV - \iint_D \vec{F}\cdot d\vec{D}$$

$$\operatorname{div}(\vec{F}) = \cancel{z\sin(yz)} + 3x^2 - \cancel{z\sin(yz)} + 3y^2 + 3z^2$$

$$= 3x^2 + 3y^2 + 3z^2 = 3\rho^2$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 3\rho^2 \rho^2 \sin(\alpha)\,d\rho\,d\alpha\,d\theta - \int_0^{2\pi}\int_0^2 \vec{F}\cdot\langle 0,0,-r\rangle\,dr\,d\theta$$

don't matter after dot product

$$= 2\pi \int_0^{\pi/2}\int_0^2 3\rho^4 \sin\alpha\,d\rho\,d\alpha - \int_0^{2\pi}\int_0^2 \langle ?, ?, -7\rangle\cdot\langle 0,0,-r\rangle\,dr\,d\theta$$

$$= 6\pi \int_0^{\pi/2} \left[\frac{\rho^5}{5}\sin\alpha\right]_0^2 d\alpha - 2\pi\int_0^2 7r\,dr$$

$$= 6\pi\cdot\frac{32}{5}\left[-\cos\alpha\right]_0^{\pi/2} - 2\pi\left[\tfrac{7}{2} r^2\right]_0^2$$

$$= 6\pi\cdot\frac{32}{5}\cdot 1 - 28\pi$$

9. (3 points) Let $\boldsymbol{f(x, y, z) = x^2 + 2y^2 + 3z^2}$. Find an equation of the tangent plane to the level surface $\boldsymbol{x^2 + 2y^2 + 3z^2 = 30}$ at the point $\boldsymbol{(1, -1, 3)}$. Circle the correct answer:

Equation of plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

(A) $\boldsymbol{2x - 4y + 18z = 0}$

$a = f_x(1,-1,3)$, $b = f_y(1,-1,3)$, $c = f_z(1,-1,3)$

$= 2(1)$ $\quad = 4(-1)$ $\quad = 6(3)$

$2(x-1) - 4(y+1) + 18(z-3) = 0$

(B) $\boldsymbol{2x + 4y + 18z = 0}$

(C) $\boldsymbol{2(x - 1) - 4(y + 1) + 18(z - 3) = 0}$

(D) $\boldsymbol{2(x + 1) - 4(y - 1) + 18(z + 3) = 0}$

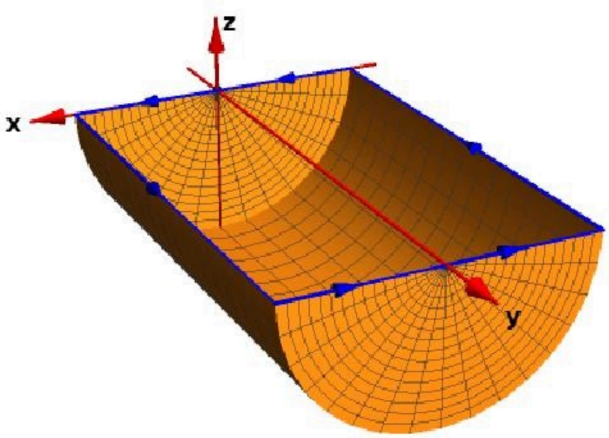
10. (10 points) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x, y, z) = \left\langle e^{x^2} + 2y,\ x - \sin^2 y,\ e^x \cos y + \sin(e^z) \right\rangle$$

and C is the boundary of the surface consisting of the half cylinder given by

$$\vec{r}(u, v) = \langle \cos u,\ v,\ -\sin u \rangle$$

where $0 \leq u \leq \pi$, $0 \leq v \leq 4$, along with the two semi-circle caps at $y = 0$ and $y = 4$. C is oriented counter-clockwise viewed from top (the positive z direction).



Green's Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_A (Q_x - P_y)\, dA$

A: $-1 \leq x \leq 1$, $0 \leq y \leq 4$, $z = 0$

$Q_x = 1$, $P_y = 2$

$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \int_0^4 (1 - 2)\, dy\, dx = -8$

Stokes' Theorem: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl}(\vec{F}) \cdot d\vec{S} \longleftarrow S$ is A, $d\vec{S} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle$

Use the simplest surface that has r as its boundary. Don't use the cylinder. Use the rectangle.

$= \int_{-1}^{1} \int_0^4 \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle\, dy\, dx$

$= \int_{-1}^{1} \int_0^4 (Q_x - P_y)\, dy\, dx$

Green's Theorem is just a special case of Stokes' Theorem when your surface is in the xy-plane

11. (10 points) Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ and f a scalar function on $\mathbb{R}^3$ such that each have continuous third order derivatives.

Determine if the following are always, sometimes, or never true.

curl $(\nabla f) = \vec{0}$

(a) If $f(x, y, z)$ is a scalar function then $\nabla \times (\nabla \times (\underbrace{\nabla f}_{\text{conservative}})) = \vec{0}$.

(A) Always true

(B) Sometimes true

(C) Never true

Curl of a conservative vector field is the zero vector field.
Curl of the zero vector field is the zero vector.

(b) If $\vec{F}(x, y, z)$ is any vector field then $\underbrace{\nabla \cdot (\nabla \times \vec{F})}_{} = 0$.

(A) Always true

(B) Sometimes true

(C) Never true

div(curl($\vec{F}$)) = 0

(c) div(grad f) $= 0$.

(A) Always true

(B) Sometimes true

(C) Never true

f(x,y,z) = x + y + z
grad(f) = <1,1,1>
div(grad(f)) = 0

f(x,y,z) = $x^2 + y^2 + z^2$
grad(f) = <2x,2y,2z>
div(grad(f)) = 6

vector field $= \vec{v}$; div(curl($\vec{v}$)) = 0 ...... NOT <0,0,0>

scalar!

(d) $\text{div}(\text{curl}(\overbrace{\text{curl}(\text{curl}\,\vec{F})}^{\text{vector field}})) = \langle 0,\ 0,\ 0 \rangle$.

(A) Always true

(B) Sometimes true

(C) Never true

(e) $\nabla \cdot (f\vec{F}) = \nabla f \cdot \vec{F} - f\, \nabla \cdot \vec{F}$, where f and div$\vec{F}$ are not zero.

(A) Always true

(B) Sometimes true

(C) Never true

$\nabla \cdot (f\vec{F}) = \nabla f \cdot \vec{F} + f(\nabla \cdot \vec{F})$ like the product rule

$\stackrel{?}{=} \nabla f \cdot \vec{F} - f(\nabla \cdot f)$ only if $f(\nabla \cdot \vec{F}) = 0$

since $f \neq 0$, $\nabla \cdot \vec{F} \neq 0$

12. (6 points) Suppose that $\boldsymbol{f}$ is a differentiable function of $\boldsymbol{x}$ and $\boldsymbol{y}$, and

$$\boldsymbol{g(u,v) = f(\underbrace{u^2 - v^3}_{x},\ \underbrace{u + 2v}_{y})}.$$

$x_u = 2u \qquad y_u = 1$

$x_v = -3v^2 \qquad y_v = 2$

	$f(x,y)$	$g(u,v)$	$f_x(x,y)$	$f_y(x,y)$
$(3,4)$	2	-1	8	6
$(2,1)$	-2	2	3	1

(a) Use the table of values to compute $\boldsymbol{g_u(2,1)}$.

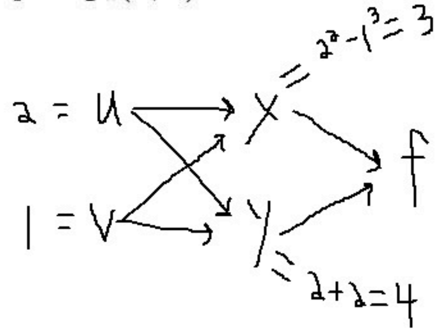
(A) -6

(B) 7

(C) 13

(D) 14

(E) 38



Chain rule: $g_u(2,1) = f_x(3,4)\cdot x_u(2,1) + f_y(3,4)\cdot y_u(2,1)$

$= 8\cdot 4 + 6\cdot 1$

(b) If $\boldsymbol{g_v(1,-1) = 5}$ and $\boldsymbol{f_x(2,-1) = 1}$, compute $\boldsymbol{f_y(2,-1)}$.

$x = 1^2 - (-1)^3 = 2$

$y = 1 + 2(-1) = -1$

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

Chain rule: $g_v(1,-1) = f_x(2,-1)\cdot x_v(1,-1) + f_y(2,-1)\cdot y_v(1,-1)$

$5 = 1\cdot -3 + f_y(2,-1)\cdot 2$

$8 = 2 f_y(2,-1)$