Solutions Math 2400, Midterm 3

April 15, 2019

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50 AM
☐	Section 002	Harrison Stalvey	8:00–8:50 AM
☐	Section 003	Daniel Martin	9:00–9:50 AM
☐	Section 004	Albert Bronstein	9:00–9:50 AM
☐	Section 005	Xingzhou Yang	10:00–10:50 AM
☐	Section 006	Mark Pullins	10:00–10:50 AM
☐	Section 007	János Englander	10:00–10:50 AM
☐	Section 008	John Willis	12:00–12:50 PM
☐	Section 009	Taylor Klotz	1:00–1:50 PM
☐	Section 010	János Englander	2:00–2:50 PM
☐	Section 011	Harrison Stalvey	2:00–2:50 PM
☐	Section 012	Xingzhou Yang	3:00–3:50 PM
☐	Section 013	Trevor Jack	4:00–4:50 PM

Question	Points	Score
1	12	
2	4	
3	15	
4	15	
5	14	
6	12	
7	15	
8	5	
9	4	
10	4	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 95 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like ln(3)/2 as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (12 points) Match the vector fields $\vec{\boldsymbol{F}}$ with the plots below.

(1) $\vec{\boldsymbol{F}} = \langle y, x \rangle$ $\underline{(A)}$

(2) $\vec{\boldsymbol{F}} = \langle x, -y \rangle$ $\underline{(C)}$

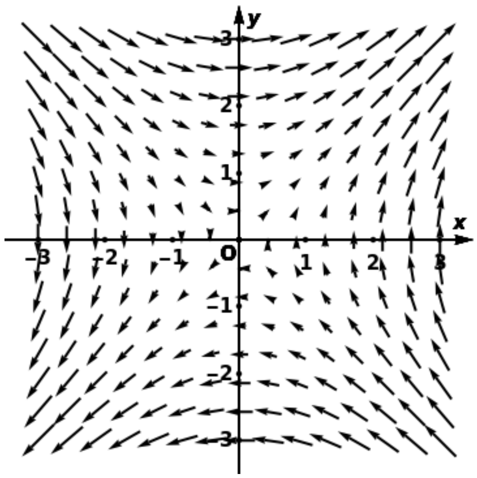
(3) $\vec{\boldsymbol{F}} = \langle y, -x \rangle$ $\underline{(D)}$

(4) $\vec{\boldsymbol{F}} = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, -\dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$ $\underline{(F)}$

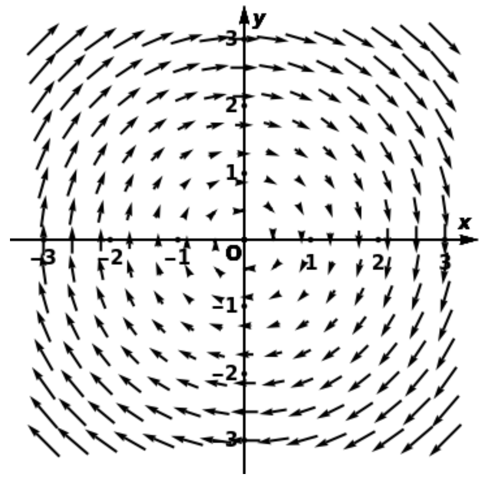
(5) $\vec{\boldsymbol{F}} = \langle 2, x-y \rangle$ $\underline{(E)}$

(6) $\vec{\boldsymbol{F}} = \langle -y, -x \rangle$ $\underline{(B)}$

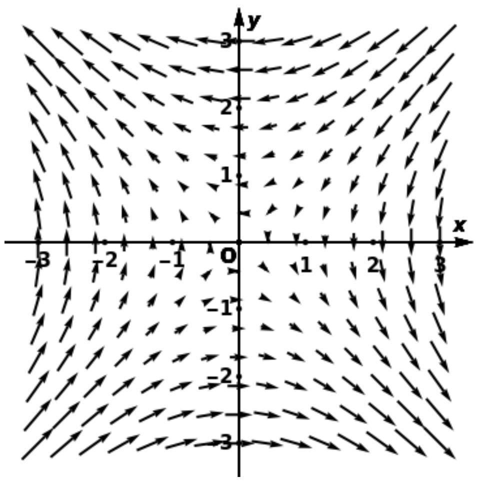
(A)



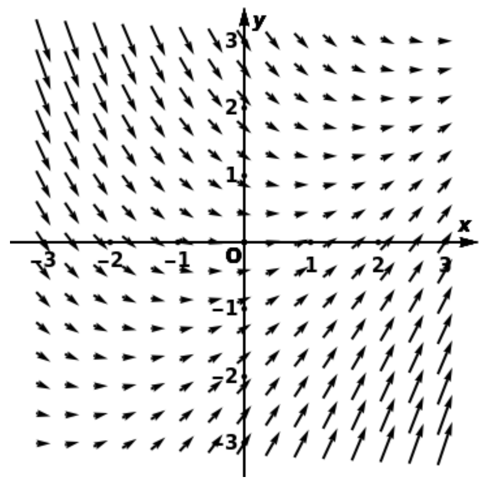
(D)



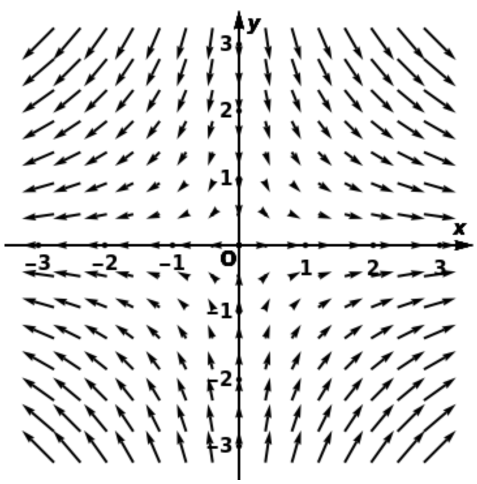
(B)



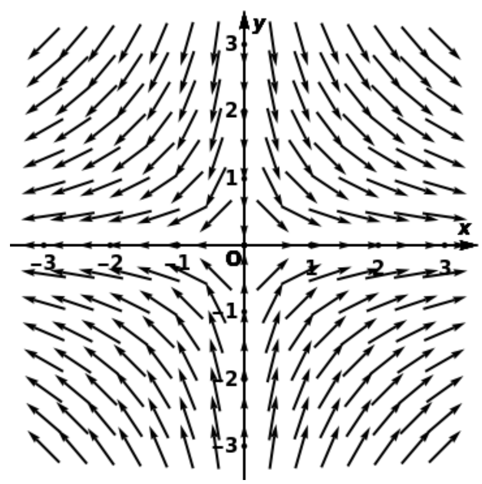
(E)



(C)



(F)



2. (4 points) **No partial credit** for this problem.

Which of the following is the center of mass of the region R bounded by $y = \sqrt{1-x^2}$ and $y = 0$ with density function $\rho(x, y) = x^2 y$? Suppose m is the mass of R.

(A) $(\bar{x}, \bar{y}) = \left(0,\ \dfrac{1}{m}\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos^2\theta\sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\right)$

(B) $(\bar{x}, \bar{y}) = \left(0,\ \dfrac{1}{m}\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1 r^4\cos^2\theta\sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\right)$

(C) $(\bar{x}, \bar{y}) = \left(0,\ \dfrac{1}{m}\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1 r^5\cos^2\theta\sin^2\theta\ \mathrm{d}r\ \mathrm{d}\theta\right)$

(D) $(\bar{x}, \bar{y}) = \left(0,\ \dfrac{1}{m}\displaystyle\int_0^{\pi}\int_0^1 r^3\cos^2\theta\sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\right)$

(E) $(\bar{x}, \bar{y}) = \left(0,\ \dfrac{1}{m}\displaystyle\int_0^{\pi}\int_0^1 r^4\cos^2\theta\sin^2\theta\ \mathrm{d}r\ \mathrm{d}\theta\right)$

(F) $\boxed{(\bar{x}, \bar{y}) = \left(0,\ \dfrac{1}{m}\displaystyle\int_0^{\pi}\int_0^1 r^5\cos^2\theta\sin^2\theta\ \mathrm{d}r\ \mathrm{d}\theta\right)}$

3. (15 points) Let $\vec{\boldsymbol{F}}(x,y) = \left\langle ye^x,\, e^x + 3y^2 \right\rangle$

$$\vec{\boldsymbol{F}}(x,y) = \left\langle ye^x,\, e^x + 3y^2 \right\rangle$$

(a) Is $\vec{\boldsymbol{F}}(x,y)$ conservative? If so, find a potential function. If not, explain why not.

Solution: Let $P(x,y) = ye^x$, $Q(x,y) = e^x + 3y^2$. $Q_x = \dfrac{\partial}{\partial x}\left(e^x + 3y^2\right) = e^x$. $P_y = \dfrac{\partial}{\partial y}\left(e^x + 3y^2\right) = e^x$. Since $Q_x = P_y$ for all $(x,y) \in \mathbb{R}^2$, $\vec{\boldsymbol{F}}$ is conservative.

Solve $\vec{\boldsymbol{F}} = \nabla f \iff \langle P, Q\rangle = \langle f_x, f_y\rangle \iff \begin{cases} f_x = ye^x & ① \\ f_y = e^x + 3y^2 & ② \end{cases}$ for f to get a potential function. Integrate both sides of ① with respect to x, and we have

$$\int f_x \,\mathrm{d}x = \int ye^x \,\mathrm{d}x \Longrightarrow f(x,y) = ye^x + C(y) \quad ③$$

To solve for $C(y)$ in ③, we plug ③ into ①, and then

$$f_y = \frac{\partial}{\partial y}\left(ye^x + C(y)\right) = e^x + C'(y) \overset{②}{=} e^x + 3y^2$$

So $C'(y) = 3y^2$ and hence $C(y) = y^3 + K$, where K is an arbitrary constant. Plug it back into ③ and we get a potential function for $\vec{\boldsymbol{F}}$ as follows,

$$\boxed{f(x,y) = ye^x + y^3 + K, \qquad \text{where } K \text{ can be any real number}}$$

(b) Suppose C is the piecewise smooth curve given by

$$\vec{\boldsymbol{r}}(t) = \begin{cases} \left\langle e^{t^2} - 1,\, e^{t^4} - 1 \right\rangle, & 0 \le t \le 2 \\ \left\langle e^{2t} - 1,\, e^{8t} - 1 \right\rangle, & 2 < t \le 3 \end{cases}$$

Calculate $\displaystyle\int_C \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}}$. Justify your answer.

Solution: From (a), $\vec{\boldsymbol{F}}$ is conservative. C is a piece-wise smooth curve with initial and terminal points at $\vec{\boldsymbol{r}}(0) = \left\langle e^0 - 1,\, e^0 - 1\right\rangle = \langle 0, 0\rangle$, $\vec{\boldsymbol{r}}(3) = \left\langle e^6 - 1,\, e^{24} - 1\right\rangle$, respectively. By the **Fundamental Theorem of Line Integrals**, $f(x,y) = ye^x + y^3$,

$$\begin{aligned}
\int_C \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_C \nabla f \cdot \mathrm{d}\vec{\boldsymbol{r}} &= f\left(\vec{\boldsymbol{r}}(3)\right) - f\left(\vec{\boldsymbol{r}}(2)\right) + f\left(\vec{\boldsymbol{r}}(2)\right) - f\left(\vec{\boldsymbol{r}}(0)\right) \\
&= f\left(\vec{\boldsymbol{r}}(3)\right) - f\left(\vec{\boldsymbol{r}}(0)\right) = f\left(e^6 - 1,\, e^{24} - 1\right) - f(0,\, 0) \\
&= \left[(e^{24} - 1)e^{e^6 - 1} + (e^{24} - 1)^3\right] - \left[(0)e^0 + (0)^3\right] \\
&= \boxed{(e^{24} - 1)e^{e^6 - 1} + (e^{24} - 1)^3}
\end{aligned}$$

4. (15 points) Evaluate

$$\iint\limits_D \cos(x^2+y^2)\,\mathrm{d}A,$$

where D is the region bounded by the circle $x^2+y^2=\dfrac{\pi}{2}$ in the **first** quadrant.

Solution: Use polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and then the region is

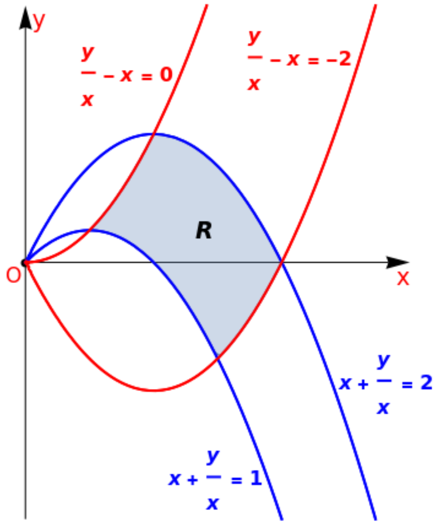
$$\left\{(r,\theta)\middle|0 \le r \le \sqrt{\frac{\pi}{2}},\ 0 \le \theta \le \frac{\pi}{2}\right\}$$

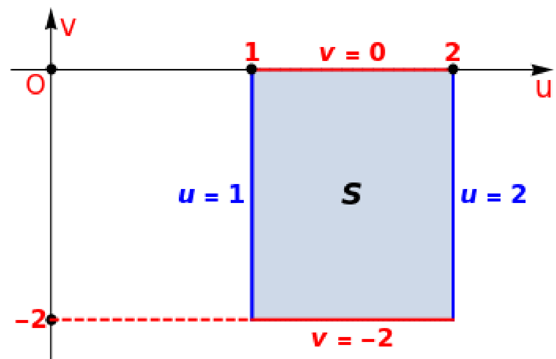
$$\begin{aligned}
\iint\limits_D \cos(x^2+y^2)\,\mathrm{d}A &= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\frac{\pi}{2}}} \cos(r^2)\, r\ \mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\sqrt{\frac{\pi}{2}}} \cos(r^2)\, r\ \mathrm{d}r \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\cos u\ \mathrm{d}u = \frac{\pi}{4}\ \sin u\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}\left(\sin\frac{\pi}{2} - \sin 0\right) \\
&= \boxed{\frac{\pi}{4}}
\end{aligned}$$

5. (14 points) Let R be the region in the xy-plane (see the graph below) bounded by the curves

$$\frac{y}{x}+x=1, \quad \frac{y}{x}+x=2, \quad \frac{y}{x}-x=-2, \quad \text{and } \frac{y}{x}-x=0.$$

(a) Find a transformation of the form $u=g(x,y)$, $v=h(x,y)$ that maps R onto a rectangle S in the uv-plane. Sketch the rectangle S in the uv-plane.





Solution: $R=\left\{(x,y)\middle|1\le \frac{y}{x}+x\le 2,\ -2\le\frac{y}{x}-x\le 0\right\}$. Let $\boxed{u=\frac{y}{x}+x,\ v=\frac{y}{x}-x}$. The transformation maps R onto a rectangle $S=\{(u,v)\big|1\le u\le 2,\ -2\le v\le 0$. The graph is plotted in the right figure above.

(b) Use the result in (a) to find an appropriate Jacobian, and use it to evaluate the integral

$$\iint_R \frac{1}{x}\sin\left(\frac{y}{x}+x\right)\,\mathrm{d}A$$

Solution: Since $u=\frac{y}{x}+x$, $v=\frac{y}{x}-x$, $u-v=2x$, and so $x=\frac{1}{2}(u-v)$. $u+v=\frac{2y}{x}$.
So $y=\frac{1}{2}(u+v)x=\frac{1}{2}(u+v)\cdot\frac{1}{2}(u-v)=\frac{1}{4}(u^2-v^2)$.

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}u\\ -\frac{1}{2} & -\frac{1}{2}v\end{vmatrix}=\left(-\frac{y}{x^2}+1\right)\frac{1}{x}-\frac{1}{x}\left(-\frac{y}{x^2}-1\right)=\frac{1}{4}(u-v)$$

$$\iint_R\frac{1}{x}\sin\left(\frac{y}{x}+x\right)\,\mathrm{d}A=\iint_S\frac{2}{u-v}\sin u\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}A'=\iint_S\frac{2}{u-v}\sin u\cdot\frac{1}{4}(u-v)\,\mathrm{d}A'$$

$$=\frac{1}{2}\int_1^2\int_{-2}^0\sin u\,\mathrm{d}v\,\mathrm{d}u=\frac{1}{2}\int_1^2\sin u\,\mathrm{d}u\int_{-2}^0\mathrm{d}v=\frac{1}{2}(-\cos u)\Big|_1^2\cdot v\Big|_{-2}^0$$

$$=\boxed{\cos 1-\cos 2}$$

Solution 2:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\dfrac{y}{x^2}+1 & \dfrac{1}{x} \\ -\dfrac{y}{x^2}-1 & \dfrac{1}{x} \end{vmatrix} = \left(-\frac{y}{x^2}+1\right)\frac{1}{x} - \frac{1}{x}\left(-\frac{y}{x^2}-1\right) = \frac{2}{x}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{\frac{2}{x}} = \frac{1}{4}(u-v)$$

$$\iint_R \frac{1}{x}\sin\left(\frac{y}{x}+x\right)\,\mathrm{d}A = \iint_S \frac{1}{x}\sin\left(\frac{y}{x}+x\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}A' = \iint_S \frac{1}{x}\sin u \cdot \frac{x}{2}\mathrm{d}A'$$

$$= \frac{1}{2}\iint_S \sin u\,\mathrm{d}A' = \frac{1}{2}\int_1^2\int_{-2}^0 \sin u\,\mathrm{d}v\,\mathrm{d}u = \frac{1}{2}\int_1^2 \sin u\,\mathrm{d}u\int_{-2}^0 \mathrm{d}v$$

$$= \frac{1}{2}(-\cos u)\Big|_1^2 \cdot v\Big|_{-2}^0 = \boxed{\cos 1 - \cos 2}$$

6. (12 points) Evaluate the following triple integral by changing the order of integration.

$$\int_0^{\sqrt{\pi}}\int_z^{\sqrt{\pi}}\int_{x^2}^{\pi}\cos(x^2+y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

Hint: You may have to switch the order of integration at some point.

Solution: $D_{xz} = \left\{(x,z)\middle|0\le z\le\sqrt{\pi},\, z\le x\le\sqrt{\pi}\right\} = \left\{(x,z)\middle|0\le z\le x,\, 0\le x\le\sqrt{\pi}\right\}$

$$\int_0^{\sqrt{\pi}}\int_z^{\sqrt{\pi}}\int_{x^2}^{\pi}\cos(x^2+y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z = \int_0^{\sqrt{\pi}}\int_z^{\sqrt{\pi}} \sin(x^2+y)\Big|_{y=x^2}^{y=\pi}\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^{\sqrt{\pi}}\int_z^{\sqrt{\pi}}\left[\sin(x^2+\pi)-\sin(2x^2)\right]\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^{\sqrt{\pi}}\int_z^{\sqrt{\pi}}\left[-\sin(x^2)-\sin(2x^2)\right]\mathrm{d}x\,\mathrm{d}z \quad \textbf{(switch order of integration)}$$

$$= \int_0^{\sqrt{\pi}}\int_0^{x}\left[-\sin(x^2)-\sin(2x^2)\right]\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^{\sqrt{\pi}}\left[-\sin(x^2)-\sin(2x^2)\right] z\Big|_{z=0}^{z=x}\mathrm{d}x$$

$$= -\int_0^{\sqrt{\pi}}\left[\sin(x^2)+\sin(2x^2)\right] x\,\mathrm{d}x \underset{\mathrm{d}u=2x\,\mathrm{d}x}{\overset{u=x^2}{=\!=\!=}} -\frac{1}{2}\int_0^{\pi}\left[\sin u+\sin(2u)\right]\mathrm{d}u$$

$$= \frac{1}{2}\left[\cos u + \frac{1}{2}\cos(2u)\right]_0^{\pi} = \frac{1}{2}\left[\cos\pi+\frac{1}{2}\cos(2\pi)\right] - \frac{1}{2}\left[\cos 0+\frac{1}{2}\cos(0)\right]$$

$$= -\frac{1}{4}-\frac{3}{4} = \boxed{-1}$$

7. (15 points) Let C be the line segment in $\mathbb{R}^3$ starting from the point $(-1,-1,-2)$ and ending at the point $(2,3,0)$.

(a) Compute $\displaystyle\int_C (x^2+y)\,\mathrm{d}s$

Solution: The line segment equation of C is

$$\vec{\boldsymbol{r}}(t) = (1-t)\langle -1,-1,-2\rangle + t\langle 2,3,0\rangle \text{ or}$$
$$\vec{\boldsymbol{r}}(t) = \langle 3t-1,\ 4t-1,\ 2t-2\rangle, \text{ where } 0 \le t \le 1$$

So $x(t) = 3t-1$, $y(t) = 4t-1$, $z(t) = 2t-2$.

$$\begin{aligned}
\vec{\boldsymbol{r}}\,'(t) &= \frac{\partial}{\partial t}\langle 3t-1,\ 4t-1,\ 2t-2\rangle = \langle 3,4,2\rangle \\
|\vec{\boldsymbol{r}}\,'(t)| &= \sqrt{(3)^2+(4)^2+(2)^2} = \sqrt{29} \\
x^2+y &= (3t-1)^2 + (4t-1) = 9t^2-2t \\
\int_C (x^2+y)\,\mathrm{d}s &= \int_0^1 \big((x(t))^2+y(t)\big)\,|\vec{\boldsymbol{r}}\,'(t)|\,\mathrm{d}t \\
&= \int_0^1 \big(9t^2-2t\big)\sqrt{29}\,\mathrm{d}t \\
&= \sqrt{29}\,\big(3t^3-t^2\big)\Big|_0^1 = \sqrt{29}(3-1) \\
&= \boxed{2\sqrt{29}}
\end{aligned}$$

(b) Compute $\displaystyle\int_C \vec{\boldsymbol{F}}\cdot \mathrm{d}\vec{\boldsymbol{r}}$, where $\vec{\boldsymbol{F}} = \langle x-y,\ z,\ -y\rangle$.

Solution: We get the line segment equation of C from (a).

$$\begin{aligned}
\vec{\boldsymbol{r}}(t) &= \langle 3t-1,\ 4t-1,\ 2t-2\rangle \\
\mathrm{d}\vec{\boldsymbol{r}}(t) &= \vec{\boldsymbol{r}}\,'(t)\,\mathrm{d}t = \langle 3,4,2\rangle\,\mathrm{d}t \\
\vec{\boldsymbol{F}} &= \langle x-y,\ z,\ -y\rangle = \langle (3t-1)-(4t-1),\ 2t-2,\ -(4t-1)\rangle \\
&= \langle -t,\ 2t-2,\ 1-4t\rangle \\
\vec{\boldsymbol{F}}\cdot \mathrm{d}\vec{\boldsymbol{r}} &= \langle -t,\ 2t-2,\ 1-4t\rangle \cdot \langle 3,4,2\rangle\,\mathrm{d}t \\
&= [(-t)(3) + (2t-2)(4) + (1-4t)(2)]\,\mathrm{d}t \\
&= (-3t-6)\mathrm{d}t = -3(t+2)\mathrm{d}t \\
\int_C \vec{\boldsymbol{F}}\cdot \mathrm{d}\vec{\boldsymbol{r}} &= \int_0^1 -3(t+2)\mathrm{d}t = -3\left(\frac{t^2}{2}+2t\right)\Big|_0^1 = \boxed{-\frac{15}{2}}
\end{aligned}$$

8. (5 points) There exists a domain R in the 4th quadrant of area 2 in the xy-plane that we are too lazy to write down. Can you still compute the surface area of the part of the plane $z = 2x - 2y$ above the domain R? If so, do so.

Solution: Yes. Denote the surface above the plane by S.

$$z_x = \frac{\partial}{\partial x}(2x - 2y) = 2,\ z_y = \frac{\partial}{\partial y}(2x - 2y) = -2.$$

Then the surface area is

$$\begin{aligned} A(S) &= \iint_R \sqrt{1 + z_x^2 + z_y^2}\, \mathrm{d}A = \iint_R \sqrt{1 + (2)^2 + (-2)^2}\, \mathrm{d}A = \iint_R \sqrt{9}\, \mathrm{d}A \\ &= 3 \iint_R \mathrm{d}A = 3A(D) = 3(2) = \boxed{6} \end{aligned}$$

9. (4 points) **No partial credit** for this problem.

Given

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x,$$

find the limits of integration when the order of integration is changed to $\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$.

(A) $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$

(B) $\boxed{\displaystyle\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x}$

(C) $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \int_{\sqrt{x}}^{1-\sqrt{z}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$

(D) $\displaystyle\int_0^1 \int_{1-\sqrt{x}}^1 \int_{\sqrt{x}}^{\sqrt{z}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$

(E) $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \int_{1-z}^{\sqrt{x}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$

(F) $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \int_{1-z}^{\sqrt{x}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$

10. (4 points) **No partial credit** for this problem.

Use spherical coordinates to express the following sum of integrals as a single integral:

$$\int_{-2\sqrt{2}}^{0}\int_{0}^{\sqrt{8-x^2}}\int_{-\sqrt{8-x^2-y^2}}^{0} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{2}\int_{x}^{\sqrt{8-x^2}}\int_{-\sqrt{8-x^2-y^2}}^{0} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

(A) $\displaystyle\int_0^{\frac{\pi}{4}}\int_{\frac{\pi}{4}}^{\pi}\int_0^{2\sqrt{2}} \rho^3\sin^2(\phi)\cos(\theta)\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

(B) $\boxed{\displaystyle\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^{2\sqrt{2}} \rho^3\sin^2\phi\cos\theta\ \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi}$

(C) $\displaystyle\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\pi}\int_0^{2\sqrt{2}} \rho^3\cos^2(\phi)\sin(\theta)\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

(D) $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^{2\sqrt{2}} \rho^3\cos^2(\phi)\sin(\theta)\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

(E) $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^{2\sqrt{2}} \rho^3\sin(\phi)\sin^2(\theta)\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$

(F) $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^{2\sqrt{2}} \rho^3\sin^3(\phi)\cos(\theta)\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$