Solutions to Math 2400, Midterm 1

February 11, 2019

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50 AM
☐	Section 002	Harrison Stalvey	8:00–8:50 AM
☐	Section 003	Daniel Martin	9:00–9:50 AM
☐	Section 004	Albert Bronstein	9:00–9:50 AM
☐	Section 005	Xingzhou Yang	10:00–10:50 AM
☐	Section 006	Mark Pullins	10:00–10:50 AM
☐	Section 007	János Englander	10:00–10:50 AM
☐	Section 008	John Willis	12:00–12:50 PM
☐	Section 009	Taylor Klotz	1:00–1:50 PM
☐	Section 010	János Englander	2:00–2:50 PM
☐	Section 011	Harrison Stalvey	2:00–2:50 PM
☐	Section 012	Xingzhou Yang	3:00–3:50 PM
☐	Section 013	Trevor Jack	4:00–4:50 PM

Question	Points	Score
1	10	
2	4	
3	4	
4	12	
5	12	
6	6	
7	11	
8	10	
9	11	
10	10	
11	10	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 95 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like ln(3)/2 as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (10 points) **Note: No partial credit for this problem.**

Let $\vec{a} = \langle -3,\ 4,\ 0\rangle$, $\vec{b} = \langle 1,\ -3,\ -1\rangle$. Compute

(a) $|\vec{a}| = \underline{5}$

> **Solution:** $|\vec{a}| = \sqrt{(-3)^2 + (4)^2 + (0)^2} = \sqrt{25} = \boxed{5}$

(b) $3\vec{a} - 2\vec{b} = \underline{\langle -11, 18, 2\rangle}$

> **Solution:**
>
> $$3\vec{a} - 2\vec{b} = 3\langle -3,\ 4,\ 0\rangle - 2\langle 1,\ -3,\ -1\rangle = \langle -9, 12, 0\rangle + \langle -2, 6, 2\rangle = \boxed{\langle -11, 18, 2\rangle}$$

(c) The angle between $\vec{a}$ and $\vec{b}$ = $\underline{\arccos\left(-\frac{3}{\sqrt{11}}\right) = \pi - \arccos\left(\frac{3}{\sqrt{11}}\right)}$

> **Solution:** $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
>
> $$\vec{a}\cdot\vec{b} = \langle -3,\ 4,\ 0\rangle \cdot \langle 1,\ -3,\ -1\rangle = (-3)(1) + (4)(-3) + (0)(-1) = -15$$
> $$|\vec{b}| = \sqrt{(1)^2 + (-3)^2 + (-1)^2} = \sqrt{1+9+1} = \sqrt{11}$$
> $$\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right) = \arccos\left(\frac{-15}{(5)\sqrt{11}}\right) = \boxed{\arccos\left(-\frac{3}{\sqrt{11}}\right) = \pi - \arccos\left(\frac{3}{\sqrt{11}}\right)}$$

(d) $\vec{a}\times\vec{b} = \underline{\langle -4, -3, 5\rangle}$

> **Solution:**
>
> $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & 0 \\ 1 & -3 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 4 & 0 \\ -3 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 0 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 4 \\ 1 & -3 \end{vmatrix}$$
> $$= \vec{i}(-4-0) - \vec{j}(3-0) + \vec{k}(9-4) = -4\vec{i} - 3\vec{j} + 5\vec{k} = \boxed{\langle -4, -3, 5\rangle}$$

(e) $\text{proj}_{\vec{a}}\vec{b} = \underline{-\frac{3}{5}\langle -3,\ 4,\ 0\rangle = \left\langle \frac{9}{5}, -\frac{12}{5}, 0\right\rangle}$

> **Solution:**
>
> $$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{-15}{5^2}\langle -3,\ 4,\ 0\rangle = \boxed{-\frac{3}{5}\langle -3,\ 4,\ 0\rangle = \left\langle \frac{9}{5}, -\frac{12}{5}, 0\right\rangle}$$

2. (4 points) **Note: No partial credit for this problem.**

The **area** of the triangle with vertices $(a, 0, 0), (0, 2a, 0)$ and $(0, 0, 3a)$ is:

(a) $\frac{3a^2}{2}$

(b) $5a^2$

(c) $\frac{7a^2}{2}$

(d) $6a^2$

(e) $\frac{3a^3}{2}$

Solution: Denote the 3 vertices by $A(a, 0, 0)$, $B(0, 2a, 0)$, and $C(0, 0, 3a)$, respectively. Then the area of the triangle is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 0, 2a, 0\rangle - \langle a, 0, 0\rangle = \langle -a, 2a, 0\rangle$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \langle 0, 0, 3a\rangle - \langle a, 0, 0\rangle = \langle -a, 0, 3a\rangle$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & 2a & 0 \\ -a & 0 & 3a \end{vmatrix} = \vec{i}\begin{vmatrix} 2a & 0 \\ 0 & 3a \end{vmatrix} - \vec{j}\begin{vmatrix} -a & 0 \\ -a & 3a \end{vmatrix} + \vec{k}\begin{vmatrix} -a & 2a \\ -a & 0 \end{vmatrix}$$
$$= \vec{i}(6a^2 - 0) - \vec{j}(-3a^2 - 0) + \vec{k}(0 + 2a^2) = \left\langle 6a^2, 3a^2, 2a^2 \right\rangle$$
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |\left\langle 6a^2, 3a^2, 2a^2 \right\rangle| = a^2 |\langle 6, 3, 2\rangle| = a^2\sqrt{6^2 + 3^2 + 2^2}$$
$$= a^2\sqrt{36 + 9 + 4} = a^2\sqrt{49} = 7a^2$$
$$\textbf{area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \boxed{\frac{7a^2}{2}}$$

3. (4 points) **Note: No partial credit for this problem.**

Let $\vec{a} = \langle -1, 2, 1\rangle$, $\vec{b} = \langle 1, -1, 1\rangle$, and $\vec{c} = \langle -2, -2, 1\rangle$. Compute the volume of the parallelepiped formed by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

(a) 9

(b) 10

(c) -10

(d) 11

(e) -11

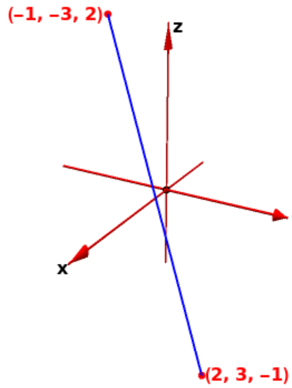
Solution: The volume of the parallelipiped determined by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\text{Volume} = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right| \overset{\text{or}}{=} |(\vec{a} \times b) \cdot \vec{c})| \overset{\text{or}}{=} \left|(\vec{c} \times a) \cdot \vec{b})\right|$$
$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -2 & -2 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ -2 & -2 \end{vmatrix}$$
$$= \vec{i}(-1 + 2) - \vec{j}(1 + 2) + \vec{k}(-2 - 2) = \vec{i} - 3\vec{j} - 4\vec{k}$$
$$= \langle 1, -3, -4\rangle$$
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle -1, 2, 1\rangle \cdot \langle 1, -3, -4\rangle = (-1)(1) + (2)(-3) + (1)(-4) = -11$$
$$\textbf{volume} = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right| = \boxed{11}$$

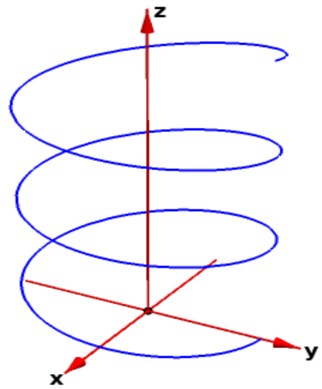
Note: $\vec{a} \times \vec{b} = \langle 3, 2, -1\rangle$, $\vec{a} \times \vec{c} = \langle 4, -1, 6\rangle$. $|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |(\vec{a} \times \vec{c}) \cdot \vec{b}| = 11$.

4. (12 points) Match each curve with one of the equations on the right side. Not all equations will be matched.

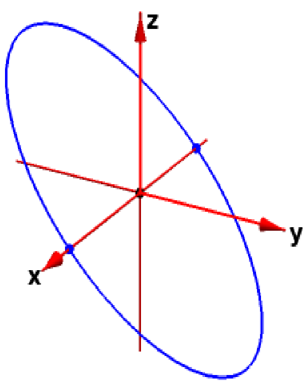
(i) **(C)**



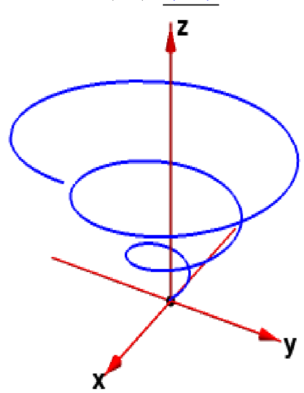
(ii) **(E)**



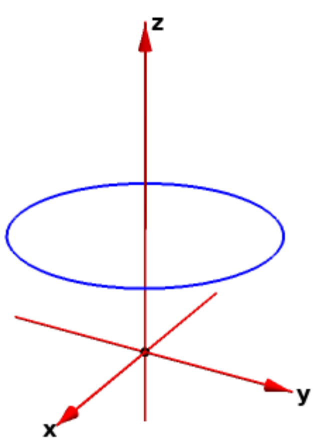
(iii) **(F)**



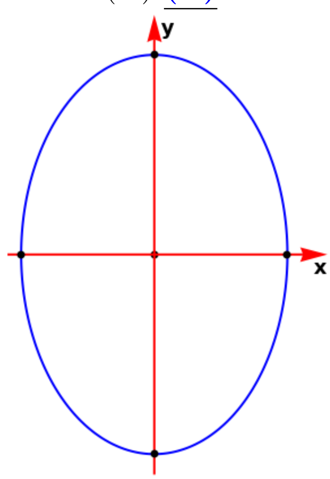
(iv) **(G)**



(v) **(H)**



(vi) **(B)**



(A) $\vec{r}(t) = \langle 3\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$

(B) $\vec{r}(t) = \langle 2\sin t, 3\cos t\rangle$, $0 \le t \le 2\pi$

(C) $\vec{r}(t) = \langle 2-3t, 3-6t, 3t-1\rangle$, $0 \le t \le 1$

(D) $\vec{r}(t) = \langle 2+3t, 3+6t, 3t-1\rangle$, $0 \le t \le 1$

(E) $\vec{r}(t) = \langle \sin t, \cos t, t\rangle$, $0 \le t \le 6\pi$

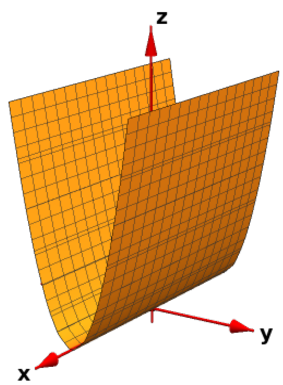
(F) $\vec{r}(t) = \langle \cos t, -\sin t, \sin t\rangle$, $0 \le t \le 2\pi$

(G) $\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$, $0 \le t \le 6\pi$

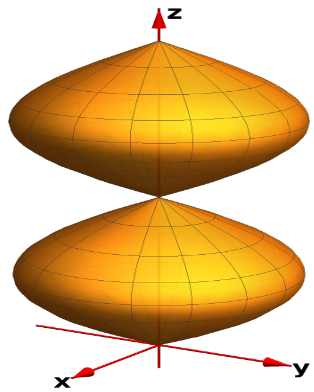
(H) $\vec{r}(t) = \langle \cos t, \sin t, 1\rangle$, $0 \le t \le 2\pi$

5. (12 points) Match each 3D surface with one of the equations on the right side. Not all equations will be matched.

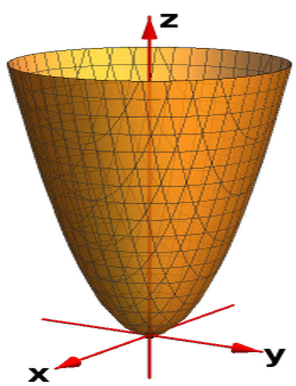
(i) **(G)**



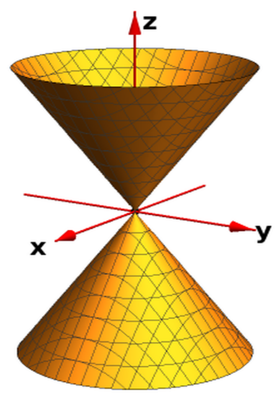
(ii) **(E)**



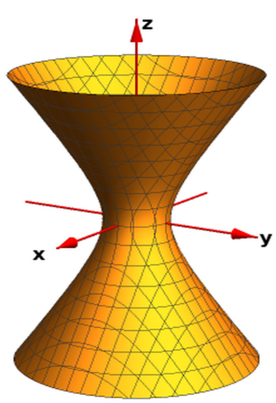
(iii) **(A)**



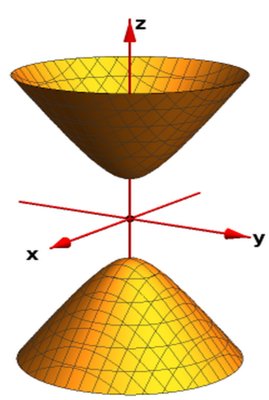
(iv) **(B)**



(v) **(C)**



(vi) **(D)**



(A) $x^2 + y^2 - z = 0$

(B) $x^2 + y^2 - z^2 = 0$

(C) $x^2 + y^2 - z^2 - 1 = 0$

(D) $x^2 + y^2 - z^2 + 1 = 0$

(E) $x^2 + y^2 - \sin^2 z = 0$

(F) $x^2 + y^2 - \cos^2 z = 0$

(G) $z - y^2 = 0$

(H) $z - x^2 = 0$

6. (6 points) Use **spherical coordinates** to describe the solid consisting of points *on and inside* the sphere of radius 3 centered at the origin, but *strictly outside* the sphere of radius 1 centered at the origin, and in the *first* octant.

Solution: The solid $E = \left\{(\rho, \theta, \phi) \middle| 1 < \rho \leq 3,\ 0 < \theta < \frac{\pi}{2},\ 0 < \phi < \frac{\pi}{2}\right\}$

7. (11 points) Suppose $\vec{r}(t)$ is a differentiable vector function with

$$\vec{r}\,'(t) = \left\langle 2te^{t^2}, \frac{2t}{1+t^2}, \sec^2(t) \right\rangle$$

and $\vec{r}(0) = \langle 0,0,0 \rangle$. Find the formula for $\vec{r}(t)$.

Solution:

$$\begin{aligned}
\vec{r}(t) &= \int \vec{r}\,'(t)\,\mathrm{d}t = \int \left\langle 2te^{t^2}, \frac{2t}{1+t^2}, \sec^2(t) \right\rangle \mathrm{d}t \\
&= \left\langle \int 2te^{t^2}\,\mathrm{d}t,\ \int \frac{2t}{1+t^2}\,\mathrm{d}t,\ \int \sec^2(t)\,\mathrm{d}t \right\rangle \\
&= \left\langle e^{t^2},\ \ln(1+t^2),\ \tan t \right\rangle + \vec{C}
\end{aligned}$$

where $\vec{C}$ is a constant vector.

Since $\vec{r}(0) = \langle 0,0,0 \rangle$,

$$\begin{aligned}
\vec{r}(0) = \left\langle e^{t^2},\ \ln(1+t^2),\ \tan t \right\rangle \Big|_{t=0} + \vec{C} &= \langle 0,0,0 \rangle \\
\langle 1,0,0 \rangle + \vec{C} &= \langle 0,0,0 \rangle \\
\vec{C} = \langle 0,0,0 \rangle - \langle 1,0,0 \rangle &= \langle -1,0,0 \rangle
\end{aligned}$$

So we get

$$\boxed{\vec{r}(t) = \left\langle e^{t^2} - 1,\ \ln(1+t^2),\ \tan t \right\rangle}$$

8. (10 points) Compute the arc length of the path parameterized by

$$\vec{r}(t) = \left\langle \cos(t), \sin(t), \frac{2}{3}t^{\frac{3}{2}} \right\rangle, \qquad 0 \leq t \leq 3.$$

Solution:

$$\vec{r}\,'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \cos(t),\ \sin(t),\ \frac{2}{3}t^{\frac{3}{2}} \right\rangle = \left\langle -\sin(t),\ \cos(t),\ t^{\frac{1}{2}} \right\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{[-\sin(t)]^2 + [\cos(t)]^2 + \left[t^{\frac{1}{2}}\right]^2} = \sqrt{1+t}$$

$$L = \int_0^3 |\vec{r}\,'(t)|\mathrm{d}t = \int_0^3 \sqrt{1+t}\ \mathrm{d}t \qquad \left[\text{let } u = \sqrt{1+t} \Rightarrow u^2 = 1+t \Rightarrow 2u\,\mathrm{d}u = \mathrm{d}t\right]$$

$$= \int_1^2 u \cdot 2u\,\mathrm{d}u = 2\int_1^2 u^2\,\mathrm{d}u = \frac{2}{3}u^3\bigg|_1^2 = \frac{2}{3}(8-1) = \boxed{\frac{14}{3}}$$

9. (11 points) Let π be the plane perpendicular to the plane given by the equation $-2x-2y+z=8$ and containing the points $(0,2,2)$ and $(4,2,4)$. Find the equation of π and express it in the form $ax+by+cz+d=0$.

Solution: Denote the two points by $A(0,2,2)$, $B(4,2,4)$, and the normal vector of the plane $-2x-2y+z=8$ by $\vec{v}=\langle -2,-2,1\rangle$. Then the plane π is parallel to $\vec{v}$, and also to $\overrightarrow{AB}$. So the normal vector of the plane π is parallel to $\vec{n}=\overrightarrow{AB}\times\vec{v}$.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\langle 4,2,4\rangle-\langle 0,2,2\rangle=\langle 4,0,2\rangle$$

$$\vec{n}=\overrightarrow{AB}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 0 & 2\\ -2 & -2 & 1\end{vmatrix}=\vec{i}(0+4)-\vec{j}(4+4)+\vec{k}(-8-0)$$

$$=4\vec{i}-8\vec{j}-8\vec{k}=\langle 4,-8,-8\rangle$$

So the equation of the plane π is

$$4(x-0)-8(y-2)-8(z-2)=0\Longleftrightarrow(x-0)-2(y-2)-2(z-2)=0$$

Simplify it and we get

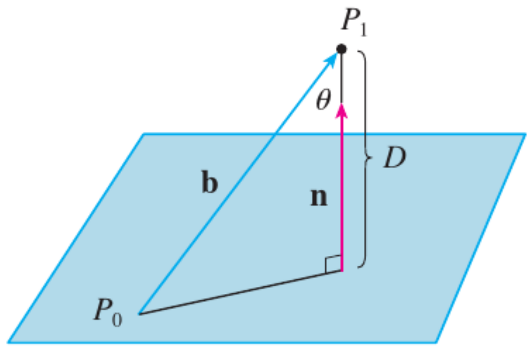
$$\boxed{4x-8y-8z+32=0\Longleftrightarrow x-2y-2z+8=0}$$

10. (10 points) Find the distance from the point $(2, -1, 5)$ to the plane $x + y + z + 1 = 0$.

Solution: By the distance formula between a point $P(x_0, y_0, z_0)$ and the plane π: $ax + by + cz + d = 0$, $\text{dist}(P, \pi) = \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

$$\textbf{distance} = \frac{|(2) + (-1) + (5) + 1|}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \boxed{\frac{7}{\sqrt{3}}} = \boxed{\frac{7\sqrt{3}}{3}}$$

Solution 2: Denote the given point by $P_1(2, -1, 5)$, and the normal vector of the plane by $\vec{n} = \langle 1, 1, 1 \rangle$. We choose a point on the plane, for example, we let $x = y = 0$, and plug them into the plane equation, and we get $z = -1$. We denote the point by $P_0(0, 0, -1)$. Let $\vec{b} = \overrightarrow{P_0P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_1} = \langle 2, -1, 5 \rangle - \langle 0, 0, -1 \rangle = \langle 2, -1, 6 \rangle$.



$$\textbf{distance} = \text{dist}(P_1, \pi) = \left|\text{comp}_{\vec{n}}\vec{b}\right| = \left|\text{proj}_{\vec{n}}\vec{b}\right| = \frac{\left|\vec{n} \cdot \vec{b}\right|}{|\vec{n}|}$$

$$= \frac{|\langle 1, 1, 1 \rangle \cdot \langle 2, -1, 6 \rangle|}{\sqrt{(1)^2 + (1)^2 + (1)^2}} = \frac{|(1)(2) + (1)(-1) + (1)(6)|}{\sqrt{3}}$$

$$= \boxed{\frac{7}{\sqrt{3}}} = \boxed{\frac{7\sqrt{3}}{3}}$$

11. (10 points) Find a parametric representation of the surface $z = x^2 + 4y^2$ within the cylinder $x^2 + 4y^2 = 4$. Include the bounds for the parameter(s).

Solution:

$$\begin{cases} x = x \\ y = y \\ z = x^2 + 4y^2 \end{cases}$$

The bounds for x and y are $\left\{(x, y)\middle| x^2 + 4y^2 \leq 4\right\}$.

Solution 2: Use cylindrical coordinates, $x = 2r\cos\theta$, $y = r\sin\theta$, $z = z$. Then $x^2 + 4y^2 = (2r\cos\theta)^2 + 4(r\sin\theta)^2 = 4r^2$. The equation of the surface is $z = x^2 + 4y^2$ and the cylinder $x^2 + 4y^2 = 4$ become $z = 4r^2$ and $4r^2 = 4$ or $r = 1$, respectively. So the parametrization of the surface is

$$\begin{cases} x = 2r\cos\theta \\ y = r\sin\theta \\ z = 4r^2 \end{cases}$$

The bounds for the parameters are $\left\{(r, \theta)\middle| 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\right\}$