1. Multiple Choice: For the following multiple choice questions, no partial credit is given. **Fill in your answer on the bubble sheet.**

(1) (3 points) **Fill in your answer on the bubble sheet.**
Suppose $f(x,y,z) = x + y^2 + z^2$, and let S be the level surface $f(x,y,z) = 8$. Find the equation of the tangent plane to S at the point $(-2,1,3)$.

(A) $(x+2) + 2(y-1) + 6(z-3) = 0$

(B) $(x+2) + 2(y-1) + 2(z-3) = 0$

(C) $(x-2) + 2(y+1) + 6(z+3) = 0$

(D) $(x-2) + 2(y+1) + 6(z+3) = 8$

(E) $(x-2) + 2(y+1) + 6(z+8) = 0$

(F) $(x+2) + 2(y-1) + 6(z-3) = 8$

$f_x = 1 \quad f_y = 2y \quad f_z = 2z$

$\vec{n} = \langle 1, 2(1), 2(3) \rangle$

$= \langle 1, 2, 6 \rangle$

$(x+2) + 2(y-1) + 6(y-3) = 0$

(2) (3 points) **Fill in your answer on the bubble sheet.**
Find the parametrization of the part of the elliptic paraboloid $y = 4x^2 + z^2 - 4$ that lies inside the cylinder $x^2 + z^2 = 4$.

$\begin{cases} x = r\cos\theta \\ z = r\sin\theta \end{cases}$

(A) $\langle x,\ 4x^2 + z^2 - 4,\ z \rangle$ for $-1 \le x \le 1$ and $-2 \le z \le 2$

(B) $\langle x,\ x^2 + z^2,\ z \rangle$ for $-2 \le x \le 2$ and $0 \le z \le 4$

(C) $\langle x,\ 4 - x^2 - z^2,\ z \rangle$ for $-2 \le x \le 2$ and $-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$

(D) $\langle r\cos\theta,\ r^2 + 3r^2\cos^2\theta - 4,\ r\sin\theta \rangle$ for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$

(E) $\langle r\cos\theta,\ r^2 - 4,\ 2r\sin\theta \rangle$ for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$

(F) $\left\langle \frac{1}{2} r\cos\theta,\ r^2 - 4,\ r\sin\theta \right\rangle$ for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$

$y = 4(r\cos\theta)^2 + (r\sin\theta)^2 - 4$

$= 4r^2\cos^2\theta + r^2\sin^2\theta - 4$

$= 3r^2\cos^2\theta + r^2 - 4$

(3) (3 points) **Fill in your answer on the bubble sheet.**
Suppose

$$f(x, y) = ye^{-x} + 3x.$$

Find the direction of the maximum rate of increase of $f(x, y)$ at $(0, 1)$.

(A) $\langle 2, 1 \rangle$

(B) $\langle -2, -1 \rangle$

(C) $\langle 3, 0 \rangle$

(D) $\langle -3, 0 \rangle$

(E) $\langle 2e^{-1}, e \rangle$

(F) $\langle -2e^{-1}, -e \rangle$

$\nabla f = \langle -ye^{-x} + 3, e^{-x} \rangle$

$\nabla f(0,1) = \langle -1e^{0} + 3, e^{0} \rangle$

$= \langle 2, 1 \rangle$

(4) (3 points) **Fill in your answer on the bubble sheet.**
Find the following limit, if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$$

(A) 0

(B) 1

(C) -1

(D) 2

(E) -2

(F) The limit does not exist.

path $x = 0$:
$\lim_{y \to 0} \frac{-2y^2}{y^2} = -2$

path $y = 0$:
$\lim_{x \to 0} \frac{x^2}{x^2} = 1$

(5) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$f(x,y) = \begin{cases} \dfrac{x+2}{x^2+y^2+1}, & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

Find a, such that the function $f(x,y)$ is continuous at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{x+2}{x^2+y^2+1} = 2 = a$$

(A) 0

(B) 1

(C) -1

(D) 2

(E) -2

(F) There is no a for which f is continuous at (0,0).

(6) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$f(x,y) = (x^3 - x)(y^2 - 1).$$

Find $f_{xy}(x,y)$.

$$f_x = (3x^2-1)(y^2-1)$$

$$f_{xy} = (3x^2-1)(2y)$$

(A) $(x^3 - x)(2y)$

(B) $(3x^2 - 1)(2y)$

(C) $(3x^2 - 1)(y^2 - 1) + (x^3 - x)(2y)$

(D) $(3x^2 - 1)(y^2 - 1)$

(E) 0

(F) $6x^2y + 2y - 3x^2 - 1$

(7) (3 points) **Fill in your answer on the bubble sheet.**

Let S be the surface parametrized by $\vec{r}(\theta, z) = \underbrace{\langle 3\cos(\theta), 3\sin(\theta), z\rangle}_{\text{cylinder radius 3}}$, for $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. Evaluate

height 2

$$\iint_S 1\, dS.$$

cylinder radius 3

(A) π

(B) 2π

(C) 3π

(D) 9π

(E) 12π

(F) 18π

2

$2\pi(3)$ ← circumference

$A = 12\pi$

(8) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$\vec{F}(x, y, z) = \langle xyz, \quad xy + yz + zx, \quad x + y + zy\rangle.$$

Find curl$\vec{F}$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy+yz+zx & x+y+zy \end{vmatrix}$$

$= (1 + z - y - x)\hat{\imath} - (1 - xz)\hat{\jmath} + (y + z - xz)\hat{k}$

(A) $\langle yz, \quad x + z, \quad 1\rangle$

(B) $\langle 1 - x - y + z, \quad -1 + xy, \quad y + z - xz\rangle$

(C) $\langle 1 + x + y, \quad xy, \quad xz\rangle$

(D) $\langle yz + y + z + 1, \quad xz + x + z + 1, \quad xy + x + y + 1\rangle$

(E) $\langle 1, \quad 1, \quad 1\rangle$

(F) $\langle y + z, \quad x + z, \quad x + y\rangle$

(9) (3 points) **Fill in your answer on the bubble sheet.**

Let

$$\vec{F}(x,y,z) = \left\langle x^3 + y^2,\ ze^{-y},\ x^2\sin(z) \right\rangle.$$

Find div$\vec{F}$

$$\frac{\partial}{\partial x}(x^3+y^2) + \frac{\partial}{\partial y}(ze^{-y}) + \frac{\partial}{\partial z}(x^2 \sin z)$$

$$3x^2 - ze^{-y} + x^2 \cos z$$

(A) $\left\langle 3x^2, -e^{-y}z, x^2\cos(z) \right\rangle$

(B) $\left\langle -e^{-y},\ -2x\sin(z),\ -2y \right\rangle$

(C) $\left\langle 3x^2 + 2y,\ -e^{-y},\ 2x\cos(z) \right\rangle$

(D) $3x^2 - ze^{-y} + x^2\cos(z)$

(E) $3x^2 + 2y - e^{-y} + 2x\cos(z)$

(F) $3x^2 + ze^{-y} - x^2\cos(z)$

(10) (3 points) **Fill in your answer on the bubble sheet.**

Let f be a scalar-valued function of three variables and $\vec{F}$ a vector field on $\mathbb{R}^3$. Which of the following must be true for all such f and $\vec{F}$? (Assume all functions and their components are polynomials.)

(A) div(div f) $= 0$

(B) div(grad f) $= 0$

(C) curl(div f) $= 0$

(D) div(curl(curl $\vec{F}$)) $= 0$

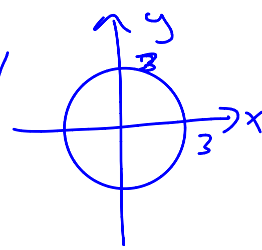
(E) curl(curl(div $\vec{F}$)) $= 0$

(F) grad(curl $\vec{F}$) $= 0$

2. (7 points) Convert the following integral from rectangular coordinates to cylindrical coordinates. Fill in all **7** blanks.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{18-2x^2-2y^2} yz \, dz \, dy \, dx$$

$$= \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{3}} \int_{\boxed{0}}^{\boxed{18-2r^2}} \boxed{r^2 z \sin\theta} \, dz \, dr \, d\theta$$



$0 \le \theta \le 2\pi$

$0 \le r \le 3$

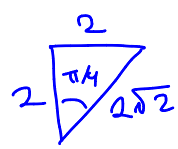
$0 \le z \le 18 - 2(x^2+y^2)$

$0 \le z \le 18 - 2r^2$

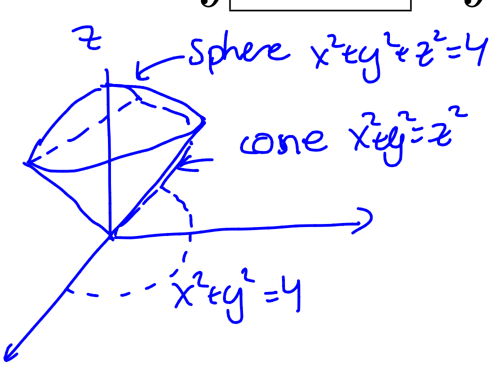
$yz = (r\sin\theta)z$

3. (7 points) Convert the following integral from spherical coordinates to rectangular coordinates. Fill in all **7** blanks.

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \sin\phi \;\, d\rho \, d\phi \, d\theta$$



$$= \int_{\boxed{-2}}^{\boxed{2}} \int_{\boxed{0}}^{\boxed{\sqrt{4-x^2}}} \int_{\boxed{\sqrt{x^2+y^2}}}^{\boxed{\sqrt{4-x^2-y^2}}} \boxed{\frac{1}{x^2+y^2+z^2}} \, dz \, dy \, dx$$



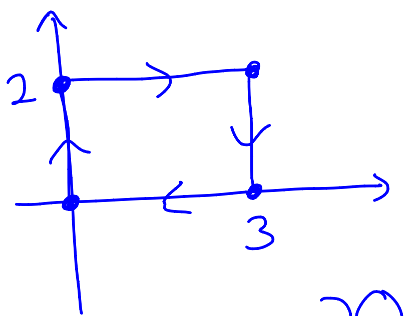
$dz\,dy\,dx = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\frac{1}{\rho^2} dz\,dy\,dx = \sin\varphi \, d\rho \, d\varphi \, d\theta$

4. (10 points) Evaluate the integral

$$\int_C \left(xy^2 + y\right)\, dx + \left(2x^2y + e^{y^2}\right)\, dy$$

where C is boundary of the rectangle in the xy-plane oriented **clockwise** with vertices $(0,0)$, $(0,3)$, $(2,3)$, and $(2,0)$.



$$P = xy^2 + y$$

$$Q = 2x^2y + e^{y^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4xy - (2xy + 1)$$

$$= 2xy - 1$$

By Green's,

$$- \int_0^2 \int_0^3 (2xy - 1)\, dx\, dy$$

$$= - \int_0^2 \left(x^2y - x \right) \Big|_{x=0}^{x=3} dy$$

$$= - \int_0^2 9y - 3\, dy$$

$$= - \left(\frac{9y^2}{2} - 3y \right) \Big|_{y=0}^{y=2}$$

$$= - (18 - 6) = -12$$

5. (8 points) For the following function, find all local maximums, local minimums, and saddle points.

$$f(x, y) = x^4 - 2x^2 + y^3 - 3y$$

$\nabla f = \langle 4x^3 - 4x, 3y^2 - 3 \rangle$

$4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$4x(x-1)(x+1) = 0$

$x = 0, 1, -1$

$3y^2 - 3 = 0$

$y^2 - 1 = 0$

$y = \pm 1$

$f_{xx} = 12x^2 - 4$

$f_{yy} = 6y$

$f_{xy} = 0$

$D(x,y) = (12x^2 - 4)(6y)$

$D(0,1) < 0 \longrightarrow (0,1)$ is a saddle point

$D(0,-1) > 0, f_{xx}(0,-1) < 0 \longrightarrow (0,-1)$ is a local max

$D(\pm 1, 1) > 0, f_{xx}(\pm 1, 1) > 0 \longrightarrow (1,1), (-1,1)$ are local mins

$D(\pm 1, -1) < 0 \longrightarrow (1,-1), (-1,-1)$ are saddle points

6. (5 points) Consider the vector field $\vec{F}$ on $\mathbb{R}^2$ given by

$$\vec{F}(x,y) = \langle \pi\cos(\pi x) + y,\ x + 2y \rangle .$$

Find a potential function $f(x,y)$ for $\vec{F}(x,y)$ such that $\nabla f = \vec{F}$.

want $\frac{\partial f}{\partial x} = \pi\cos(\pi x) + y$ $\quad \frac{\partial f}{\partial y} = x + 2y$

$f(x,y) = xy + y^2 + g(x)$

$f_x = y + g'(x) = \pi\cos(\pi x) + y$

$\rightarrow g'(x) = \pi\cos(\pi x)$

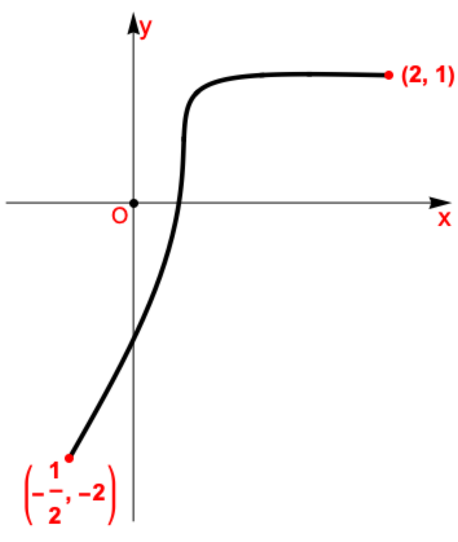
$\rightarrow g(x) = \sin(\pi x)$

$f(x,y) = xy + y^2 + \sin(\pi x)$

7. (3 points) Let $\vec{F} = \nabla g$ where $g(x,y) = e^{\cos(\pi x)} + xy$. Evaluate the integral

$$\int_C \vec{F} \cdot d\vec{r},$$

where C is the path pictured below from $\left(-\frac{1}{2}, -2\right)$ to $(2,1)$.



FTLI,

$\int_C \vec{F} \cdot d\vec{r} = g(2,1) - g(-\frac{1}{2}, -2)$

$= \left(e^{\cos 2\pi} + 2\right) - \left(e^{\cos -\frac{\pi}{2}} + 1\right)$

$= e$

8. (10 points) Let S be the helicoid parameterized by

$$\vec{r}(u,v) = \langle u\sin v,\ 2v,\ u\cos v\rangle \quad \text{for } 0 \le u \le 1, \quad 0 \le v \le \pi,$$

oriented in the direction of the positive y-axis. Let $\vec{F}$ be a vector field given by

$$\vec{F} = xy\vec{i} + (y^2+1)\vec{j} + yz\vec{k}.$$

Evaluate $\displaystyle\iint_S \vec{F}\cdot d\vec{S}$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin v & 0 & \cos v \\ u\cos v & 2 & -u\sin v \end{vmatrix} = \langle -2\cos v, u, 2\sin v\rangle$$

$$\vec{F}(\vec{r}(u,v)) = \langle 2uv\sin v,\ 4v^2+1,\ 2uv\cos v\rangle$$

$$\vec{F}(\vec{r}(u,v))\cdot\langle \vec{r}_u \times \vec{r}_v\rangle = \cancel{-4uv\sin v\cos v} + (4v^2+1)u$$
$$\cancel{+4uv\sin v\cos v}$$

$$\iint_S \vec{F}\cdot dS = \int_0^1\int_0^\pi (4v^2+1)(u)\ dv\,du$$
$$= \left(\int_0^1 u\,du\right)\left(\int_0^\pi 4v^2+1\,dv\right)$$
$$= \left(\frac{u^2}{2}\Big|_0^1\right)\left(\frac{4v^3}{3}+v\Big|_0^\pi\right)$$
$$= \frac{1}{2}\left(\frac{4\pi^3}{3}+\pi\right)$$

9. (10 points) Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ given by

$$\vec{F} = (\cos x + y)\vec{i} + (e^y + xz^2)\vec{j} + (2z^2 + yx)\vec{k}.$$

Let C be a circle of radius 1 centered at $(0,0,2)$ lying on the plane $z = 2$, which is oriented **counterclockwise** when viewed from above. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

Stoke's Theorem

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x + y & e^y + xz^2 & 2z^2 + yx \end{vmatrix}$$

$$= \langle x - 2xz, \ -y, \ z^2 - 1 \rangle$$

disk with bondary C:

$$x^2 + y^2 \le 1 \quad z = 2$$

$$x = r\cos\theta \qquad 0 \le r \le 1$$

$$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$$

$$z = 2$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r \rangle$$ upward orientation ✓

$$\int_0^{2\pi} \int_0^1 \langle r\cos\theta - 4r\cos\theta, \ -r\sin\theta, 3 \rangle \cdot \langle 0, 0, r \rangle \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^1 3r \, dr \, d\theta = 2\pi \left(\frac{3r^2}{2} \right) \Big|_0^1$$

$$= 3\pi$$

10. (10 points) Let

$$\vec{F}(x,y,z) = \left(x^3 + e^{y^2+z^2}\right)\vec{i} + \left(\cos(x^4) + y^3\right)\vec{j} + \left(\ln(x^2+4) + z^3\right)\vec{k}$$

be a vector field on $\mathbb{R}^3$, region E be the part of the solid sphere $x^2+y^2+z^2 \le 4$ in the **first octant**, and S be the boundary of E oriented outward. Find the total flux of $\vec{F}$ through S:

$$\iint_S \vec{F} \cdot d\vec{S}.$$

Divergence Theorem

region E in spherical:

$x = \rho \sin\varphi \cos\theta \qquad 0 \le \rho \le 2$

$y = \rho \sin\varphi \sin\theta \qquad 0 \le \varphi \le \pi/2$

$z = \rho \cos\varphi \qquad 0 \le \theta \le \pi/2$

$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$

$$\iint_S \vec{F} \cdot dS = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 (3\rho^2)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \frac{\pi}{2}\left(\int_0^{\pi/2} \frac{3\rho^5}{5} \sin\varphi \Big|_0^2 \, d\varphi\right)$$

$$= \frac{48\pi}{5}\int_0^{\pi/2} \sin\varphi \, d\varphi$$

$$= \frac{48\pi}{5}\left(-\cos\varphi\right)\Big|_0^{\pi/2}$$

$$= \frac{48\pi}{5}$$