1. (8 points) Match the vector fields $\vec{F}$ with the plots below.

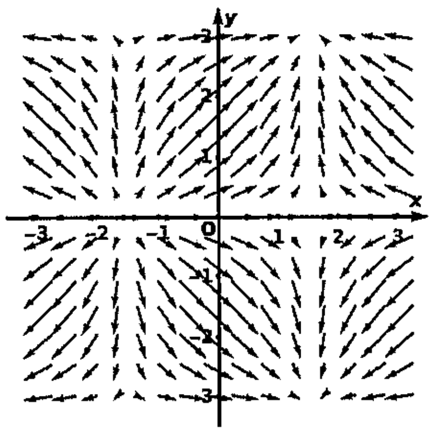
(i) $\vec{F} = \langle x, x+y \rangle$ D

(ii) $\vec{F} = \langle x, x-y \rangle$ C

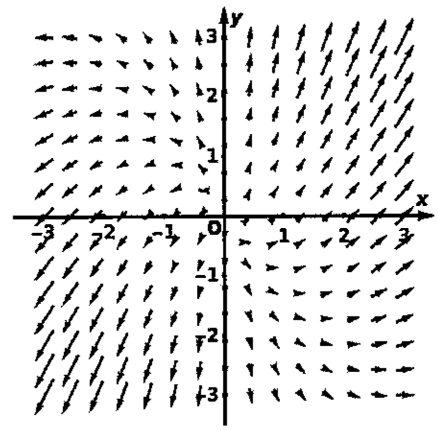
(iii) $\vec{F} = \langle 1, xy \rangle$ E

(iv) $\vec{F} = \langle y^2, x^2 \rangle$ B

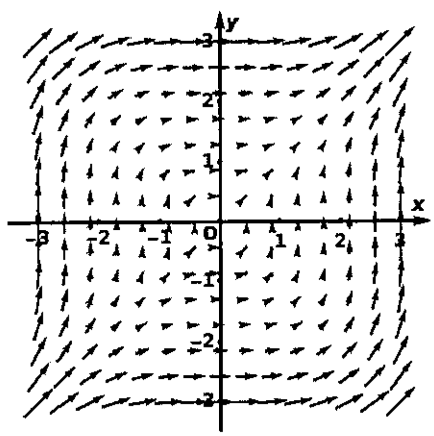
(A)



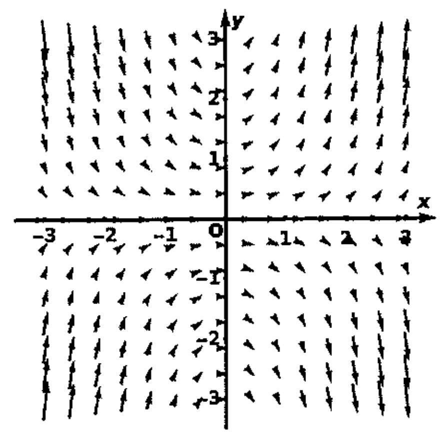
(D)



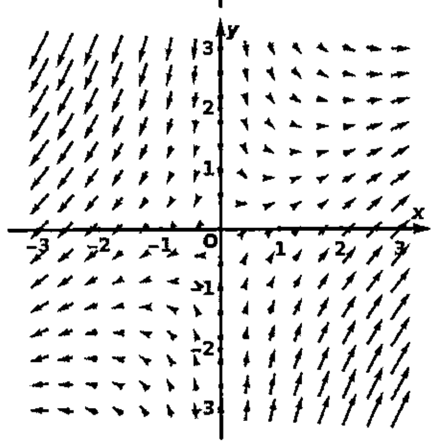
(B)



(E)



(C)



2. Let R be the region in the xy-plane enclosed by the lines $y = \frac{1}{4}x$, $y = \frac{1}{2}x$ and the hyperbolas $y^2 = x^2 - 1$, $y^2 = x^2 - 3$ in the first quadrant.

(i) (3 points) Which of the following is a transformation that maps R onto a rectangle S in the uv-plane?

(A) $u = y$, $v = x^2 - y^2$

(B) $u = x$, $v = x^2 - y^2$

(C) $u = x + y$, $v = x - y$

(D) $u = xy$, $v = y^2 - x^2$

(E) $u = \frac{y}{x}$, $v = y^2 + x^2$

(F) $u = \frac{x}{y}$, $v = x^2 - y^2$

The lines: $\frac{x}{y} = 4$, $\frac{x}{y} = 2$

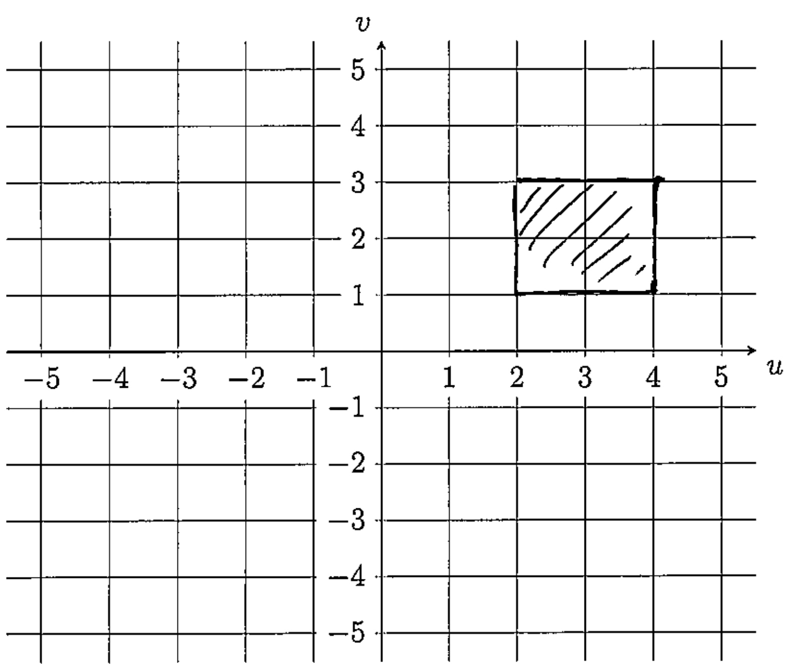
Hyperbolas: $x^2 - y^2 = 1$, $x^2 - y^2 = 3$

$u = \frac{x}{y}$, $v = x^2 - y^2$

$2 \le u \le 4$

$1 \le v \le 3$

(ii) (4 points) Sketch and shade the rectangle S in the uv-plane.



3. (4 points) Which of the following is equivalent to $\int_0^1 \int_0^{x^2} \int_0^{2y} f(x,y,z)\,dz\,dy\,dx$?

(A) $\int_0^1 \int_0^{x^3} \int_0^{x} f(x,y,z)\,dy\,dz\,dx$

(B) $\int_0^1 \int_0^{\sqrt[3]{y}} \int_0^{\frac{2z}{3}} f(x,y,z)\,dy\,dz\,dx$

(C) $\int_0^1 \int_0^{\frac{x^2}{2}} \int_{2z}^{x} f(x,y,z)\,dy\,dz\,dx$

(D) $\int_0^1 \int_1^{x^2} \int_1^{1-\frac{z}{2}} f(x,y,z)\,dy\,dz\,dx$

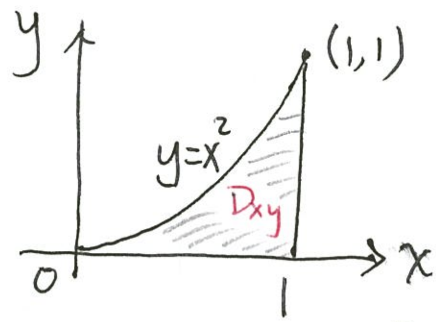
(E) $\int_0^1 \int_0^{2x^2} \int_{\frac{z}{2}}^{x^2} f(x,y,z)\,dy\,dz\,dx$ $= \iiint_E f(x,y,z)\,dV = \iint_{D_{xz}} \int_{\frac{z}{2}}^{x^2} f(x,y,z)\,dy\,dA$

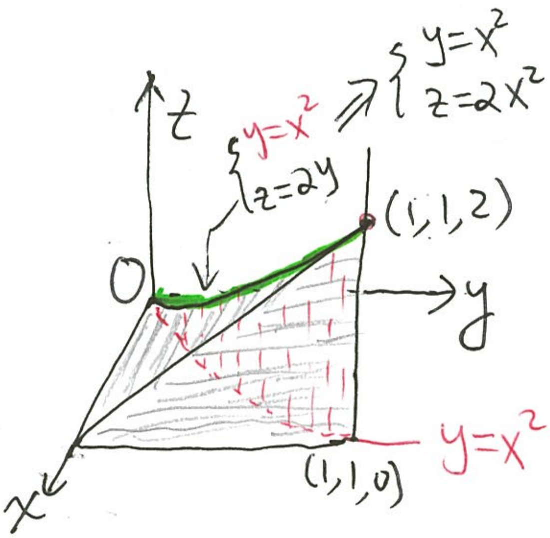
(F) $\int_0^1 \int_{x^2}^{2} \int_{\frac{z}{2}}^{x^2} f(x,y,z)\,dy\,dz\,dx$

$E=\{(x,y,z)\,|\,0\le x\le 1,\ 0\le y\le x^2,\ 0\le z\le 2y\}$

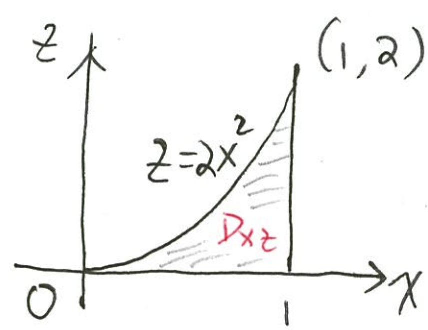
$D_{xy}=\{(x,y)\,|\,0\le x\le 1,\ 0\le y\le x^2\}$

D_{xy} is the projection of the solid E onto the xy-plane.





D_{xz} is the projection of E onto the xz-plane



4. (4 points) Which of the following represents the volume of the solid in the **first octant** with coordinates satisfying $x + y + z \leq 3$?

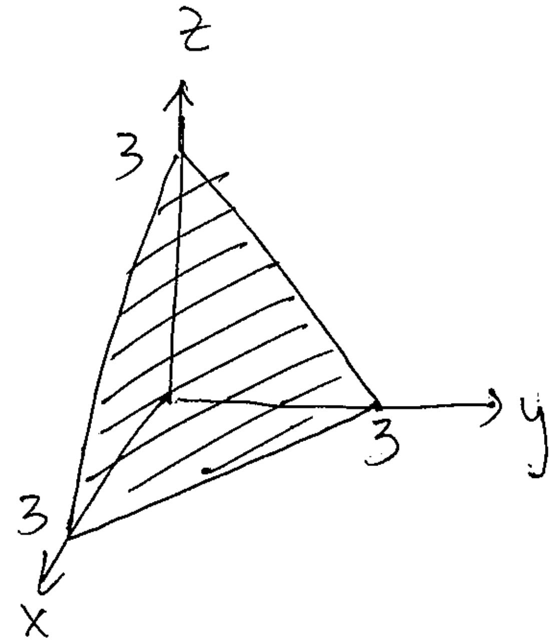
(A) $\displaystyle\int_0^3 \int_0^3 \int_0^3 1 \, dz \, dy \, dx$

(B) $\displaystyle\int_0^3 \int_0^{3-z} \int_0^{3+y-z} 1 \, dy \, dx \, dz$

(C) $\displaystyle\int_0^3 \int_0^3 \int_0^{3-y-z} 1 \, dx \, dy \, dz$

(D) $\displaystyle\int_0^3 \int_0^{3-y} \int_0^{3-y-z} 1 \, dx \, dy \, dz$

(E) $\displaystyle\int_0^3 \int_0^{3-x} \int_0^{3-x-y} 1 \, dz \, dy \, dx$

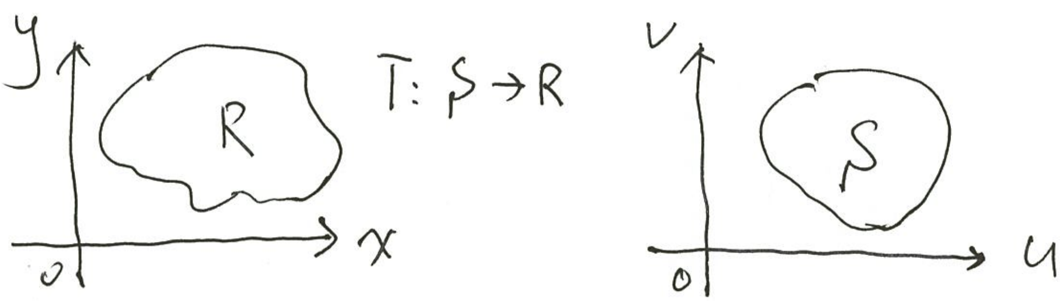


5. Suppose $T(u, v) = (x, y)$ is the one-to-one transformation given by the equations $x = 4u + 3v$ and $y = 2u + v$.

(i) (5 points) What is the Jacobian of the transformation T?

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= (4)(1) - (3)(2) = 4 - 6 = \boxed{-2}$$



(ii) (3 points) Let S be a region in the uv-plane. If T transforms S into a region in the xy-plane that has an area of 10, what is the area of S?

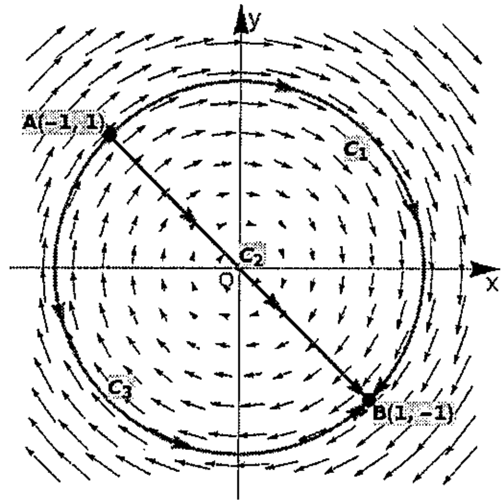
$$10 = \iint_R dA \quad \text{R is in xy-plane}$$

$$= \iint_S |J|\, dA' = 2 \iint_S dA' \quad \text{S is in uv-plane}$$

$$= 2A(S)$$

$$\text{So } A(S) = \frac{10}{2} = \boxed{5}$$

6. A force field $\vec{F}$ and three paths C_1, C_2, and C_3 from $A(-1,1)$ to $B(1,-1)$ are shown in the figure below.



Consider a particle with the force $\vec{F}$ moving from $A(-1,1)$ to $B(1,-1)$ along three paths C_1, C_2 and C_3.

(i) (3 points) Circle the correct answer to fill the blanks in the following sentence.

The work on the particle moving along the path $\underline{C_1}$ is the largest and the work on the particle moving along the path $\underline{C_3}$ is the smallest.

(A) C_1, C_2	(C) C_1, C_3	(E) C_2, C_3
(B) C_2, C_1	(D) C_3, C_1	(F) C_3, C_2

(ii) (3 points) Explain your reasoning for your answer to part (i).

Along C_1, $\vec{F}$ and $d\vec{r} = \vec{r}'(t)dt$ have the same direction,
$\vec{F} \cdot d\vec{r} > 0$, so the work $= \int_C \vec{F} \cdot d\vec{r} > 0$
Along C_2, $\vec{F} \perp d\vec{r}$, the work $= \int_C \vec{F} \cdot d\vec{r} = 0$
Along C_3, $\vec{F}$ and $d\vec{r}$ have the opposite direction,
$\vec{F} \cdot d\vec{r} < 0$, so the work $= \int_C \vec{F} \cdot d\vec{r} < 0$

7. (13 points) Compute $\int_C x^2\cos(x^2y)\,dx + y\sin(xy)\,dy$, where C is the line segment from $A(-1,1)$ to $B(1,-1)$.

Solution 1: The eqn of the line segment AB is

$y = -x,\ -1 \le x \le 1.$ so $dy = -dx$, and

the integral $= \int_{-1}^{1} x^2\cos(x^2(-x))\,dx - y\sin(-x^2)(-1)dx$

$= \int_{-1}^{1} x^2\cos(x^3)\,dx - \int_{-1}^{1} x\sin(x^2)\,dx$

$\leftarrow$ odd function $\Rightarrow$ the 2nd int $= 0$

let $u = x^3$. $du = 3x^2dx$

$\int_{-1}^{1} x^2\cos(x^3)dx = \frac{1}{3}\int_{-1}^{1}\cos(u)du = \frac{1}{3}\sin(u)\Big|_{-1}^{1} = \frac{2}{3}\sin(1)$

So the integral $= \frac{2}{3}\sin(1) - 0 = \boxed{\frac{2}{3}\sin(1)}$

Solution 2: Parametrize AB: $\vec{r}(t) = (1-t)\langle -1,1\rangle + \langle 1,-1\rangle t$

$= \langle 2t-1, 1-2t\rangle \quad 0 \le t \le 1$

or $\begin{cases} x = 2t-1 \\ y = 1-2t \end{cases} \Rightarrow \begin{cases} dx = 2dt \\ dy = -2dt \end{cases}$

So the integral $= \int_0^1 (2t-1)^2\cos[(2t-1)^2(1-2t)]2dt$

$+ (1-2t)\sin[(2t-1)(1-2t)](-2)\,dt$

$= \int_0^1 [(2t-1)^2\cos((2t-1)^3) - (2t-1)\sin((2t-1)^2)]2dt$

$\overset{u=2t-1}{=\!=} \int_{-1}^{1} [u^2\cos(u^3) - u\sin(u^2)]du$

$= \left[\frac{1}{3}\sin(u^3) + \frac{1}{2}\cos(u^2)\right]\Big|_{-1}^{1} = \left[\frac{1}{3}\sin(1) + \frac{1}{2}\cos(1)\right] - \left[\frac{1}{3}\sin(-1) + \frac{1}{2}\cos(-1)\right]$

$= \boxed{\frac{2}{3}\sin(1)}$

Solution 3:

$$\int_C x^2\cos(x^2y)\,dx + y\sin(xy)\,dy = \int_C \vec{F}\cdot d\vec{r}$$

where $\vec{F} = \langle x^2\cos(x^2y),\ y\sin(xy)\rangle$.

Parametrize the line segment $\overrightarrow{AB}$

$$\vec{r}(t) = (1-t)\overrightarrow{OA} + t\,\overrightarrow{OB} = (1-t)\langle -1,1\rangle + t\langle 1,-1\rangle = \langle 2t-1,\ 1-2t\rangle$$

$$\vec{r}\,'(t) = \langle 2,-2\rangle. \qquad 0\le t\le 1$$

The integral $= \int_0^1 \langle (2t-1)^2\cos((2t-1)^2(1-2t)),\ (1-2t)\sin((2t-1)(1-2t))\rangle\cdot\langle 2,-2\rangle\,dt$

$$= \int_0^1 \left[2(2t-1)^2\cos((2t-1)^3) - 2(2t-1)\sin((2t-1)^2)\right]dt$$

$$\underline{\underline{u=2t-1}}\ \int_{-1}^1 \left[u^2\cos(u^3) - u\sin(u^2)\right]du$$

$$= \left[\tfrac{1}{3}\sin(u^3) + \tfrac{1}{2}\cos(u^2)\right]\Big|_{-1}^{1} = \left[\tfrac{1}{3}\sin(1)+\tfrac{1}{2}\cos(1)\right] - \left[\tfrac{1}{3}\sin(-1)+\tfrac{1}{2}\cos(-1)\right]$$

$$= \boxed{\tfrac{2}{3}\sin(1)}$$

Solution 4: AB: $y=-x$, $-1\le x\le 1$, or $x=-y$, y is from 1 to -1

$$\int_C x^2\cos(x^2y)\,dx + y\sin(xy)\,dy = \int_{-1}^1 x^2\cos(-x^3)\,dx + \int_1^{-1} y\sin(-y^2)\,dy \equiv I_1 + I_2$$

$$I_1 = \int_{-1}^1 x^2\cos(x^3)\,dx \ \underline{\underline{u=x^3}}\ \tfrac{1}{3}\int_{-1}^1 \cos(u)\,du = \tfrac{1}{3}\sin(u)\Big|_{-1}^{1} = \tfrac{2}{3}\sin(1)$$

$$I_2 = -\int_1^{-1} y\sin(y^2)\,dy = \int_{-1}^1 y\sin(y^2)\,dy = 0$$ b/c $y\sin(y^2)$ is an odd function of y.

$$\underline{\underline{\text{or}}}\ \tfrac{1}{2}\int_1^1 \sin(u)\,du = -\tfrac{1}{2}\cos(u)\Big|_1^1 = 0$$

So the integral $= I_1 + I_2 = \boxed{\tfrac{2}{3}\sin(1)}$

8. (13 points) Let C be the space curve given by the parametric equation

$$\vec{r}(t) = \langle \cos(t), 2\sin(t), t \rangle$$

for $0 \le t \le 2\pi$. Compute $\int_C y\sqrt{3x^2+2}\, ds$.

$$\vec{r}\,'(t) = \langle -\sin(t), 2\cos(t), 1 \rangle$$

$$ds = |\vec{r}\,'(t)|\, dt = \sqrt{(-\sin(t))^2 + (2\cos(t))^2 + (1)^2}\, dt$$

$$= \sqrt{\sin^2(t) + 4\cos^2(t) + 1}\, dt = \sqrt{\sin^2(t) + \cos^2(t) + 3\cos^2(t) + 1}\, dt$$

$$= \sqrt{1 + 3\cos^2(t) + 1}\, dt = \sqrt{3\cos^2(t) + 2}\, dt$$

$$\int_C y\sqrt{3x^2+2}\, ds = \int_0^{2\pi} 2\sin(t)\sqrt{3\cos^2(t)+2}\sqrt{3\cos^2(t)+2}\, dt$$

$$= \int_0^{2\pi} 2\sin(t)\left(3\cos^2(t)+2\right) dt$$

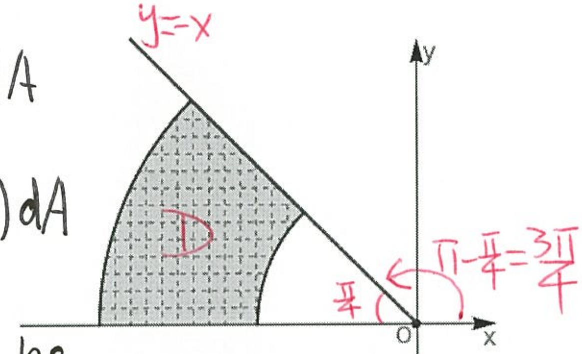
$$\underset{du = -\sin(t)dt}{\overset{u=\cos(t)}{=}} \int_1^1 -2(3u^2+2)\, du = \boxed{0}$$

$$\underline{\underline{\text{Or}}}\quad -2(u^3+2u)\Big|_1^1 = \boxed{0}$$

9. (12 points) Compute the total mass of the lamina that occupies the region in the **second quadrant** bounded by

$$x^2 + y^2 = 1, \quad x^2 + y^2 = 4, \quad y = 0, \quad y = -x$$

and has density $\rho(x, y) = \arctan\left(\frac{y}{x}\right)$, measured in g/m^2. *State the units of your final answer.*



mass $= \iint_D \rho(x,y)\,dA$

$= \iint_D \arctan\left(\frac{y}{x}\right) dA$

The integral can be evaluated by using polar coordinates or change of variables

* In polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$, $\Rightarrow \frac{y}{x} = \tan\theta$

So $\theta = \arctan\left(\frac{y}{x}\right)$

mass $= \int_{\frac{3\pi}{4}}^{\pi} \int_1^2 \theta\, r\,dr\,d\theta$

$= \int_{\frac{3\pi}{4}}^{\pi} \theta\,d\theta \int_1^2 r\,dr$

$= \frac{\theta^2}{2}\Big|_{\frac{3\pi}{4}}^{\pi} \frac{r^2}{2}\Big|_1^2$

$= \frac{1}{2}\left[\pi^2 - \left(\frac{3\pi}{4}\right)^2\right] \frac{1}{2}\left[(2)^2 - 1^2\right]$

$= \frac{1}{2}\left(\pi^2 - \frac{9}{16}\pi^2\right) \frac{1}{2}(4-1)$

$= \frac{1}{2} \cdot \frac{7}{16}\pi^2 \cdot \frac{3}{2} = \boxed{\frac{21\pi^2}{64} g}$

* let $u = x^2 + y^2$, $v = \frac{y}{x}$. Then

$D_{uv} = \{(u,v) \mid 1 \le u \le 4, -1 \le v \le 0\}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2x\left(\frac{1}{x}\right) + 2y \cdot \frac{y}{x^2}$

$= 2 + 2\frac{y^2}{x^2} = 2 + 2v^2$

So $J = (2+2v^2)^{-1}$. So

mass $= \iint_{D_{uv}} \frac{\arctan(v)}{2+2v^2}\,dA'$

$= \int_1^4 \int_{-1}^0 \frac{\arctan(v)}{2(1+v^2)}\,dv\,du$

$= \frac{3}{2}\int_{-1}^0 \frac{\arctan(v)}{1+v^2}\,dv$

$= \frac{3}{2} \cdot \frac{1}{2}\arctan^2(v)\Big|_{-1}^0 = \frac{3}{4}\left[0 - \arctan^2(-1)\right]$

$= \frac{3}{4}\left[0 - \left(\frac{3}{4}\pi\right)^2\right] = \boxed{\frac{21\pi^2}{64} g}$

10. (12 points) Compute the surface area of the part of the plane $x + \frac{1}{2}y - z = 0$ inside the cylinder $x^2 + z^2 = 4$.

Solution 1: The equation of the surface S_1: $y = 2(z - x)$

$y_x = -2$, $y_z = 2$, $\sqrt{y_x^2 + y_z^2 + 1} = \sqrt{(-2)^2 + (2)^2 + 1} = 3$

The surface area $= \iint_{D_{xz}} \sqrt{y_x^2 + y_z^2 + 1}\, dA = \iint_{D_{xz}} 3\, dA = 3 \iint_{D_{xz}} dA$

where D_{xz} is the projection of S onto the xz-plane, and is a disc with radius 2

So the surface area $= 3 \iint_{D_{xz}} dA = 3\pi(2)^2 = \boxed{12\pi}$

Solution 2: The plane S_1: $x + \frac{1}{2}y - z = 0$ can be parametrized as $\vec{r}(r, \theta) = \langle r\cos\theta, 2r(\sin\theta - \cos\theta), r\sin\theta \rangle$, $0 \le \theta \le 2\pi$, $0 \le r \le 2$

$\vec{r}_r = \langle \cos\theta, 2(\sin\theta - \cos\theta), \sin\theta \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, 2r(\cos\theta + \sin\theta), r\cos\theta \rangle = r\langle -\sin\theta, 2(\cos\theta + \sin\theta), \cos\theta \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & 2(\sin\theta - \cos\theta) & \sin\theta \\ -r\sin\theta & 2r(\cos\theta + \sin\theta) & r\cos\theta \end{vmatrix}$$

$= \vec{i}\left[2r(\cos\theta\sin\theta - \cos^2\theta) - 2r(\sin\theta\cos\theta + \sin^2\theta)\right] - \vec{j}\left(r\cos^2\theta + r\sin^2\theta\right)$
$+ \vec{k}\left[2r(\cos^2\theta + \sin\theta\cos\theta) + 2r(\sin^2\theta - \sin\theta\cos\theta)\right]$

$= -2r(\cos^2\theta + \sin^2\theta)\vec{i} - r(\cos^2\theta + \sin^2\theta)\vec{j} + 2r(\cos^2\theta + \sin^2\theta)\vec{k} = \langle -2r, -r, 2r \rangle$

$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{(-2r)^2 + (-r)^2 + (2r)^2} = 3r$

The surface area $= \iint_{D_{r\theta}} 3r\, dA = \int_0^{2\pi}\int_0^2 3r\, dr\, d\theta$

$= \int_0^{2\pi} d\theta \int_0^2 3r\, dr = 2\pi \left.\frac{3r^2}{2}\right|_0^2 = 2\pi \frac{3(2)^2}{2} = \boxed{12\pi}$

10. (12 points) Compute the surface area of the part of the plane $x + \frac{1}{2}y - z = 0$ inside the cylinder $x^2 + z^2 = 4$.

Solution 3: Let $F(x,y,z) = x + \frac{1}{2}y - z$. Then the plane eqn is $F(x,y,z) = 0$.

$F_x = 1$, $F_y = \frac{1}{2}$, $F_z = -1$

$\vec{n} = \langle F_x, F_y, F_z \rangle = \langle 1, \frac{1}{2}, -1 \rangle$

$|\vec{n}| = \sqrt{(1)^2 + (\frac{1}{2})^2 + (-1)^2} = \frac{3}{2}$

The surface area is

$$\iint_S dS = \iint_{D_{xz}} \frac{|\vec{n}|}{|F_y|} dA = \iint_{D_{xz}} \frac{\frac{3}{2}}{\frac{1}{2}} dA$$

$$= 3 \iint_{D_{xz}} dA = 3\,(\text{area of } D_{xz})$$

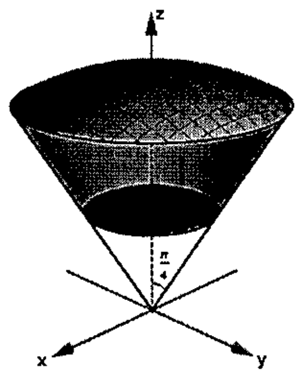
$$= 3\pi(2)^2 = \boxed{12\pi}$$

D_{xz} is the projection of the surface onto xz-plane.

11. (13 points) Evaluate the triple integral

$$\iiint_E z\,\mathrm{d}V$$

over the solid E bounded by the sphere $x^2+y^2+z^2=1$, the sphere $x^2+y^2+z^2=4$, and the cone $z=\sqrt{x^2+y^2}$ as in the figure below.



In spherical coordinates

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

$$\iiint_E z\,dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_1^2 \rho\cos\phi\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{4}} \cos\phi\sin\phi\, d\phi \int_0^{2\pi} d\theta \int_1^2 \rho^3 d\rho$$

$$= \frac{1}{2}\sin^2\phi\Big|_0^{\frac{\pi}{4}} (2\pi)\, \frac{\rho^4}{4}\Big|_1^2$$

$$= \frac{1}{2}\left[\left(\frac{\sqrt{2}}{2}\right)^2 - 0^2\right](2\pi)\,\frac{1}{4}\left[(2)^4-(1)^4\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)(2\pi)\left(\frac{1}{4}\right)(16-1) = \boxed{\frac{15\pi}{8}}$$