1. (3 points) Evaluate the following limit. You do not need to show any work.

$$\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$$

(A) 0

(B) $\frac{1}{e}$

(C) $e\pi$

(D) e

(E) 1

(F) Does not exist

2. (5 points) Consider the following integral

$$\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$$

Select the answer that is equivalent to the integral above. You do not need to show any work.

(A) $\int_0^2 \int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)\,dx\,dy$

(B) $\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)\,dx\,dy$

(C) $\int_{\frac{y}{2}}^{\sqrt{y}} \int_0^2 f(x,y)\,dx\,dy$

(D) $\int_0^4 \int_{\sqrt{y}}^{\frac{y}{2}} f(x,y)\,dx\,dy$

(E) $\int_0^2 \int_{\sqrt{y}}^{\frac{y}{2}} f(x,y)\,dx\,dy$

3. Let $f(x, y) = x^3 + e^{xy}$.

(i) (4 points) Evaluate $\nabla f(-1, 3)$.

$f_x = \frac{\partial}{\partial x}(x^3 + e^{xy}) = 3x^2 + ye^{xy},$

$f_y = \frac{\partial}{\partial y}(x^3 + e^{xy}) = xe^{xy}.$

$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 + ye^{xy}, xe^{xy} \rangle$

$\nabla f(-1,3) = \langle 3(-1)^2 + (3)e^{(-1)(3)}, (-1)e^{(-1)(3)} \rangle$

$= \boxed{\langle 3 + 3e^{-3}, -e^{-3} \rangle}$

(ii) (4 points) What is the value of the directional derivative of f at the point $P(-1, 3)$ in the direction of $\vec{v} = \langle -2, 5 \rangle$?

The directional derivative is

$D_{\vec{v}} f(-1,3) = \nabla f(-1,3) \cdot \frac{\vec{v}}{|\vec{v}|}$

$= \langle 3 + 3e^{-3}, -e^{-3} \rangle \cdot \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + (5)^2}}$

$= \frac{1}{\sqrt{29}} \left[(3 + 3e^{-3})(-2) + (-e^{-3})(5) \right]$

$= \frac{1}{\sqrt{29}} (-6 - 6e^{-3} - 5e^{-3})$

$= \boxed{-\frac{1}{\sqrt{29}}(6 + 11e^{-3})} = \boxed{-\frac{11 + 6e^{3}}{\sqrt{29}\, e^{3}}}$

4. Suppose that g is a differentiable function of x and y, and

$$f(u,v) = g(x,\, y), \text{ where } x = u^2 - 2v,\; y = 3u - v.$$

Suppose that the following facts are given:

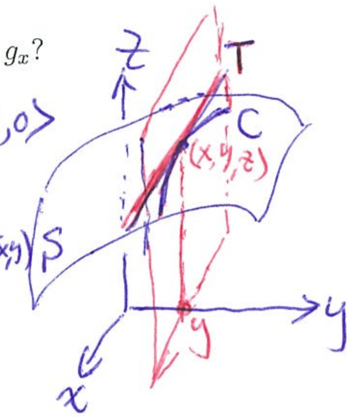
	$g(x,y)$	$f(u,v)$	$g_x(x,y)$	$g_y(x,y)$
$(1,0)$	4	3	4	5
$(1,3)$	3	5	-1	2

(i) (2 points) What is the geometric meaning of the partial derivative g_x?

The rate of change of g along direction $\langle 1,0\rangle$

or

the slope of the tangent line T to the curve C through (x,y,z). See the graph →

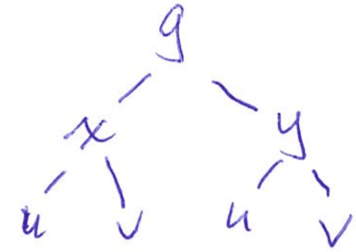


(ii) (5 points) Find $f_u(1,0)$.

as $u=1$, $v=0$, $x=(1)^2-2(0)=1$, $y=3(1)-0=3$

$$f_u(u,v) = g_x(x,y)\frac{\partial x}{\partial u} + g_y(x,y)\frac{\partial y}{\partial v}$$

$$= g_x(x,y)(2u) + g_y(x,y)(3)$$



$$f_u(1,0) = g_x(1,3)(2)(1) + g_y(1,3)(3)$$

$$= (-1)(2) + (2)(3)$$

$$= -2 + 6 = \boxed{4}$$

5. (13 points) Consider the function $f(x, y) = x^3 + 2y^2 - 3x - 4y$. Find all critical points and classify each as a local maximum, a local minimum, or a saddle point or say there is not enough information.

$$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 - 3,\ 4y - 4 \rangle$$

$$\nabla f = \vec{0} \iff \begin{cases} 3x^2 - 3 = 0 & (1) \\ 4y - 4 = 0 & (2) \end{cases}$$

$$(1) \iff x^2 = 1 \iff x = \pm 1$$

$$(2) \iff y = 1$$

There are 2 critical points $\boxed{(1,1),\ (-1,1)}$.

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 - 3) = 6x, \quad f_{xy} = \frac{\partial}{\partial y}(f_x) = 0$$

$$f_{yy} = \frac{\partial}{\partial y}(4y - 4) = 4, \quad f_{yx} = f_{xy} = 0$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx} = (6x)(4) - 0$$

$$= 24x.$$

At $(1,1)$, $D(1,1) = 24(1) = 24 > 0$

$f_{xx}(1,1) = 6(1) = 6 > 0$

So $\boxed{(1,1) \text{ is a local minimum point}}$.

At $(-1,1)$, $D(-1,1) = 24(-1) < 0$, $(\cancel{f_{xx}(-1,1) = 6(-1) < 0})$

So $\boxed{(-1,1) \text{ is a saddle point}}$.

6. (12 points) Consider the parametric surface $\vec{r}(s,t) = \langle s+3t,\ st^2+s,\ \ln(t)\rangle$. Find an equation for the tangent plane at $P(1,-4,0)$.

To find the parameters corresponding to $P(1,-4,0)$, solve $\vec{r}(s,t) = \vec{OP} = \langle 1,-4,0\rangle$, or

$$\begin{cases} s+3t=1 & ① \\ st^2+s=-4 & ② \\ \ln(t)=0 & ③ \end{cases}$$

③ $\Rightarrow t=1$

① $\Rightarrow s+3(1)=1 \Rightarrow s=1-3=-2.$

$\vec{r}_s = \frac{\partial}{\partial s}\langle s+3t, st^2+s, \ln(t)\rangle = \langle 1, t^2+1, 0\rangle$, $\vec{r}_s(-2,1)=\langle 1,2,0\rangle$

$\vec{r}_t = \frac{\partial}{\partial t}\langle s+3t, st^2+s, \ln(t)\rangle = \langle 3, 2st, \frac{1}{t}\rangle$, $\vec{r}_t(-2,1)=\langle 3,-4,1\rangle$

$$\vec{r}_s(-2,1)\times\vec{r}_t(-2,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -4 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 0 \\ -4 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} = 2\vec{i} - \vec{j} - 10\vec{k} = \langle 2,-1,-10\rangle$$

So the equation for the tangent plane at $P(1,-4,0)$ is

$$\boxed{2(x-1)-1(y+4)-10(z-0)=0}$$

or

$$\boxed{2x-y-10z-6=0}$$

7. (12 points) Find the limit if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^6}{(x^2+y^2)^4}$$

let $f(x,y)=\frac{x^2y^6}{(x^2+y^2)^4}$

Choose a path $C_1: y=x$. Then

$$L_1=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1: y=x}} f(x,y)=\lim_{x\to 0}\frac{x^2x^6}{(x^2+x^2)^4}=\lim_{x\to 0}\frac{x^{2+6}}{(2x^2)^4}=\lim_{x\to 0}\frac{x^8}{16x^8}=\frac{1}{16}$$

Choose another path $C_2: y=0$. Then

$$L_2=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2: y=0}} f(x,y)=\lim_{x\to 0}\frac{x^2(0)^6}{(x^2+0^2)^4}=\lim_{x\to 0} 0=0.$$

$L_1\neq L_2$. So the limit does not exist.

Solution 2: let $x=r\cos\theta$, $y=r\sin\theta$

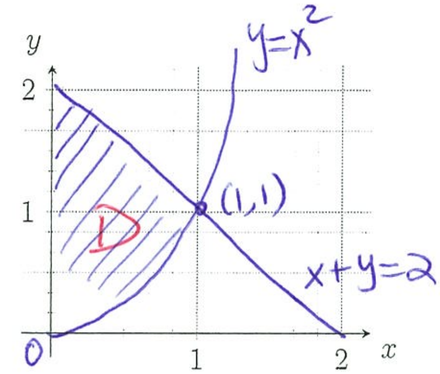
Then $$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{(r\cos\theta)^2(r\sin\theta)^6}{((r\cos\theta)^2+(r\sin\theta)^2)^4}$$

$$=\lim \frac{r^2\cos^2\theta\, r^6\sin^6\theta}{[r^2(\cos^2\theta+\sin^2\theta)]^4}=\lim \cos^2\theta\sin^6\theta$$

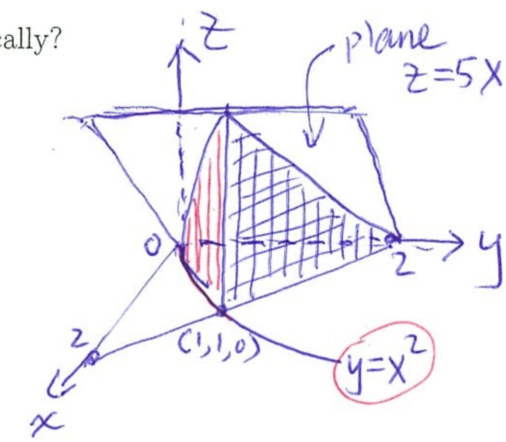
This limit varies as θ varies. For example, along rays, $\theta=\frac{\pi}{4}$, $\theta=\frac{\pi}{6}$, the limits are different. So the limit does not exist.

8. Consider the integral $\iint_D 5x\,dA$, where D is the region bounded by the y-axis, the curves $y + x = 2$ and $y - x^2 = 0$ in the first quadrant.

(i) (4 points) Sketch and shade the region D. Label any intersection points.



(ii) (3 points) What does $\iint_D 5x\,dA$ represent geometrically?



it represents the volume of the solid under the plane $z = 5x$ above the region D.

(iii) (8 points) Evaluate the integral.

$$\iint_D 5x\,dA = \int_0^1 \int_{x^2}^{2-x} 5x\,dy\,dx$$

$$= \int_0^1 5xy \Big|_{y=x^2}^{y=2-x} dx = \int_0^1 5x(2-x-x^2)\,dx$$

$$= 5\int_0^1 (2x - x^2 - x^3)\,dx = 5\left(x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= 5\left(1 - \frac{1}{3} - \frac{1}{4}\right) = (5)\,\frac{12-4-3}{12} = (5)\left(\frac{5}{12}\right) = \boxed{\frac{25}{12}}$$

or $$\iint_D 5x\,dA = \int_0^1 \int_0^{\sqrt{y}} 5x\,dx\,dy + \int_1^2 \int_0^{2-y} 5x\,dx\,dy$$

$$= \int_0^1 \frac{5x^2}{2}\Big|_{x=0}^{x=\sqrt{y}} dy + \int_1^2 \frac{5x^2}{2}\Big|_{x=0}^{x=2-y} dy$$

$$= \int_0^1 \frac{5y}{2}\,dy + \frac{5}{2}\int_1^2 (2-y)^2\,dy = \frac{5y^2}{4}\Big|_0^1 - \frac{5}{6}(2-y)^3\Big|_1^2$$

$$= \frac{5}{4} + \frac{5}{6} = \boxed{\frac{25}{12}}$$

9. (13 points) Use **the method of Lagrange multipliers** to find the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $3x^2 + y^2 = 6$.

Let $g(x,y) = 3x^2 + y^2$

$\nabla f = \langle f_x, f_y \rangle = \langle y, x \rangle$, $\nabla g = \langle g_x, g_y \rangle = \langle 6x, 2y \rangle$

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 6 \end{cases} \Leftrightarrow \begin{cases} \langle y, x \rangle = \lambda \langle 6x, 2y \rangle \\ 3x^2 + y^2 = 6 \end{cases} \Leftrightarrow \begin{cases} y = 6\lambda x & ① \\ x = 2\lambda y & ② \\ 3x^2 + y^2 = 6 & ③ \end{cases}$

① → ② $\Rightarrow x = (2\lambda)(6\lambda x) \Rightarrow x = 12\lambda^2 x$

$\Rightarrow x(1 - 12\lambda^2) = 0 \Rightarrow x = 0$, or $\lambda = \pm\frac{1}{\sqrt{12}} = \pm\frac{1}{2\sqrt{3}} = \pm\frac{\sqrt{3}}{6}$

(i) If $x = 0$, $\overset{①}{\Rightarrow} y = 6\lambda(0) = 0$. $\overset{③}{\Rightarrow} 3(0)^2 + (0)^2 = 6$ NOT possible!

(ii) If $\lambda = \pm\frac{1}{2\sqrt{3}} = \pm\frac{\sqrt{3}}{6}$ ① $\Rightarrow y = \pm\sqrt{3}x$

② $\Rightarrow x = \pm\frac{1}{\sqrt{3}}y \Leftrightarrow y = \pm\sqrt{3}x$

$\overset{③}{\Rightarrow} 3x^2 + (\pm\sqrt{3}x)^2 = 6 \Rightarrow 6x^2 = 1 \Rightarrow x = \pm 1$.

So we get

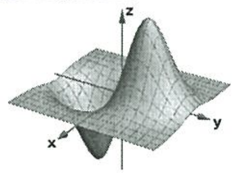
(x, y, λ)	$f(x,y) = xy$	max/min
$(1, -\sqrt{3}, -\frac{\sqrt{3}}{6})$	$(1)(-\sqrt{3}) = -\sqrt{3}$	min
$(-1, \sqrt{3}, -\frac{\sqrt{3}}{6})$	$(-1)(\sqrt{3}) = -\sqrt{3}$	min
$(1, \sqrt{3}, \frac{\sqrt{3}}{6})$	$(1)(\sqrt{3}) = \sqrt{3}$	max
$(-1, -\sqrt{3}, \frac{\sqrt{3}}{6})$	$(-1)(-\sqrt{3}) = \sqrt{3}$	max

By comparison, f attains the $\boxed{\text{max value } \sqrt{3} \text{ at } (1, \sqrt{3}), (-1, -\sqrt{3})}$

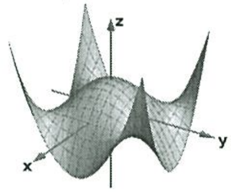
and the $\boxed{\text{min value } -\sqrt{3} \text{ at } (-1, \sqrt{3}), (1, -\sqrt{3})}$.

10. (12 points) Match each 3D surface with one of the contour plots, and one of the equations. Not all the equations will be matched.

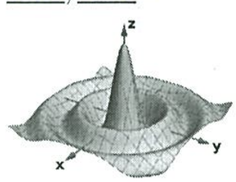
(i) (4), (E)



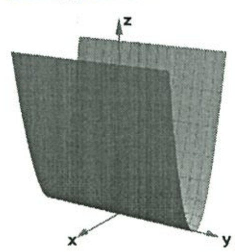
(ii) (3), (D)



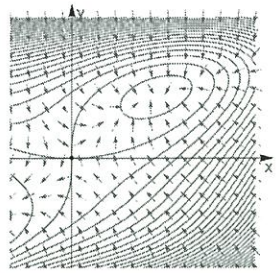
(iii) (5), (F)



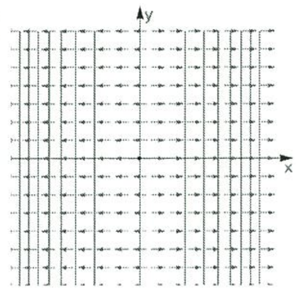
(iv) (2), (A)



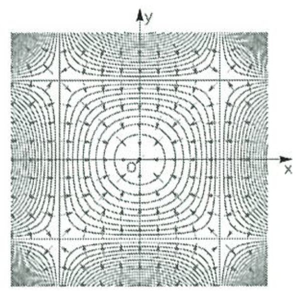
(1)



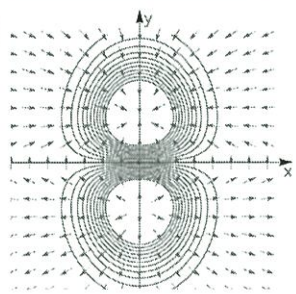
(2)



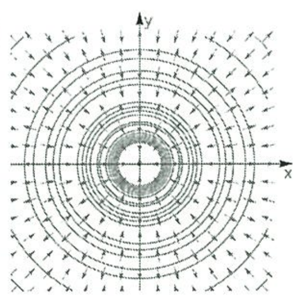
(3)



(4)



(5)



(A) $z = x^2$

(B) $z = y^2$

(C) $z = x^2 + y^2$

(D) $z = (1 - x^2)(1 - y^2)$

(E) $z = ye^{-2x^2 - y^2}$

(F) $z = \dfrac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}$

(G) $z = \sin(xy)$

(H) $z = \sin x \sin y$