1. (8 points) **Note: No partial credit for this problem.**

Let $\vec{v} = \langle 2, 1, 3\rangle$ and $\vec{w} = \langle 2, 0, -1\rangle$. Compute the following:

(a) $2\vec{v} - 3\vec{w} = \boxed{\langle -2, 2, 9\rangle}$

$= \langle 4, 2, 6\rangle - \langle 6, 0, -3\rangle$

$= \langle 4-6, 2-0, 6-(-3)\rangle = \langle -2, 2, 9\rangle$

(b) $|\vec{v}| = \boxed{\sqrt{14}}$

$|\vec{v}| = \sqrt{(2)^2 + (1)^2 + (3)^2} = \sqrt{4+1+9} = \sqrt{14}$

(c) $\vec{v} \cdot \vec{w} = \boxed{1}$

$\vec{v} \cdot \vec{w} = (2)(2) + (1)(0) + (3)(-1)$

$= 4 + 0 - 3 = 1$

(d) $\vec{v} \times \vec{w} = \boxed{\langle -1, 8, -2\rangle}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 2 & 0 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix}$$

$= \vec{i}\,[(-1) - 0] - \vec{j}(-2-6) + \vec{k}(0-2)$

$= -\vec{i} + 8\vec{j} - 2\vec{k}$

$= \langle -1, 8, -2\rangle$

2. (6 points) Use the graphs to select the best possible answer. All graphed vectors lie in the xy-plane.

(a) If $|\vec{a}| = 2$, then $\vec{a} \cdot \vec{b}$ is

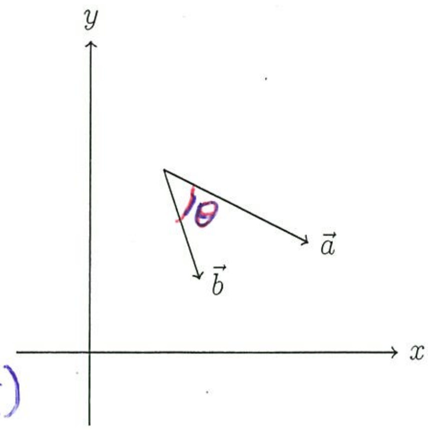
(A) -4

(B) -2

(C) 0

(D) 2

(E) 4



$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$

$= 2|\vec{b}|\cos\theta$

From the graph,

$|\vec{b}| < |\vec{a}| = 2$

$0 < \cos\theta < 1 \quad (0 < \theta < \frac{\pi}{2})$

so $0 < \vec{a}\cdot\vec{b} = 2|\vec{b}|\cos\theta < (2)(2)(1) = 4$

(b) If $|\vec{c}| = 3$, then $\vec{c} \times \vec{d}$ is

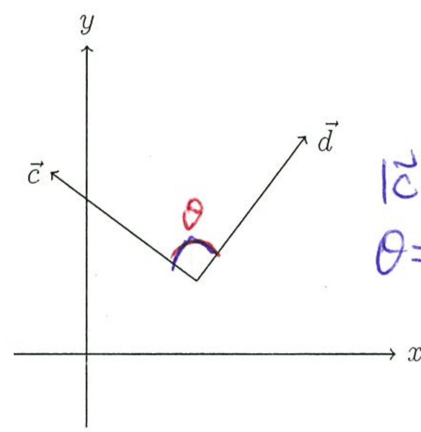
(A) $-9\vec{k}$

(B) $-3\vec{k}$

(C) $\langle 0, 0, 0 \rangle$

(D) $3\vec{k}$

(E) $9\vec{k}$



$|\vec{c}| = |\vec{d}| = 3$

$\theta = \frac{\pi}{2}$

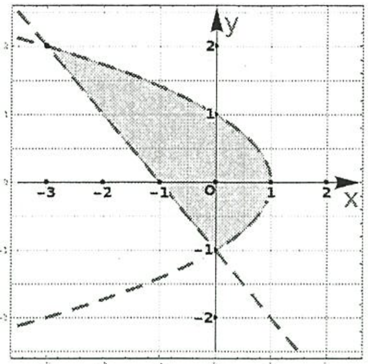
$\vec{c}\times\vec{d} = (|\vec{c}|\cdot|\vec{d}|\sin\theta)\vec{n}$,

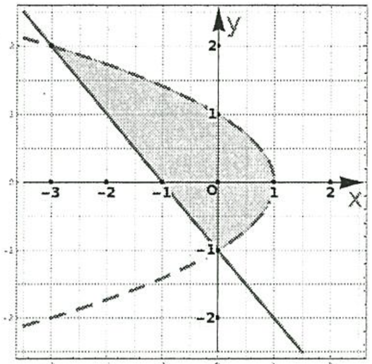
$= (3)(3)\vec{n} = 9\vec{n}$

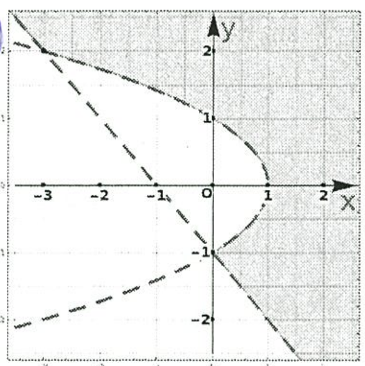
$= -9\vec{k}$

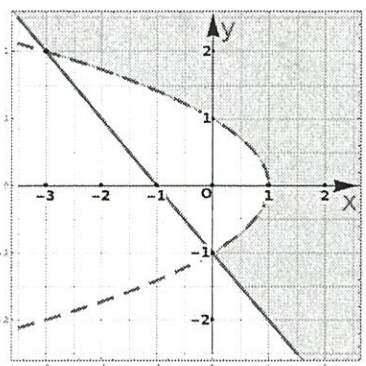
$\vec{n}$ unit vector pointing downward (negative z-axis) by the right hand rule.

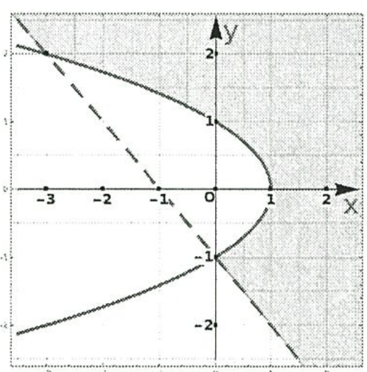
3. (3 points) Let $f(x,y) = \dfrac{\ln(x+y^2-1)}{\sqrt{x+y+1}}$. Which one of the shaded regions below is the domain of f? A dotted line indicates the curve is not part of the domain and a solid line indicates the curve is part of the domain.

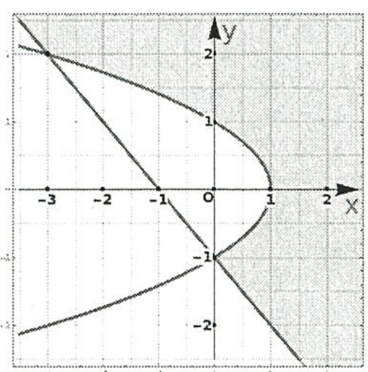
(A)


(D)


(B)


(E)


(C)


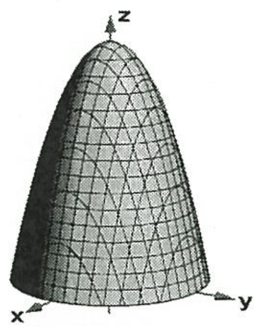
(F)


$D = \{(x,y) \mid x+y+1 > 0,\ x+y^2-1 > 0\}$

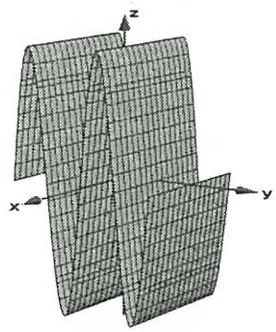
$= \{(x,y) \mid y > -x-1,\ x > 1-y^2\}$

4. (8 points) Match each 3D surface with one of the equations on the right side. Not all equations will be matched.

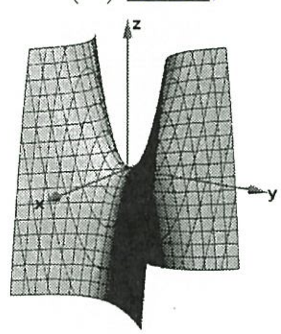
(i) (C)



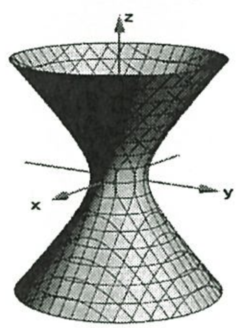
(ii) (G)



(iii) (B)



(iv) (D)



(A) $z - x^2 + y^2 = 0$

(B) $z + x^2 - y^2 = 0$

(C) $z + x^2 + y^2 - 1 = 0$

(D) $x^2 + y^2 - z^2 - 1 = 0$

(E) $x^2 + y^2 - z^2 + 1 = 0$

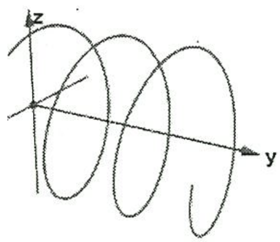
(F) $z - \sin x = 0$

(G) $z - \sin y = 0$

It's easy to match by using traces.

5. (6 points) Match the vector equation with the curve that it parametrizes. Not all equations will be matched.

(i) (C)



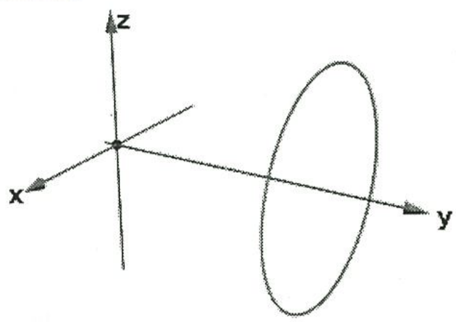
(A) $\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t\rangle,\ 0 \le t \le 6\pi$

(iii) $x^2+y^2=t^2=z^2$ (cone)

The curve is on the cone.

(B) $\vec{r}(t) = \langle t\cos t,\ t, t\sin t\rangle,\ 0 \le t \le 6\pi$

(ii) (F)



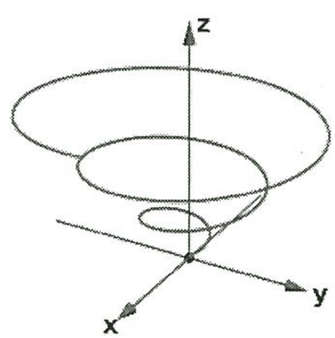
(C) $\vec{r}(t) = \langle \cos t,\ t,\ \sin t\rangle,\ 0 \le t \le 6\pi$

(i) $x^2+z^2=1$ (cylinder)

the curve is on the cylinder.

(D) $\vec{r}(t) = \langle \cos t,\ \sin t,\ 2\rangle,\ 0 \le t \le 6\pi$

(iii) (A)



(E) $\vec{r}(t) = \langle 2,\ \cos t,\ \sin t\rangle,\ 0 \le t \le 6\pi$

(F) $\vec{r}(t) = \langle \cos t,\ 2,\ \sin t\rangle,\ 0 \le t \le 6\pi$

(ii) The curve is on the plane $y=2$, and on the cylinder $x^2+z^2=1$

6. (a) (5 points) Find a rectagular (Cartesian) equation for the surface whose **cylindrical** equation is

$$r = 2\cos\theta. \qquad (\Delta)$$

Solution: Multiplying both sides of (Δ) by r gives

$$r^2 = 2r\cos\theta \qquad ①$$

In Cylindrical coordinates,

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

and $x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

So the eqn ① becomes

$$\boxed{x^2 + y^2 = 2x}$$

(b) (5 points) Which of the following equations in **spherical** coordinates is the surface given by the rectangular (Cartesian) equation

$$x^2 + y^2 + z^2 - 2z = 0? \qquad (\ast)$$

(A) $\rho - 2\sin\phi = 0$

(B) $\rho - 2\cos\phi = 0$

(C) $\rho - 2\sin\theta = 0$

(D) $\rho - 2\cos\theta = 0$

(E) $\rho^2 - 2\sin\theta = 0$

(F) $\rho^2 - 2\cos\theta = 0$

In Spherical Coordinates,

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$x^2 + y^2 + z^2 = \rho^2$.

So ($\ast$) becomes $\rho^2 - 2\rho\cos\phi = 0$

$\Rightarrow \rho = 0$ or $\rho - 2\cos\phi = 0$

or simply $\boxed{\rho - 2\cos\phi = 0}$

7. Suppose a particle is moving along a path. The particle's velocity vector is

$$\vec{v}(t) = \sin(t)\,\vec{i} + t^3\,\vec{j} + e^t\,\vec{k}$$

(a) (4 points) Find the function representing the particles acceleration.

$$\vec{a}(t) = \vec{v}\,'(t) = \boxed{\cos t\,\vec{i} + 3t^2\,\vec{j} + e^t\,\vec{k}}$$

(b) (9 points) Suppose the particle's initial position is at $P(2,\,1,\,-1)$. Find the particle's position vector $\vec{r}(t)$.

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left(\sin t\,\vec{i} + t^3\,\vec{j} + e^t\,\vec{k}\right) dt$$

$$= -\cos t\,\vec{i} + \frac{1}{4}t^4\,\vec{j} + e^t\,\vec{k} + \vec{C}$$

$$\vec{r}(0) = -\cos(0)\,\vec{i} + \frac{1}{4}(0)^4\,\vec{j} + e^0\,\vec{k} + \vec{C}$$

$$= \langle -1, 0, 1\rangle + \vec{C}$$

On the other hand,

$$\vec{r}(0) = \overrightarrow{OP} = \langle 2, 1, -1\rangle$$

So $\langle -1, 0, 1\rangle + \vec{C} = \langle 2, 1, -1\rangle$

$$\vec{C} = \langle 2, 1, -1\rangle - \langle -1, 0, 1\rangle = \langle 3, 1, -2\rangle$$

So $\vec{r}(t) = \langle -\cos t, \frac{1}{4}t^4, e^t\rangle + \langle 3, 1, -2\rangle$

$$= \boxed{\langle 3 - \cos t,\ \frac{1}{4}t^4 + 1,\ e^t - 2\rangle}$$

8. (10 points) Compute the area of triangle with vertices at $(-1, 0, 2)$, $(2, 1, 6)$ and $(1, 1, 2)$.

Solution: let $A(-1,0,2)$, $B(2,1,6)$, $C(1,1,2)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 2,1,6\rangle - \langle -1,0,2\rangle = \langle 3,1,4\rangle$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \langle 1,1,2\rangle - \langle -1,0,2\rangle = \langle 2,1,0\rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 4 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= \vec{i}(0-4) - \vec{j}(0-8) + \vec{k}(3-2)$$

$$= -4\vec{i} + 8\vec{j} + \vec{k} = \langle -4, 8, 1\rangle$$

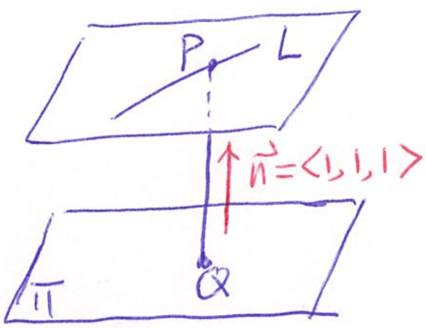
The area of the triangle $\triangle ABC$

$$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(-4)^2 + (8)^2 + (1)^2}$$

$$= \frac{1}{2}\sqrt{16+64+1} = \frac{1}{2}\sqrt{81} = \boxed{\frac{9}{2}}$$

$\Pi: x+y+z-5=0$

9. (12 points) Calculate the distance between the plane $x + y + z = 5$ and the line $\vec{r}(t) = \langle t,\, 3 - 2t,\, t + 1\rangle$, $t \in \mathbb{R}$. The plane and the line do **not** intersect.



Solution 1: Choose any point on the line L: $\vec{r}(t)=\langle t, 3-2t, t+1\rangle$

For example, let $t=0 \Rightarrow P(0,3,1) \in L$.

Then $\text{dist}(L,\Pi) = \text{dist}(P,\Pi)$

$$= \frac{|(0)+(3)+(1)-5|}{\sqrt{(1)^2+(1)^2+(1)^2}} = \frac{|4-5|}{\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{3}}.$$

Note: The formula between a point $P_0(x_0,y_0,z_0)$ and a plane $\Pi: ax+by+cz+d=0$ is $\text{dist}(P_0,\Pi) = \dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

Solution 2: If $PQ \perp \Pi$, Q is the intersection of PQ and Π, then $\text{dist}(L,\Pi) = |\vec{PQ}|$. $\vec{PQ} \parallel \vec{n} = \langle 1,1,1\rangle$ the normal dir. of Π

The eqn of PQ: $\vec{r}(t) = \vec{OP} + \vec{n}t = \langle 0,3,1\rangle + \langle 1,1,1\rangle t$

$$= \langle t, 3+t, 1+t\rangle$$

or $x=t$, $y=3+t$, $z=1+t$, $t \in \mathbb{R}$.

To find the coordinates of Q, plug them into the equ of Π

$$t + (3+t) + (1+t) = 5 \Leftrightarrow 4+3t=5, \text{ so } t=\tfrac{1}{3}$$

so $Q\left(\frac{1}{3}, 3+\frac{1}{3}, 1+\frac{1}{3}\right) = \left(\frac{1}{3}, \frac{10}{3}, \frac{4}{3}\right)$.

$\vec{PQ} = \vec{OQ} - \vec{OP} = \left\langle \frac{1}{3}, \frac{10}{3}, \frac{4}{3}\right\rangle - \langle 0,3,1\rangle = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle$

$$\text{dist}(L,\Pi) = |\vec{PQ}| = \sqrt{\left(\tfrac{1}{3}\right)^2+\left(\tfrac{1}{3}\right)^2+\left(\tfrac{1}{3}\right)^2} = \sqrt{\tfrac{1}{3}} = \boxed{\frac{1}{\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{3}}.$$

Solution 3: Choose any point $R \in \Pi$, for example $R(0,0,5)$.

$\vec{PR} = \vec{OR} - \vec{OP} = \langle 0,-3,4\rangle$. Then

$$\text{dist}(L,\Pi) = |\text{Proj}_{\vec{n}}\vec{PR}| = \frac{|\vec{PR}\cdot\vec{n}|}{|\vec{n}|} = \frac{|\langle 0,-3,4\rangle\cdot\langle 1,1,1\rangle|}{\sqrt{(1)^2+(1)^2+(1)^2}} = \boxed{\frac{1}{\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{3}}$$

10. A particle begins at $A(1,0,0)$ and follows a path C: $\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$, where $t \geq 0$ represents time.

(a) (9 points) Reparametrize the curve with respect to arc length measured from the point $A(1,0,0)$ in the direction of increasing t.

Solution: solve $\vec{r}(t) = \overrightarrow{OA}$ to get the initial value t.

$$\begin{cases} \cos(3t) = 1 \\ \sin(3t) = 0 \\ 4t = 0 \end{cases} \Rightarrow t = 0$$

$$\vec{r}'(t) = \langle -3\sin(3t), 3\cos(3t), 4 \rangle$$

$$|\vec{r}'(t)| = \sqrt{(-3\sin(3t))^2 + (3\cos(3t))^2 + (4)^2} = \sqrt{9+16} = 5$$

$$s(t) = \int_0^t |\vec{r}'(u)|du = \int_0^t 5\,du = 5t$$

solve $s = 5t$ for t and we get $t = \frac{s}{5}$

$$\boxed{\vec{r}(t(s)) = \left\langle \cos\left(\frac{3s}{5}\right), \sin\left(\frac{3s}{5}\right), \frac{4s}{5} \right\rangle, \quad s \geq 0}$$

(b) (3 points) Determine at which value of t the particle has traveled 20 units along the curve from $A(1,0,0)$.

Solution: From (a), $s(t) = 5t$.

Solve $5t = 20$. $t = \frac{20}{5} = \boxed{4}$

11. (12 points) Consider a surface given as a part of the sphere $x^2 + y^2 + z^2 = 16$ that lies inside the cylinder $x^2 + y^2 = 4$ and above the xy-plane ($z \geq 0$). Find a parametric representation for the surface. Specify the bounds for the parameter(s).

Solution 1.
$$\boxed{\begin{cases} x = x \\ y = y \\ z = \sqrt{16 - x^2 - y^2} \end{cases}} \qquad \boxed{x^2 + y^2 \leq 4}$$

Solution 2: Use cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

plug them into the surface eqn $x^2+y^2+z^2=16$

$r^2 + z^2 = 16$. so $z = \sqrt{16 - r^2}$ $(z \geq 0)$

$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{16 - r^2} \end{cases}} \qquad \boxed{\begin{array}{l} 0 \leq \theta < 2\pi \\ 0 \leq r \leq 2 \end{array}}$$

Solution 3: Use spherical coordinates

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

plug them into the surface eqn $x^2+y^2+z^2=16$

$\Rightarrow \rho = 4$. So we have

$$\boxed{\begin{cases} x = 4\cos\theta\sin\phi \\ y = 4\sin\theta\sin\phi \\ z = 4\cos\phi \end{cases}} \qquad \boxed{\begin{array}{l} 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \frac{\pi}{6} \end{array}}$$