Math 2400, Midterm 2

March 12, 2018

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Xingzhou Yang	8:00–8:50
☐	Section 003	Albert Bronstein	9:00–9:50
☐	Section 004	Cliff Blakestad	10:00–10:50
☐	Section 005	Albert Bronstein	10:00–10:50
☐	Section 006	Mark Pullins	11:00–11:50
☐	Section 009	Taylor Klotz	11:00–11:50
☐	Section 007	Albert Bronstein	12:00–12:50
☐	Section 008	Martin Walter	1:00–1:50
☐	Section 010	Braden Balentine	2:00–2:50
☐	Section 011	Pedro Berrizbeitia	3:00–3:50
☐	Section 012	Pedro Berrizbeitia	4:00–4:50

Question	Points	Score
1	12	
2	16	
3	13	
4	14	
5	8	
6	14	
7	14	
8	9	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like $\mathbf{100/7}$ or expressions like $\mathbf{\ln(3)/2}$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (12 points) Determine whether the following function is continuous at $(\mathbf{0}, \mathbf{0})$. Explain why or why not.

$$f(x, y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^4 + y^8}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Take a path $C: x = y^2$ through $(0,0)$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } C}} f(x,y) = \lim_{y\to 0} \frac{(y^2)\,y^2}{\sqrt{(y^2)^4 + y^8}} = \lim_{y\to 0} \frac{y^4}{\sqrt{2}\,y^4} = \frac{1}{\sqrt{2}} \neq f(0,0) = 0$$

So the function f is NOT continuous at $(0,0)$.

2. (a) (8 points) Evaluate the double integral $\iint\limits_R [x + \sin(y)]\, dA$, where R is a rectangle $R = [0, 3] \times [0, 3\pi]$.

$$\iint\limits_R (x+\sin y)\,dA = \int_0^3\int_0^{3\pi}(x+\sin y)\,dy\,dx = \int_0^3 (xy-\cos y)\Big|_{y=0}^{y=3\pi}dx$$

$$= \int_0^3 (3\pi x - \cos(3\pi) + \cos(0))\,dx = \int_0^3 (3\pi x + 2)\,dx$$

$$= \left(\frac{3\pi}{2}x^2 + 2x\right)\Big|_0^3 = \boxed{\frac{27\pi}{2} + 6}$$

$\boxed{\text{Or}}$

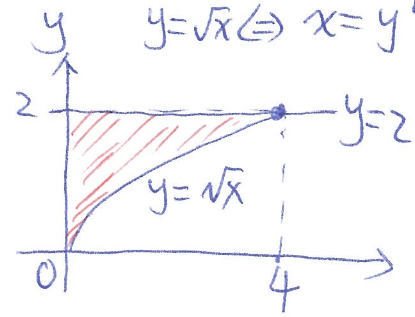
$$= \int_0^{3\pi}\int_0^3 (x+\sin y)\,dx\,dy = \int_0^{3\pi}\left(\frac{x^2}{2} + x\sin y\right)\Big|_{x=0}^{x=3}dy$$

$$= \int_0^{3\pi}\left(\frac{9}{2} + 3\sin y\right)dy = \left(\frac{9}{2}y - 3\cos y\right)\Big|_0^{3\pi}$$

$$= \frac{9}{2}(3\pi) - 3\cos(3\pi) + 3\cos(0) = \boxed{\frac{27}{2}\pi + 6}$$

(b) (8 points) Compute the following double integral by changing the order of integration.

$$\int_0^4 \int_{\sqrt{x}}^2 y\cos\left(y^4\right)\, dy\, dx$$



$$D = \{(x,y) \mid 0 \le x \le 4,\ \sqrt{x} \le y \le 2\}$$

$$= \{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le y^2\}$$

$$\text{Integral} = \int_0^2\int_0^{y^2} y\cos(y^4)\,dx\,dy = \int_0^2 xy\cos(y^4)\Big|_{x=0}^{x=y^2}dy$$

$$= \int_0^2 y^3\cos(y^4)\,dy \xrightarrow[du=4y^3dy]{u=y^4} \int_0^{16}\frac{1}{4}\cos(u)\,du$$

$$= \frac{1}{4}\sin(u)\Big|_0^{16} = \boxed{\frac{\sin(16)}{4}}$$

y	0	2
u	0	$2^4=16$

3. (13 points) Find an equation of the tangent plane to the surface given by

$$\vec{r}(u,v) = \langle u+v,\ u-v,\ u^2-v^3\rangle$$

at the point $(2,0,0)$.

$\vec{r}_u = \langle 1, 1, 2u\rangle$, $\vec{r}_v = \langle 1, -1, -3v^2\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2u \\ 1 & -1 & -3v^2 \end{vmatrix} = \begin{vmatrix} 1 & 2u \\ -1 & -3v^2 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2u \\ 1 & -3v^2 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\vec{k}$$

$$= (-3v^2+2u)\vec{i} - (-3v^2-2u)\vec{j} + (-2)\vec{k}$$

$$= \langle -3v^2+2u,\ 3v^2+2u,\ 2\rangle$$

At $(2,0,0)$, $u+v=2$, $u-v=0$, $u^2-v^3=0$

So $u=v=1$, and the normal direction of the tangent plane is

$$\vec{n} = \vec{r}_u(1,1)\times\vec{r}_v(1,1) = \langle -3(1)^2+2(1),\ 2(1)^2+2(1),\ 2\rangle$$

$$= \langle -1, 5, -2\rangle$$

So the equation of the tangent plane is

$$\boxed{(-1)(x-2) + 5(y-0) + (-2)(z-0) = 0}$$

or $-x+2+5y-2z=0$

or $\boxed{x-5y+2z-2=0}$

4. Given function $z = f(x, y) = x^3 - 5x + y^2 + 3$,

(a) (7 points) Find the gradient of f at $(-1, 1)$.

$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 - 5, 2y \rangle$

$\nabla f(-1,1) = \langle 3(-1)^2 - 5, 2(1) \rangle = \boxed{\langle -2, 2 \rangle}$

(b) (7 points) Find the directional derivative of f at $(-1, 1)$ in the direction of $\vec{v} = \langle 4, -3 \rangle = 4\vec{i} - 3\vec{j}$.

$\vec{v}^0 = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 4, -3 \rangle}{\sqrt{4^2 + (-3)^2}} = \frac{1}{5}\langle 4, -3 \rangle$

$D_{\vec{u}} f(-1,1) = \nabla f(-1,1) \cdot \vec{v}^0$

$= \langle -2, 2 \rangle \cdot \frac{1}{5}\langle 4, -3 \rangle$

$= \frac{1}{5}[(-2)(4) + (2)(-3)]$

$= \frac{1}{5}(-8-6) = \boxed{-\frac{14}{5}}$

5. (8 points) Let $z = f(x, y) = x^2 + 2y^2 + xy^2$. How many critical points does f have?

A. **0**

B. **1**

C. **2**

D. **3**

E. **4**

$\nabla f = \langle f_x, f_y \rangle = \langle 2x + y^2, 4y + 2xy \rangle$

Solve $\nabla f = \langle 0, 0 \rangle$ for x, y

or $\begin{cases} 2x + y^2 = 0 \\ 4y + 2xy = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + y^2 = 0 & (1) \\ 2y(2+x) = 0 & (2) \end{cases}$

From (2) $y = 0$, or $x = -2$

(a) if $y = 0$, plug it into (1) and we get $x = 0$

$(0, 0)$ is a critical point.

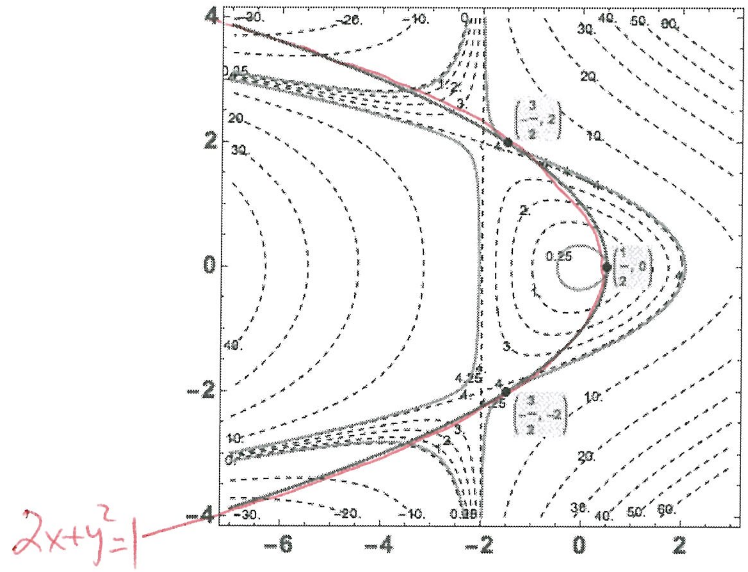
(b) if $x = -2$, from (1) $2(-2) + y^2 = 0$

$y^2 = 4$, $y = \pm 2$.

$(-2, 2)$ and $(-2, -2)$ are the other two critical points.

So there are total $\boxed{\text{3 critical points}}$.

6. (14 points) Use the **method of Lagrange multipliers** to find the maximum and minimum values of $\boldsymbol{f(x,y) = x^2 + 2y^2 + xy^2}$ subject to the constraint $\boldsymbol{g(x,y) = 2x + y^2 = 1}$.



Solution:

$\nabla f = \langle f_x, f_y \rangle$

$= \langle 2x+y^2,\ 4y+2xy \rangle$

$\nabla g = \langle g_x, g_y \rangle = \langle 2, 2y \rangle$

$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=1 \end{cases} \Leftrightarrow \begin{cases} 2x+y^2 = 2\lambda & ① \\ 4y+2xy = 2\lambda y & ② \\ 2x+y^2 = 1 & ③ \end{cases}$

$① - ③ \Rightarrow 2\lambda = 1$ or $\lambda = \frac{1}{2}$

plug it into ②: $4x+2xy = y \Rightarrow 3y+2xy=0$

$y(3+2x)=0$. So $y=0$ or $x=-\frac{3}{2}$

(a) If $y=0 \xrightarrow{③} 2x+0^2=1$, $x=\frac{1}{2}$. the corresponding pt $(\frac{1}{2}, 0)$

$f(\frac{1}{2}, 0) = (\frac{1}{2})^2 + 2(0)^2 + \frac{1}{2}(0)^2 = \boxed{\frac{1}{4}}$

(b) If $x = -\frac{3}{2} \xrightarrow{③} 2(-\frac{3}{2}) + y^2 = 1 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$f(-\frac{3}{2}, \pm 2) = (-\frac{3}{2})^2 + 2(\pm 2)^2 + (-\frac{3}{2})(\pm 2)^2 = \frac{9}{4} + 8 - 6 = \boxed{\frac{17}{4}}$

By Method of Lagrange Multipliers, the max of f is ~~$\frac{1}{4}$~~ $\frac{17}{4}$ and is attained at $(-\frac{3}{2}, \pm 2)$.

However $\frac{1}{4}$ is not the min value of f, b/c $2x+y^2=1$ is not compact. It's easy to check as $(x,y) = (-\frac{15}{2}, -4)$, which is on the curve of $g(x,y) = 2x+y^2=1$,

but $f(-\frac{15}{2}, 4) = -\frac{127}{4} < \frac{1}{4}$.

7. Suppose that $z = F(x, y)$ and that $x = X(u, w)$ and $y = Y(u, w)$, where F, X and Y all have continuous partial derivatives at all points.

Caution: one can view z as a function of x and y, and one can view z as a function of u and w.

Suppose that the following facts are given:

$$\frac{\partial z}{\partial x}(3,4) = q \quad \frac{\partial z}{\partial x}(a,b) = 10 \quad \frac{\partial z}{\partial y}(3,4) = 5 \quad \frac{\partial z}{\partial y}(a,b) = 14$$

$$\frac{\partial x}{\partial u}(a,b) = 10 \quad \frac{\partial x}{\partial w}(a,b) = 8 \quad \frac{\partial y}{\partial u}(a,b) = p \quad \frac{\partial z}{\partial w}(a,b) = 0$$

$$X(p,q) = -2 \quad Y(p,q) = -75 \quad X(a,b) = 3 \quad Y(a,b) = 4$$

(a) (7 points) Use the Chain Rule to find $\frac{\partial z}{\partial u}$ when $u = a$ and $w = b$.

(Your answer may depend on a, b, p, and/or q.)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \qquad X(a,b)=3,\ Y(a,b)=4$$

$$\left.\frac{\partial z}{\partial u}\right|_{\substack{u=a\\w=b}} = \frac{\partial z}{\partial x}(X(a,b),Y(a,b))\cdot\frac{\partial X}{\partial u}(a,b) + \frac{\partial z}{\partial y}(X(a,b),Y(a,b))\frac{\partial y}{\partial u}(a,b)$$

$$= \frac{\partial z}{\partial x}(3,4)\cdot\frac{\partial x}{\partial u}(a,b) + \frac{\partial z}{\partial y}(3,4)\cdot\frac{\partial y}{\partial u}(a,b)$$

$$= (q)(10) + (5)\cdot(p) = \boxed{10q + 5p}$$

(b) (7 points) Use the Chain Rule to find $\frac{\partial y}{\partial w}$ when $u = a$ and $w = b$.

(Your answer may depend on a, b, p, and/or q.)

$$z = F(X(u,w), Y(u,w))$$

$$\frac{\partial z}{\partial w} = F_x \cdot X_w + F_y \cdot Y_w$$

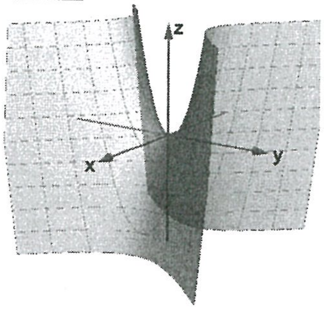
$$\left.\frac{\partial z}{\partial w}\right|_{\substack{u=a\\w=b}} = z_x(3,4)\cdot X_w(a,b) + z_y(3,4)\cdot Y_w(a,b)$$

$$0 = (q)(8) + (5)\frac{\partial y}{\partial w}(a,b)$$

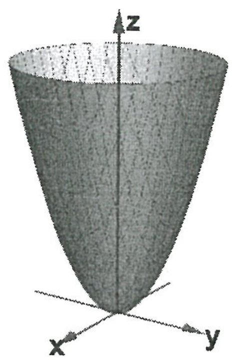
$$\Rightarrow \frac{\partial y}{\partial w}(a,b) = \boxed{-\frac{8q}{5}}$$

8. (9 points) Match each 3D surface with one of the contour plots.

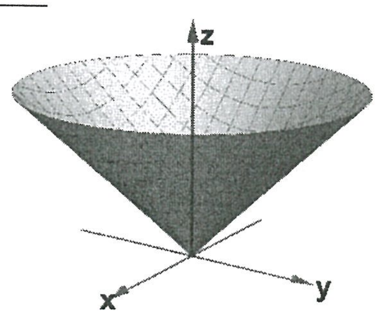
A (5)



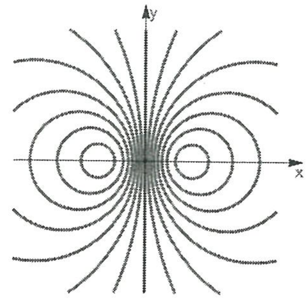
B (2)



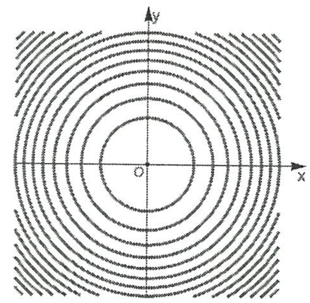
C (3)



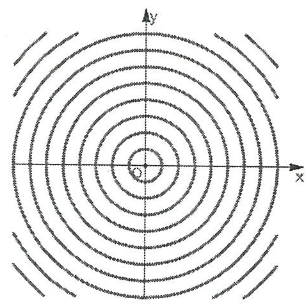
(1)



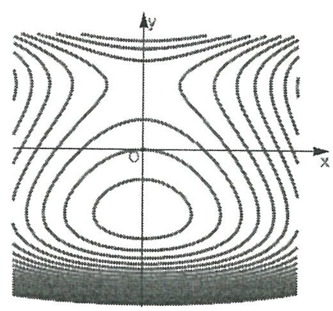
(2)



(3)



(4)



(5)