ANSWER KEY

Math 2400 Final Exam Spring 2017

1. (16 points) You do not have to justify your answers in this question.

(a) (2 pts.) True/False: Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

True

(b) (2 pts.) Complete the statement: If a curve C is parametrized with respect to arc length, then

$\int_0^t |\mathbf{r}'(u)| du =$ arc length of curve from $\vec{r}(0)$ to $r(t)$

(c) (4pts.) Use spherical coordinates to set up the triple integral for the volume of the unit sphere. Do not evaluate.

$$Vol(S) = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

2. (18 points) Short answer questions: show work here and in the rest of the exam.

(a) (4 pts.) Find an equation of a plane that contains a point $(2, 3, 4)$ and is perpendicular to the vector $\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$.

$$i - 4j + 3k = \langle 1, -4, 3 \rangle$$

Equation of plane is:

$$(x-2)\cdot 1 + (y-3)(-4) + (z-4)3 = 0$$

$$x - 2 \quad -4y + 12 \quad + 3z - 12 = 0$$

$$\boxed{x - 4y + 3z - 2 = 0}$$

(b) (4 pts.) Let $f(x, y)$ be given, where $x = u + v$ and $y = u - v$. Use the Chain Rule to compute

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial x}{\partial v} = 1 \quad , \frac{\partial y}{\partial v} = -1$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}}$$

(d) (2 pts.) Let C be a unit circle oriented counterclockwise, centered at $(0,0)$, and let $f(x,y) = \cos x + \ln(2-y)$. Evaluate

$$\int_C \nabla f \cdot d\mathbf{r} = \quad 0$$

(f is continuously differentiable inside an open region containing C and its interior and C is a simple closed curve,

(e) (2 pts.) True/False: Let f be a scalar valued function, then the following expression is meaningful:

curl div f.

False div(f) doesn't make sense

(f) (2 pts.) Complete the statement: Let $\mathbf{F}$ be a conservative vector field on $\mathbb{R}^3$, then

curl $\mathbf{F} =$ curl $\nabla f = \vec{0}$

(Do not give the definition of curl.)

(g) (2 pts.) Finish the formula for the Green's Theorem: Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, if D is a region bounded by C, where C is a piecewise-smooth, positively oriented, simple closed curve in the plane, then

$$\int_C P\ dx + Q\ dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

3. (5 points) Find a vector equation of the tangent line to the helix with parametric equations

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t,$$

at the point $(0, 1, \frac{\pi}{2})$.

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ At $t = \frac{\pi}{2}$, $\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\vec{r}'(\frac{\pi}{2}) = \langle -\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle = \langle -1, 0, 1 \rangle \leftarrow$ Tangent vector to helix at $(0, 1, \frac{\pi}{2})$

Equation of line $\vec{\gamma}(t)$ is

$\vec{\gamma}(t) = \langle 0, 1, \frac{\pi}{2} \rangle + t\langle -1, 0, 1 \rangle \quad t \in \mathbb{R}$

4. (5 points) Demonstrate that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}$$

$x = r\cos\theta$

$y = r\sin\theta$

$\lim\limits_{(x,y)\to(0,0)} = \lim\limits_{r\to 0^+} \frac{(r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2}$

$= \lim\limits_{r\to 0^+} \frac{r^2 \sin^2\theta}{r^2(\cos^2\theta + \sin^2\theta)}$

$= \lim\limits_{r\to 0^+} (\sin^2\theta)$ no limit

Limit Doesn't exist

(c) (6 pts.) Use Lagrange multipliers to set up a system of four equations to find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$. Do not solve for x, y, z, λ.

$$f(x,y,z) = (x-2)^2 + (y-0)^2 + (z+3)^2$$

Distance Squared from (x,y,z) to $(2,0,-3)$

Constraint

$$g(x,y,z) = x+y+z-1 = 0$$

$$\nabla f = \lambda \nabla g$$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}$$

$$\boxed{\begin{aligned} 2(x-2) &= \lambda \cdot 1 \\ 2y &= \lambda \cdot 1 \\ 2(z+3) &= \lambda \cdot 1 \\ x+y+z &= 1 \end{aligned}}$$

Uses fact that Distance is at min when $[\text{Distance}]^2$ is at min

(d) (4 pts.) Compute the Jacobian of the change of variables (x, y) to polar coordinates (r, θ):

$$x = r\cos\theta, \; y = r\sin\theta.$$

Show work.

$$\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial r} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$= \boxed{r}$$

5. (8 points) Suppose you are climbing a hill whose shape is given by $z = 6 - x^2 - y^2$, and you are standing at the point $(1, 2, 1)$.

(a) (3 pts.) In which direction should you walk to go up with the most difficulty?

Let $z = f(x, y) = 6 - x^2 - y^2$. $\nabla f(x, y) = \langle f_x, f_y \rangle$

$= \langle -2x, -2y \rangle$. when $\langle x, y \rangle = \langle 1, 2 \rangle$

$\nabla f(1,2) = \langle -2, -4 \rangle$ At ~~[illegible]~~ $(x, y) = (1, 2)$

to ascend most steeply go in direction of $\nabla(f(1,2))^{\wedge}$: unit vector

is $\dfrac{\langle -2, -4 \rangle}{|\langle -2, -4 \rangle|} = \boxed{\left\langle \dfrac{-2}{\sqrt{20}}, \dfrac{-4}{\sqrt{20}} \right\rangle}$

(b) (2 pts.) What is the rate of ascent in that direction?

rate of ascent is (directional derivative?) : $|\nabla f(1, 2)| = |\langle -2, -4 \rangle|$

$= \boxed{\sqrt{20}}$

(c) (3 pts.) You are still at the point $(1, 2, 1)$. If you walk in the direction of the vector $\mathbf{i} + \sqrt{3}\mathbf{j}$, will you ascend or descend? With what rate?

Directional derivative of f at $(1, 2, 1)$ in direction of $\vec{i} + \sqrt{3}\,\vec{j} = \langle 1, \sqrt{3} \rangle$ is

$\nabla f(1, 2) \cdot \dfrac{\langle 1, \sqrt{3} \rangle}{|\langle 1, \sqrt{3} \rangle|} = \langle -2, -4 \rangle \cdot \dfrac{\langle 1, \sqrt{3} \rangle}{\sqrt{4}}$

$= \dfrac{-2 - 4\sqrt{3}}{2} = \boxed{-1 - 2\sqrt{3}}$

descending!

6. (12 points) Let $f(x, y) = 3x^2 + 3x^2y + 6y^2$. Find all local minima, maxima and saddle points if any. Give both location and value for each.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 6x + 6xy, 3x^2 + 12y \rangle = \langle 0, 0 \rangle$$

$6x(1+y) = 0$ $\quad x = 0$ or $y = -1$

$3x^2 + 12y = 0$ $\quad$ If $x = 0$, $y = 0$

If $y = -1$, $x = \pm 2$

Critical points: $(0,0)$

$(2,-1)$

$(-2,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 6 + 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 12$$

$$H_f(x,y) = \det \begin{bmatrix} 6+6y & 6x \\ 6x & 12 \end{bmatrix}$$

$$= 72 + 72y - 36x^2$$

$H_f(0,0) = 72 > 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) = 6 > 0$

Local min at $(0,0)$ and $f(0,0) = 0$

$H_f(2,-1) = 72 - 72 - 36 \cdot 4 = -144 < 0$

$(x,y) = (2,-1)$ saddle pt.

Similarly, $(x,y) = (-2,-1)$ is saddle pt

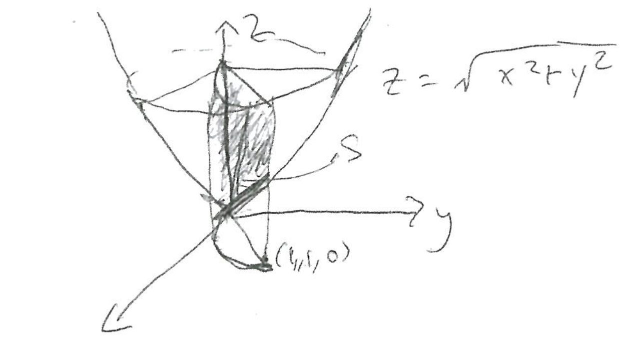
$f(2,-1) = 6,$

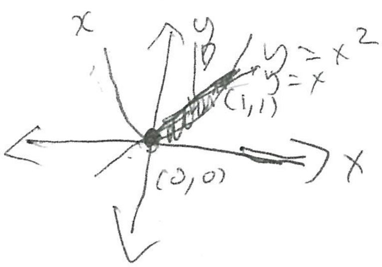
$f(-2,-1) = 6.$

7. (12 points) Evaluate the surface integral

$$\iint_S x \, dS,$$

where S is the part of the cone $z = \sqrt{x^2 + y^2}$ in the first octant that lies between the plane $y = x$ and the parabolic cylinder $y = x^2$.





$$\vec{r}(x,y) = \{(x, y, \sqrt{x^2+y^2})\}$$

$$0 \le x \le 1$$

$$x^2 \le y \le x$$

$$\vec{r}_x = \langle 1, 0, \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x \rangle$$

$$= \langle 1, 0, \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \rangle$$

$$\vec{r}_y = \langle 0, 1, \frac{y}{(x^2+y^2)^{\frac{1}{2}}} \rangle$$

$$\vec{r}_x \times \vec{r}_y = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \\ 0 & 1 & \frac{y}{(x^2+y^2)^{\frac{1}{2}}} \end{bmatrix}$$

$$= \langle \frac{-x}{[x^2+y^2]^{\frac{1}{2}}}, \frac{-y}{[x^2+y^2]^{\frac{1}{2}}}, 1 \rangle$$

$$\therefore |\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1^2} = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{2}$$

$$\therefore \iint_S x \, dS = \iint_D x \, |\vec{r}_x \times \vec{r}_y| \, dA = \iint_D x \cdot \sqrt{2} \, dA$$

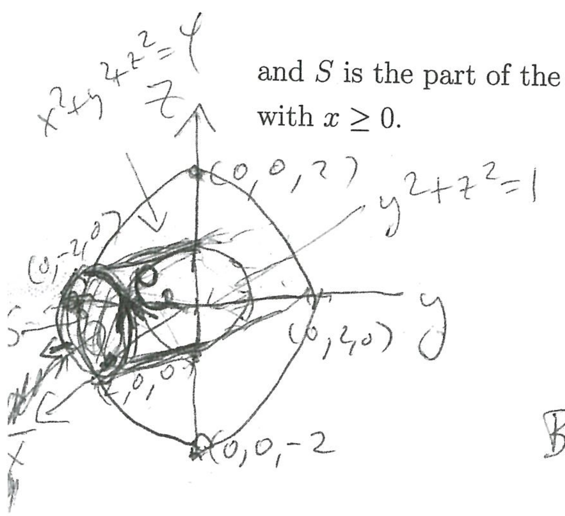
$$= \sqrt{2} \int_0^1 \int_{x^2}^{x} x \, dy \, dx = \sqrt{2} \int_0^1 xy \Big]_{y=x^2}^{y=x} dx = \sqrt{2} \int_0^1 (x^2 - x^3) \, dx$$

$$= \sqrt{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_{x=0}^{x=1} = \sqrt{2} \cdot \left[\frac{1}{3} - \frac{1}{4} \right] = \boxed{\frac{\sqrt{2}}{12}}$$

8. (12 points) Use Stokes' Theorem to show the surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ where

$$\mathbf{F}(x, y, z) = \langle yz, xy, xz \rangle,$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $y^2 + z^2 = 1$ with $x \geq 0$.



$x^2+y^2+z^2=4$ and $y^2+z^2=1$

$\Rightarrow x^2+1=4 \Rightarrow x^2=3$

$\Rightarrow \sqrt{x^2} = \sqrt{3} > 0$ ($x \geq 0$ is given)

$\text{Boundary}(S) = \mathscr{C} = \{(\sqrt{3}, y, z) : y^2+z^2=1\}$

Orient $\mathscr{C}$ positively with respect to outward pointing normal on S:

$\vec{r}(t) = \langle \sqrt{3}, \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi$

$\phantom{\vec{r}(t) = \langle} \underset{x(t)}{\shortparallel} \quad \underset{y(t)}{\shortparallel} \quad \underset{z(t)}{\shortparallel}$

By Stokes Theorem, $\oint_{\mathscr{C}} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$

$\vec{F}(\vec{r}(t)) = \langle (\cos t)(\sin t), \sqrt{3} \cdot (\cos t), \sqrt{3}\sin t \rangle \quad 0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle 0, -\sin t, \cos t \rangle \quad 0 \leq t \leq 2\pi$

$\oint_{\mathscr{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \langle \cos t \sin t, \sqrt{3}\cos t, \sqrt{3}\sin t \rangle \cdot \langle 0, -\sin t, \cos t \rangle\, dt$

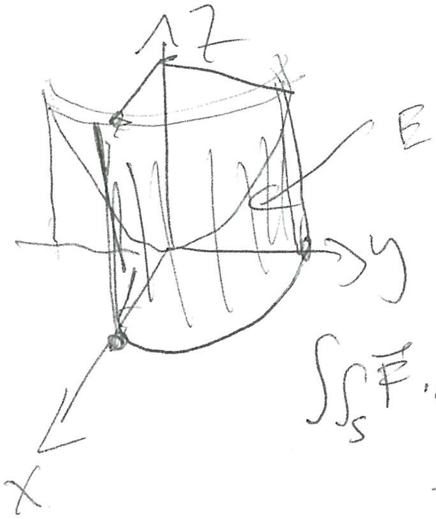
$= \int_0^{2\pi} (0 + -\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t)\,dt = \int_0^{2\pi} 0\,dt = 0,$

So, by Stokes Theorem $\iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S} = \oint_{\mathscr{C}} \vec{F} \cdot d\vec{r} = 0.$

9. (12 points) Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle z^3, (\pi - x)^4, \frac{z}{9\sqrt{x^2+y^2}} \rangle,$$

and S is the positively (outward) oriented boundary of E, where E is the solid in the first octant beneath the paraboloid $z = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 9$.



S has 4 sides,
It is easier to compute by
the divergence Thm

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_E \frac{\partial}{\partial x}(z^3) + \frac{\partial}{\partial y}((\pi - x)^4) + \frac{\partial}{\partial z}\left(\frac{z}{9\sqrt{x^2+y^2}}\right) dV$$

$$= \iiint_E \frac{1}{9\sqrt{x^2+y^2}}\, dV$$

use cylindrical coordinates

first octant

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{r^2} \frac{1}{9 \cdot r}\, dz\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \frac{1}{9}\, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \left[\frac{1}{9} r^2\right] dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{27}\right]_0^3 d\theta = \frac{27}{27} \cdot \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$