Midterm 3 – Math 2400 – December 4, 2017

On my honor as a University of Colorado at Boulder student I have neither given nor received unauthorized assistance on this exam (**please print your name**).

Name: Answer Key

Please select your section:

- ◯ **001** K. Berg .... (8 AM)
- ◯ **002** P. Lessard .... (8 AM)
- ◯ **003** H. Stalvey .... (9 AM)
- ◯ **004** C. Blakestad .... (9 AM)
- ◯ **005** L. Roberson .... (10 AM)
- ◯ **006** H. Stalvey .... (11 AM)
- ◯ **007** T. Klotz .... (8 AM)
- ◯ **008** J. Belcher .... (12 NOON)
- ◯ **009** J. Packer .... (1 PM)
- ◯ **010** A. Bronstein .... (1 PM)
- ◯ **011** A. Healy .... (2 PM)
- ◯ **012** T. Davison .... (2 PM)
- ◯ **013** S. Weinell .... (3 PM)
- ◯ **014** T. Davison .... (3 PM)
- ◯ **015** A. Bronstein .... (4 PM)

In order to receive full credit your answer must be **complete**, **legible** and **correct**. You should show all of your work, and give clear explanations, except for the multiple-choice or true-false questions. This is a closed-book exam. **Papers, note cards, books, calculators, phones, headsets, or other electronic devices are not allowed.**

Problem	Max. points	Points
1	11	
2	11	
3	8	
4	9	
5	11	
6	12	
7	13	
8	13	
9	12	
Total	100	

(1) Let S be the surface lying in the first octant of the xyz-plane parametrized by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle \;=\; \langle u^2, uv, \frac{v^2}{2}\rangle,$$

where $(u,v) \in \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$.

(a) (4 pt.) One calculates the surface area of S by computing the following integral (**Circle** the correct answer):

(i) $\int_0^1 \int_0^1 \mathbf{r}_u \cdot \mathbf{r}_v \, dudv$

(ii) $\int_0^1 \int_0^1 |\mathbf{r}_u \times \mathbf{r}_v|^2 \, dudv$

(iii) $\int_0^1 \int_0^1 |\mathbf{r}_u \times \mathbf{r}_v| \, dudv$

(iv) None of the above.

(b) (7 pt.) The surface area of S is equal to (**Circle** the correct answer):

(i) $\frac{2}{3}$

(ii) 1

(iii) $\sqrt{3}$

(iv) $\frac{\sqrt{2}}{3}$

$\vec{r}_u = \langle 2u, v, 0\rangle$

$\vec{r}_v = \langle 0, u, v\rangle$

$\vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & v \end{bmatrix}$

$= i\, v^2 - j\, 2uv + k\, 2u^2 = \langle v^2, -2uv, 2u^2\rangle$

$|\vec{r}_u \times \vec{r}_v| = |\langle v^2, -2uv, 2u^2\rangle| = \sqrt{v^4 + 4u^2v^2 + 4u^4}$

$= \sqrt{(v^2+2u^2)(v^2+2u^2)} = v^2 + 2u^2$

Surface area $S = \int_0^1 \int_0^1 |\vec{r}_u \times \vec{r}_v| \, dudv =$

$\int_0^1 \int_0^1 [v^2 + 2u^2] dudv = \int_0^1 \left[v^2u + \frac{2u^3}{3}\right]_0^1 dv$

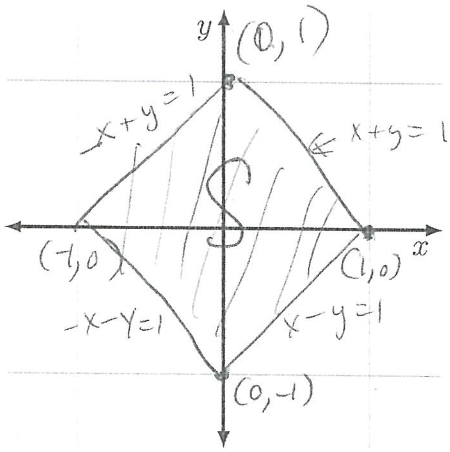
$= \int_0^1 [v^2 + \frac{2}{3}] dv = \left[\frac{v^3}{3} + \frac{2v}{3}\right]_0^1 = \frac{1}{3} + \frac{2}{3} = \boxed{1}$

(2) Let S be the lamina $\{(x, y) \; : \; |x| + |y| \leq 1\}$ with density function

$$\rho(x, y) = |xy| \text{ kg per square unit.}$$

(a) (4 pt.) On the grid below, draw the lamina S, showing its boundary and shading the lamina.

$|x| + |y| \leq 1$



Note S is symmetric about x-axis, about y-axis, and about origin. $\rho(x,y) = |xy|$ is even in both x and y.

(b) (7 pt.) Let S be the lamina described in part (a) above, with the same density function ρ as above. Given that the mass of the lamina S is equal to $\frac{1}{6}$ kg, the center of mass $(\overline{x}, \overline{y})$ of the lamina S is equal to (**circle** the correct answer):

(i) $\left(\frac{1}{6}, 0\right)$

(ii) $(0, 0)$

(iii) $\left(\frac{1}{6}, \frac{1}{6}\right)$

(iv) $\left(0, -\frac{1}{6}\right)$

(v) None of the above.

$$\bar{x} = \frac{\int_{-1}^{1} \int_{|x|-1}^{1-|x|} x \, \rho(x,y) \, dy \, dx}{\frac{1}{6}} = 0$$

(Symmetry)

$$\bar{y} = \frac{\int_{-1}^{1} y \int_{|y|-1}^{1-|y|} \rho(x,y) \, dx \, dy}{\frac{1}{6}} = 0$$

(Symmetry

(3) (8 pt.) A solid tetrahedron is in the first octant of xyz–space and is bounded by the coordinate planes, i.e. the planes $x = 0$, $y = 0$, and $z = 0$, and the plane $x + 2y + z = 2$. Express the volume of the solid as a triple integral. **Circle** the correct answer.

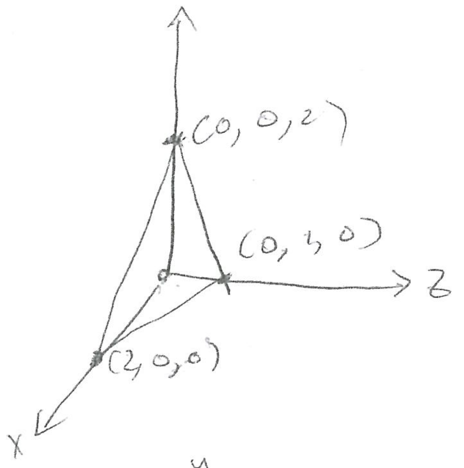
(a) $\displaystyle\int_0^1 \int_0^2 \int_0^{2-x-2y} 1\, dz\, dx\, dy$

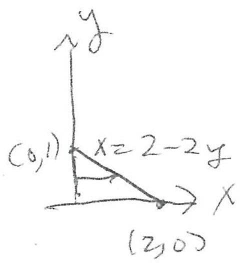
(b) $\displaystyle\int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 1\, dz\, dx\, dy$

(c) $\displaystyle\int_0^2 \int_0^{2-2y} \int_0^{2-x-2y} 1\, dz\, dx\, dy$

(d) $\displaystyle\int_0^1 \int_0^2 \int_0^{2-x-2y} (2 - 2y - z)\, dz\, dx\, dy$

(e) $\displaystyle\int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} (2 - 2y - z)\, dz\, dx\, dy$

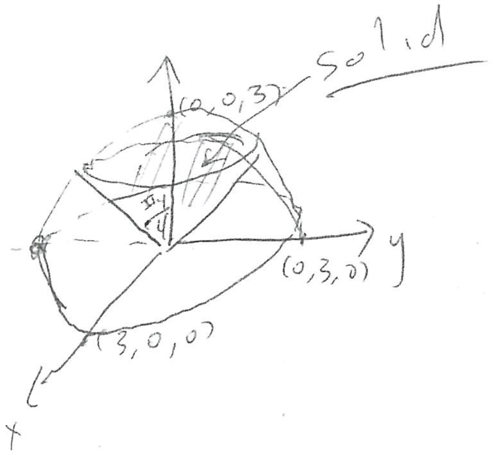




(4) (9 pt.) Using spherical coordinates, set up, but do not evaluate, an integral which finds the mass of a solid of constant density δ that lies within the sphere $x^2 + y^2 + z^2 = 9$, above the xy-plane, and also inside the cone $x^2 + y^2 = z^2$. **Draw a picture** of the solid region to justify your set–up of the integral.

$$\int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{\pi/4}} \int_{\boxed{0}}^{\boxed{3}} \boxed{\delta \cdot \rho^2 \sin\varphi}\, d\boxed{\rho}\, d\boxed{\varphi}\, d\boxed{\theta}$$

$z \geq 0$ since above xy-pla



$x^2+y^2=z^2$, $\sqrt{x^2+y^2}=z$

$\rho^2\sin^2\varphi=\rho^2\cos^2\varphi$

$\sin^2\varphi=\cos^2\varphi$

$|\sin\varphi=\cos\varphi|$

$\varphi=\frac{\pi}{4}$ is eq of top part of cone in spherical coords.

$x^2+y^2+z^2=9$

$\rho^2=9$

$\rho=3$

(5) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by

$$x(u, v) = u + 2v$$
$$y(u, v) = uv.$$

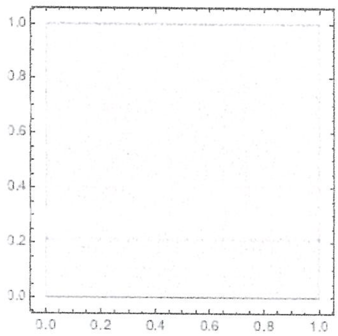
(a) (5 pt.) Compute the Jacobian, $\dfrac{\partial(x, y)}{\partial(u, v)}$, of this transformation.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 1 & 2 \\ v & u \end{bmatrix}$$
$$= \boxed{u - 2v}$$

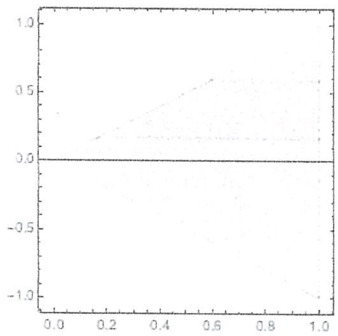
(b) (6 pt.) Match each region in uv-space to its corresponding image, under T, in xy-space, **by writing the correct capital letter in the blank parentheses on the right–hand column.** Hint: consider where vertices in uv-space map under T.

uv-space

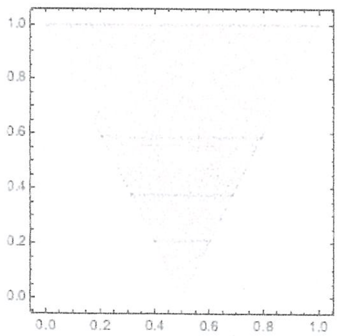
(A)



(B)

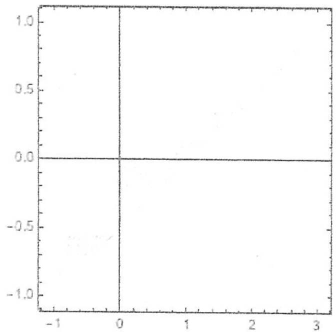


(C)

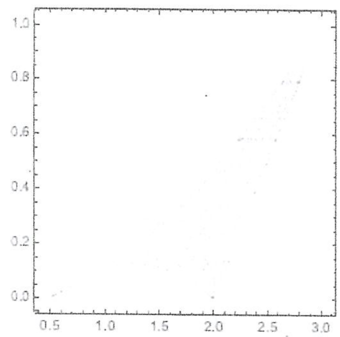


xy-space

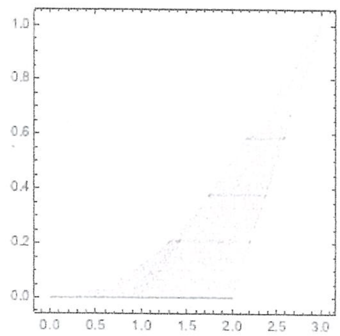
(B)



(C)

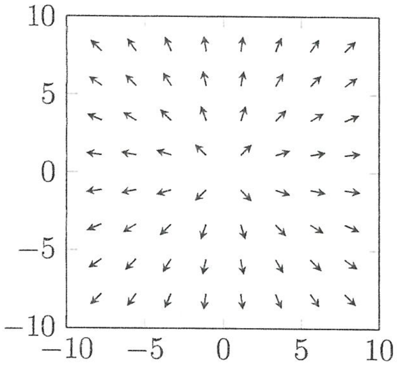


(A)



(6) (3 pt. each) Match the following vector field plots with the corresponding vector functions, **by writing the correct capital letter for the vector field plot in the right-hand column under the matching mathematical formula.**

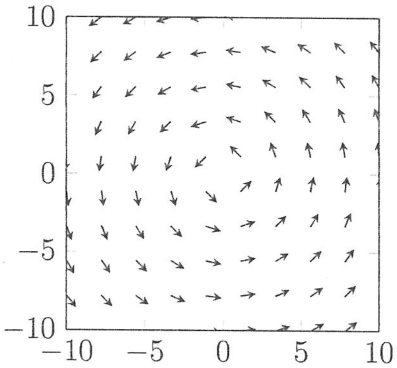
(A)



(I) $\mathbf{F}(x,y) = \langle \frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \rangle$

Vector field plot = C

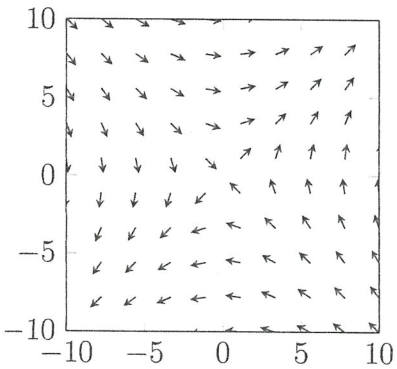
(B)



(II) $\mathbf{F}(x,y) = \langle 1, x/|x| \rangle$

Vector field plot = D

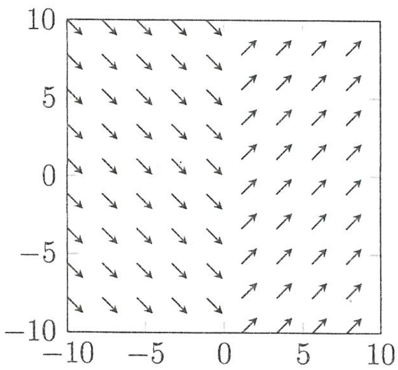
(C)



(III) $\mathbf{F}(x,y) = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$

Vector field plot = A

(D)



(IV) $\mathbf{F}(x,y) = \langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \rangle$

Vector field plot = B

(7) (a) (6 pt.) Evaluate the line integral

$$\int_C ye^x\,dy,$$

where C is the arc of the curve $y = e^x$ from $(0,1)$ to $(1,e)$.

$\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t, e^t\rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \langle 1, e^t\rangle$

$dy = y'(t)\,dt = e^t\,dt$

$\therefore \int_C ye^x\,dy = \int_0^1 y(t)e^{x(t)}y'(t)\,dt$

$= \int_0^1 e^t e^t e^t\,dt = \int_0^1 e^{3t}\,dt$

$= \left.\dfrac{e^{3t}}{3}\right|_{t=0}^{t=1} = \dfrac{e^3 - e^0}{3} = \boxed{\dfrac{e^3-1}{3}}$

(b) (7 pt.) Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F}(x,y,z) = 2xz\mathbf{i} + x^2\mathbf{k}$ and C is the line segment from $(-1,2,0)$ to $(3,0,1)$.

Method 1 Parametrize C: $\vec{r}(t) = (1-t)\langle -1, 2, 0\rangle + t\langle 3, 0, 1\rangle$

$= \langle -1+4t, 2-2t, t\rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 4, -2, 1\rangle \qquad \vec{F}(x(t), y(t), z(t)) = \langle 2(-1+4t)\cdot t, 0, (-1+4t)^2\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(x(t), y(t), z(t))\cdot \vec{r}\,'(t)\,dt = \int_0^1 8(-1+4t)\cdot t + 2\cdot 0 + 1\cdot(-1+4t)^2\,dt$

$= \int_0^1 (-1+4t)(8t + (-1+4t))\,dt = \int_0^1 (-1+4t)(-1+12t)\,dt$

$= \int_0^1 [1 - 16t + 48t^2]\,dt = \left. t - 8t^2 + 16t^3\right|_{t=0}^{t=1} = 1 - 8 + 16 = \boxed{9}$

Method 2: Note $\vec{F}$ is conservative with $\nabla f = \vec{F}$ for $f(x,y,z) = x^2 z$. (Either by observation, or note $\operatorname{curl}\vec{F} = \vec{0}$)

By Fundamental Theorem for line integrals,

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(3,0,1) - f(-1,2,0) = 9\cdot 1 - 1\cdot 0$

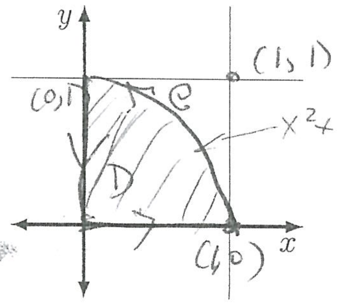
$= \boxed{9}$

(8) Let D be the unit disk restricted to the first quadrant,

$$D = \{(x,y) : x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$$

with C its boundary curve.

(a) (4 pt.) Draw a picture of the region D and its boundary curve C on the grid below, making sure to shade and label D and to label C.



(b) (9 pt.) Use Green's theorem to evaluate the line integral of

$$\mathbf{F}(x,y) = \left(y^2+e^{-x^2}\right)\mathbf{i} + \left(xy^2 + x^3/3\right)\mathbf{j}$$

over the curve C, oriented positively.
Hint: at some stage you should use polar coordinates.

Let $P(x,y) = y^2 + e^{-x^2}$

Let $Q(x,y) = xy^2 + x^3/3$

By Green's Theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$\frac{\partial Q}{\partial x} = y^2 + \frac{3x^2}{3} = y^2+x^2$

$\frac{\partial P}{\partial y} = 2y$

$\Rightarrow \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$= \iint_D (y^2+x^2) - 2y\, dA = \int_0^{\frac{\pi}{2}} \int_0^1 (r^2 - 2r\sin\theta)\, r\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4} - \frac{2}{3} r^3 \sin\theta \Big|_0^1 d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} - \frac{2}{3}\sin\theta\right] d\theta$

$= \frac{1}{4}\theta + \frac{2}{3}\cos\theta \Big|_0^{\frac{\pi}{2}} = \left(\frac{\pi}{8} + 0\right) - \left(0 + \frac{2}{3} \cdot 1\right) = \boxed{\frac{\pi}{8} - \frac{2}{3}}$

(9) Below are a series of statements concerning gradients, vector fields, curl and divergence. Assume all functions and/or components of vector fields have continuous second–order partial derivatives. **Circle** the answer that best describe each statement.

(a) (3 pt.) If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a vector field, and $P_y(x, y) = Q_x(x, y)$ for every point in the domain of $\mathbf{F}$, then $\mathbf{F}$ is conservative.

(i) Always

(ii) Sometimes

(iii) Never

only if domain F is simply connected

(b) (3 pt.) If $f : \mathbb{R}^3 \to \mathbb{R}$, then $\nabla \times (\nabla f) = \mathbf{0}$.

curl grad $f = \bar{0}$

(i) Always

(ii) Sometimes

(iii) Never

(c) (3 pt.) If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

(i) Always

(ii) Sometimes

(iii) Never

divergence curl $\bar{F} = 0$

(d) (3 pt.) If $\mathbf{F}(x, z) = P(x, z)\mathbf{i} - 2\mathbf{j} + R(x, z)\mathbf{k}$, and if $\text{curl}(\mathbf{F}) \neq \mathbf{0}$, then $\text{curl}(\mathbf{F})$ is parallel to the x-axis.

(i) Always

(ii) Sometimes

(iii) Never

$$\text{curl } F = \nabla \times F$$

$$= \det\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & -2 & R \end{bmatrix} =$$

$$\vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial(-2)}{\partial z}\right) - \vec{j}\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + \vec{k}\left[\frac{\partial(-2)}{\partial x} - \right.$$

$$= \vec{i}\,(0-0) - \vec{j}\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + \vec{k}\cdot[0-0]$$

$$= \left\langle 0, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, 0\right\rangle \neq \vec{0}\,!$$

$\vec{i} \cdot \text{curl}(\vec{F}) = \bar{0}$ so curl $\vec{F}$ is never parallel to $\bar{x}$ axis