Midterm 2 – Math 2400 – October 23, 2017

On my honor as a University of Colorado at Boulder student I have neither given nor received unauthorized assistance on this exam (**please print your name**).

Name: Answer Key

Please select your section:

◯ **001** K. Berg .... (8 AM)	◯ **009** J. Packer .... (1 PM)
◯ **002** P. Lessard .... (8 AM)	◯ **010** A. Bronstein .... (1 PM)
◯ **003** H. Stalvey .... (9 AM)	◯ **011** A. Healy .... (2 PM)
◯ **004** C. Blakestad .... (9 AM)	◯ **012** T. Davison .... (2 PM)
◯ **005** L. Roberson .... (10 AM)	◯ **013** S. Weinell .... (3 PM)
◯ **006** H. Stalvey .... (11 AM)	◯ **014** T. Davison .... (3 PM)
◯ **007** T. Klotz .... (8 AM)	◯ **015** A. Bronstein .... (4 PM)
◯ **008** J. Belcher .... (12 NOON)	

In order to receive full credit your answer must be **complete**, **legible** and **correct**. You should show all of your work, and give clear explanations, except for the multiple-choice or true-false questions. This is a closed-book exam. **Papers, note cards, books, calculators, phones, headsets, or other electronic devices are not allowed.**

Problem	Max. points	Points
1	12	
2	14	
3	8	
4	13	
5	12	
6	10	
7	9	
8	6	
9	16	
Total	100	

(1) Let f be a real-valued function of two variables x and y such that $f(0,0) = 0$. In each of the following questions, deduce the correct conclusion. **Circle** the correct answer in each case.

(a) (3 pt.) We calculate that for every m in $\mathbb{R}$,

$$\lim_{x\to 0} f(x, mx) = m.$$

From this we may conclude:

(A) $\lim_{(x,y)\to(0,0)} f(x,y) = m$
(B) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$
(C) The function f is continuous at $(0,0)$
(D) The limit, $\lim_{(x,y)\to(0,0)} f(x,y)$, does not exist
(E) none of the above

(b) (3 pt.) We now choose a particular fixed m in $\mathbb{R}$ and calculate that

$$\lim_{x\to 0} f(x, mx) = 3.$$

From this we may conclude:

(A) $\lim_{(x,y)\to(0,0)} f(x,y) = 3$
(B) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$
(C) The function f is continuous at $(0,0)$
(D) The limit, $\lim_{(x,y)\to(0,0)} f(x,y)$, does not exist
(E) none of the above

(c) (3 pt.) We calculate that

$$\lim_{r\to 0} f(r\cos(\theta), r\sin(\theta)) = 3.$$

From this we may conclude:

(A) $\lim_{(x,y)\to(0,0)} f(x,y) = 3$
(B) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$
(C) The function f is continuous at $(0,0)$
(D) The limit, $\lim_{(x,y)\to(0,0)} f(x,y)$, does not exist
(E) none of the above

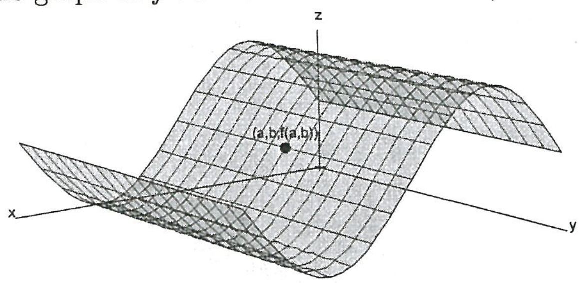
(d) (3 pt.) We calculate that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

From this we may conclude:

(A) $\lim_{(x,y)\to(0,0)} f(x,y) = 3$
(B) $\lim_{(x,y)\to(0,0)} f(x,y) = 1$
(C) The function f is continuous at $(0,0)$
(D) The limit, $\lim_{(x,y)\to(0,0)} f(x,y)$, does not exist
(E) none of the above

(2) The graph of $f : D \to \mathbb{R}$ is shown below, where $D \subset \mathbb{R}^2$ is the domain of the function f.



(a) Determine the sign of the partial and directional derivatives for the function f at (a, b) from the graph of f. **Circle** the correct answer in each case.

(i) (2 pt.) $f_x(a, b)$
(A) positive
(B) negative
(C) zero

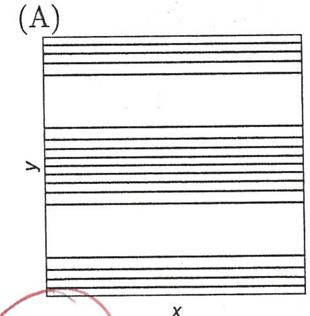
(ii) (2 pt.) $f_y(a, b)$
(A) positive
(B) negative
(C) zero

(iii) (2 pt.) $D_{\mathbf{u}}f(a, b)$ for $\mathbf{u} = \langle -1, 0 \rangle$
(A) positive
(B) negative
(C) zero.

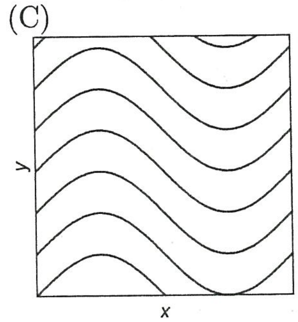
(iv) (2 pt.) $D_{\mathbf{u}}f(a, b)$ for $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$
(A) positive
(B) negative
(C) zero

(b) (3 pt.) One of the pictures below is a **contour map** of f, that is, one picture is a graph consisting of plottings of various level curves for the function f. **Circle** the correct contour map.

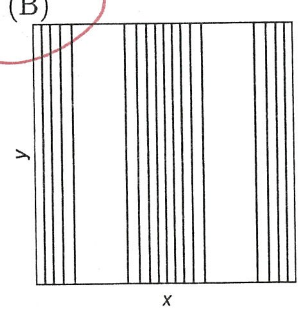
(A)



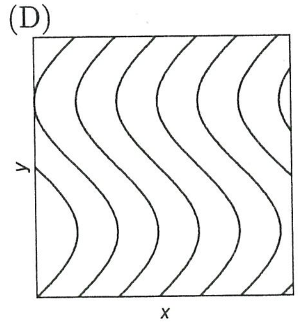
(C)



(B)



(D)



(c) (3 pt.) Assuming the entire graph of f is shown in the picture at the top of the page, the domain D of f is (**Circle** the correct answer):

(A) a line
(B) a rectangle
(C) a circular disk
(D) a sine curve

(3) Let f be a continuously differentiable function of two variables x and y defined on $\mathbb{R}^2$. Suppose

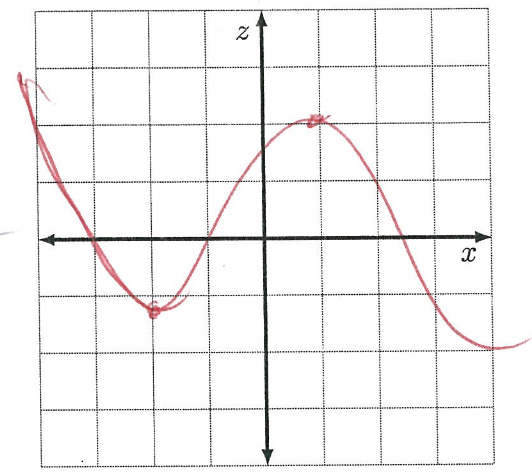
$$\nabla f(-2,0) = \nabla f(1,0) = \langle 0,0\rangle, \text{ and } f_{xx}(1,0) = -2, \text{ and } f_{xx}(-2,0) = 5.$$

(a) (5 pt.) Sketch a possible graph of the curve $z = f(x,0)$ in the xz-plane given below. You need not justify your reasoning.

$f(x,0)$ has local min at $x=-2$

local max at 1

$f(x,0)$ is a continuously differentiable (smooth) function



(b) (3 pt.) Let f be as defined in part 3(a). Which of the following statements about f **cannot be true**? **Circle** the correct answer.

(A) f attains a local maximum at $(1,0)$

(B) f attains an absolute maximum at $(1,0)$

(C) f has a saddle point at $(-2,0)$

(D) f attains a local maximum at $(-2,0)$

(E) f attains a local minimum at $(-2,0)$

For f attaining a local max at $(-2,0)$ we need

$f_x(-2,0) = f_y(-2,0) = 0$ and $f_{xx}(-2,0) < 0$ (among other things)

but we have $f_{xx}(-2,0) = 5 > 0$,

(4) (i) (10 pt.) Suppose

$$u(x,y,z) = xy + xz + yz,\ x(s,t) = st,\ y(s,t) = e^{st}, \text{ and } z(s,t) = t^2.$$

Calculate $\frac{\partial u}{\partial t}$ at $s = 0$ and $t = 1$. **Be sure to justify your reasoning.**

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

u

x y z

s t s t s t

$$\frac{\partial u}{\partial t} = (y+z)s + (x+z)se^{st} + (x+y)2t = 2$$

$$x(0,1) = 0,\ y(0,1) = e^0 = 1,\ z(0,1) = 1$$

$$\Rightarrow \frac{\partial u}{\partial t}(0,1) = (1+1)\cdot 0 + (0+1)\cdot 0 + (0+1)\cdot 2$$

$$= \boxed{2}$$

(ii) (3 pt.) Which of the following is the equation of the tangent plane to the surface

$$2x + y + 2z = 5e^{xyz} - 5$$

at the point $(-1, 0, 1)$? **Circle** the correct answer.

(A) $x + y + 2z = 1$

(B) $x + 6y + 2z = 1$

(C) $\boxed{x + 3y + z = 0}$

(D) $2x + 2y - 3z = 1$

(E) $2x + 6y + 2z = 5$

$$2x + y + 2z - 5e^{xyz} = -5$$

normal vec. is $\vec{n} = \langle 2 - 5yze^{xyz},\ 1 - 5xze^{xyz},\ 2 - 5xye^{xyz} \rangle \Big|_{(-1,0,1)}$

$$= \langle 2, 6, 2 \rangle$$

$$2(x+1) + 6y + 2(z-1) = 0$$

$$2x + \cancel{2} + 6y + 2z - \cancel{2} = 0$$

$$\boxed{x + 3y + z = 0}$$

(5) (i) (3 pt.) At what point in $\mathbb{R}^3$ is the tangent plane to the graph of the function

$$f(x, y) = x^2 + y^2 + xy - x + 4y$$

horizontal? **Circle** the correct answer.

(A) $(2, 3, 29)$

(B) $(2, -3, -7)$

(C) $(-2, 3, 21)$

(D) $(-2, -3, 9)$

(E) $(3, 2, 24)$

Tangent plane is horizontal for $\nabla f(a,b) = \langle 0, 0\rangle$

$\nabla f(x,y) = \langle 2x+y-1,\ 2y+x+4\rangle$

$= \langle 0\quad 0\rangle$

for $x = 2,\ y = -3.$

$f(2,-3) = -7$

(ii) (9 pt.) Over a certain region of space the electrical potential V is given by

$$V(x, y, z) = 2x^2 - 2xy + xyz.$$

Let P be the point with coordinates $(1, 2, 3)$. In which direction does V increase most rapidly at P, and what is the maximum rate of increase? **Be sure to justify your reasoning.**

$\nabla V = \langle V_x, V_y, V_z\rangle = \langle 4x - 2y + yz, -2x + xz, xy\rangle$

$\nabla V(1,2,3) = \langle 4 - 4 + 6, -2+3, 2\rangle$

$= \langle 6, 1, 2\rangle$

$|\nabla V(1,2,3)| = |\langle 6, 1, 2\rangle| = \sqrt{36+1+4} = \sqrt{41}$

V has max rate of increase in direction of gradient; corresponding unit vector at $(1,2,3)$ is

$\frac{1}{\sqrt{41}}\langle 6, 1, 2\rangle = \left\langle \frac{6}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{2}{\sqrt{41}}\right\rangle$

and max rate of increase is $|\nabla V(1,2,3)| = \sqrt{41}$, by theory.

(6) (10 pt.) Find the local maximum and minimum value(s) and the saddle point(s) of the function

$$f(x,y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x.$$

Be sure to justify your reasoning.

Set $\nabla f = \langle 0,0 \rangle$; $f_x = \frac{1}{2}2x - 3y - 9 = x - 3y - 9 = 0$ ①

$f_y = 9y^2 + 18y - 3x + 9 = 0$ ②

① $\Rightarrow$ $x = 3y + 9$ Sub for x into ②

$9y^2 + 18y - 3(3y+9) + 9 = 0$

$9y^2 + 9y - 18 = 0$, $y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$ $y = -2$ or $y = 1$

$y = -2 \Rightarrow x = 3$, $y = 1 \Rightarrow x = 12$

$(3,-2)$ and $(12,1)$ are critical pts,

Use 2nd Deriv. test

$f_{xx} = 1$ $f_{xy} = -3$

$f_{yx} = -3$ $f_{yy} = 18y + 18$

$\det \begin{bmatrix} 1 & -3 \\ -3 & 18y+18 \end{bmatrix}_{(x,y)=(3,-2)} = -18 - 9 = -27 < 0$

$\Rightarrow (3,-2)$ is a saddle pt. for f

$\det \begin{bmatrix} 1 & -3 \\ -3 & 18y+18 \end{bmatrix}_{(x,y)=(12,1)} = 36 - 9 = 27 > 0$

$f_{xx}(12,1) = 1 > 0 \Rightarrow f$ attains local min at $(12,1)$ and.

$f(12,1) = -51$

$(3,-2)$ is saddle pt for f

(7) The quantity Q of a product manufactured depends on the amount of capital K and labor L used according to the equation

$$Q(K, L) = 10K^{\frac{3}{5}}L^{\frac{2}{5}},$$

where $K \geq 0$, and $L \geq 0$. Suppose that capital costs \$20 dollars per unit and labor costs \$50 per unit, and that the total budget is \$1000. Therefore the constraint is

$$20K + 50L = 1000.$$

In each question below, **circle** the correct answer.

(a) (3 pt.) The gradient of the function Q is:

(A) $\langle 6K^{\frac{-2}{5}}L^{\frac{2}{5}}, 4K^{\frac{3}{5}}L^{\frac{-3}{5}} \rangle$

(B) $6K^{\frac{-2}{5}}L^{\frac{2}{5}} + 4K^{\frac{3}{5}}L^{\frac{-3}{5}}$

(C) $\langle 6K^{\frac{3}{5}}L^{\frac{2}{5}}, 4K^{\frac{3}{5}}L^{\frac{2}{5}} \rangle$

(D) $\langle 10K^{\frac{-2}{5}}L^{\frac{2}{5}}, 10K^{\frac{3}{5}}L^{\frac{-3}{5}} \rangle$

(E) none of the above

(b) (3 pt.) The Lagrange Multiplier equation is:

(A) $\lambda \left(6K^{\frac{-2}{5}}L^{\frac{2}{5}} + 4K^{\frac{3}{5}}L^{\frac{-3}{5}}\right) = 70$

(B) $\langle 6K^{\frac{-2}{5}}L^{\frac{2}{5}}, 4K^{\frac{3}{5}}L^{\frac{-3}{5}} \rangle = \lambda\langle 20, 50\rangle$

(C) $\langle 6K^{\frac{3}{5}}L^{\frac{2}{5}}, 4K^{\frac{3}{5}}L^{\frac{2}{5}} \rangle = \lambda\langle 50, 20\rangle$

(D) $\langle 10K^{\frac{-2}{5}}L^{\frac{2}{5}}, 10K^{\frac{3}{5}}L^{\frac{-3}{5}} \rangle = \lambda\langle 20, 50\rangle$

(E) none of above

(c) (3 pt.) The maximum of the quantity Q of the product manufactured occurs when:

(A) $L = 30$ and $K = 8$

(B) $L = 8$ and $K = 30$

Use constraint eq!

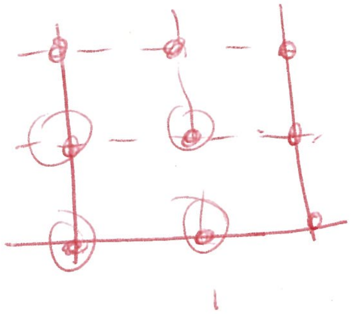
(C) $L = 20$ and $K = 0$

(D) $L = 30$ and $K = 2$

(E) none of the above

(8) Let
$$f(x,y) = 2x + 3y^2.$$

(a) (4 pts) Let $R = [0,2] \times [0,2]$. Use a Riemann sum with $m = n = 2$ to estimate the value of $\int\int_R f(x,y)dA$. Take the sample points to be the lower left corners of the sub–rectangles in the Riemann sum.



2	4	10	16
1	2	5	14
0	6	3	12
y/x	0	1	2

1 pt

Right Fnct Vals

1-pt

2	12	14	16
1	3	5	7
0	0	2	4
y/x	0	1	2

$$(0 + 2 + 5 + 3)\underset{1}{\Delta x}\,\underset{1}{\Delta y} = \boxed{10}$$

(a) (2 pts) Now calculate the actual value of the double integral
$$\int\int_{[0,2]\times[0,2]} \left[2x + 3y^2\right]\, dA.$$

$$\int_0^2 \int_0^2 2x + 3y^2 \, dx\,dy$$

$$\int_0^2 x^2 + 3y^2x \Big|_0^2 \, dy$$

$$= \int_0^2 4 + 6y^2 \, dy$$

$$4y + 2y^3 \Big|_0^2 = 8 + 16 = \boxed{24}$$

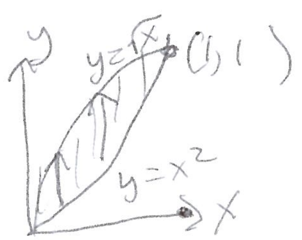
(9) (i) (7 pt.) Set up, but **do not evaluate**, the double integral to compute the volume of the solid S bounded by the cylinders

$$x = y^2 \text{ and } y = x^2 \text{ and the planes } z = 0 \text{ and } z = 3x + 2y.$$

Be sure to justify your reasoning.

$$\mathrm{Vol}(S) = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{x^2}}^{\boxed{\sqrt{x}}} \boxed{3x+2y}\, d\boxed{y}\, d\boxed{x}.$$

Draw region !!



Find Intersection pts of curve

$x^2 = x$

$\Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 1$ or $x = 0$

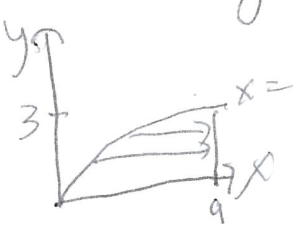
or: $\int_0^1 \int_{y^2}^{\sqrt{y}} (3x+2y)\,dx\,dy$ is OK (problem is symmetric in x and y)

(ii) (9 pt.) Evaluate the double integral

$$\int_0^3 \int_{y^2}^9 y \cos x^2 \, dx dy.$$

Be sure to justify your reasoning.

Draw region!!



← Diagram I

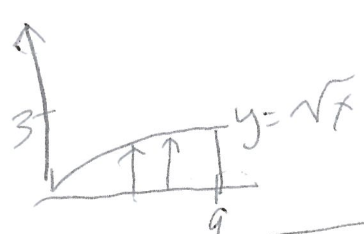


Diagram II

Change the order of the iterated integration using diagram 2. Boundary conditions change.

We set:

$$\int_0^9 \int_0^{\sqrt{x}} y \cos x^2 \, dy\, dx$$

$$= \int_0^9 \frac{y^2}{2} \cos(x^2) \Big|_{y=0}^{y=\sqrt{x}} dx = \int_0^9 \frac{x}{2} \cos(x^2)\, dx$$

$$= \frac{\sin(x^2)}{4} \Big|_{x=0}^{x=9} = \frac{\sin(81) - \sin(0)}{4} = \frac{\sin(81) - 0}{4}$$