Midterm 1 – Math 2400 – September 25, 2017

On my honor as a University of Colorado at Boulder student I have neither given nor received unauthorized assistance on this exam (**please print your name**).

Name: Answer Key

Please select your section:

- ◯ **001** K. Berg ....(8 am)
- ◯ **002** P. Lessard ....(8 am)
- ◯ **003** H. Stalvey ....(9 am)
- ◯ **004** C. Blakestad ....(9 am)
- ◯ **005** L. Roberson ....(10 am)
- ◯ **006** H. Stalvey ....(11 am)
- ◯ **007** T. Klotz ....(8 am)
- ◯ **008** J. Belcher ....(12 noon)
- ◯ **009** J. Packer ....(1 pm)
- ◯ **010** A. Bronstein ....(1 pm)
- ◯ **011** A. Healy ....(2 pm)
- ◯ **012** T. Davison ....(2 pm)
- ◯ **013** S. Weinell ....(3 pm)
- ◯ **014** T. Davison ....(3 pm)
- ◯ **015** A. Bronstein ....(4 pm)

In order to receive full credit your answer must be **complete**, **legible**, and **correct**. You should show all of your work, and give clear explanations, except for the multiple-choice or true-false questions. This is a closed-book exam. **Papers, note–cards, books, calculators, phones or other electronic devices are not allowed.**

Problem	Max. points	Points
1	15	
2	10	
3	10	
4	15	
5	10	
6	15	
7	10	
8	15	
Total	100	

(1) Determine whether the following statements are true or false. **Circle** the correct answer; you do not need to justify your reasoning.

(a) (3 pt.) In $\mathbb{R}^3$, any two lines perpendicular to a third line are parallel to one another.

T: TRUE

F: FALSE

No, take $\vec{b}, \vec{c} \in \mathbb{R}^3$, let $\vec{a} = \vec{b} \times \vec{c}$. $\vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$.

(b) (3 pt.) For any vectors $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$, the expression $\mathbf{v} \times (\mathbf{w} \cdot \mathbf{u})$ is meaningful.

T: TRUE

F: FALSE

No, $\vec{v}$ is vector
$\vec{w} \cdot \vec{u}$ is scalar

(c) (3 pt.) For any vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,

$$(-\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = \frac{|\mathbf{v} \times \mathbf{w}|}{|\mathbf{v}|}.$$

T: TRUE

F: FALSE

No, $-\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$
so $-\vec{v} \times \vec{w} \cdot \vec{v} = 0$.

(d) (3 pt.) There exists a $c \in \mathbb{R}$ such that $3c\mathbf{i} - 10\mathbf{j}$ and $5\mathbf{i} + 3c\mathbf{j}$ are orthogonal in $\mathbb{R}^2$.

T: TRUE

F: FALSE

$\langle 3c, -10 \rangle \cdot \langle 5, 3c \rangle =$
$15c - 30c = 0$ for $c = 0$

(e) (3 pt.) If $\mathbf{a},\ \mathbf{b}, \mathbf{c}, \mathbf{d}\ \in \mathbb{R}^3$, the expression $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ is meaningful.

T: TRUE

F: FALSE

Both $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are vectors in $\mathbb{R}^3$

(2) Let $\mathcal{P}$ denote the plane in three-dimensional space containing both the point

$$(1,1,1)$$

and the line ℓ whose parametric form is given by

$$\langle x,y,z\rangle \;=\; \langle 2,0,0\rangle \;+\; t\langle -1,0,-1\rangle.$$

(a) (6 pt.) Write an equation for the plane $\mathcal{P}$ of the form

$$ax+by+cz+d=0.$$

We note $\langle 1,1,1\rangle - \langle 2,0,0\rangle = \langle -1,1,1\rangle = \vec{a}$ is a vector parallel to the plane P, as is $\vec{b} = \langle -1,0,-1\rangle$. Therefore $\vec{a}\times\vec{b} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

$= \vec{i}\,(1(-1) - 1\cdot 0) - \vec{j}\,[(-1)(-1) - 1(-1)] + \vec{k}\,[(-1)\cdot 0 - 1\cdot(-1)]$

$= -\vec{i} - 2\vec{j} - \vec{k} = \langle -1,-2,1\rangle$ is a normal vector to P

$\therefore$ the equation for the plane P is:

$-1(x-1) - 2(y-1) + 1(z-1) = 0$

$\Rightarrow x - 1 + 2(y-1) - (z-1) = 0$

$\Rightarrow \boxed{x + 2y - z - 2 = 0}$

(b) (4 pt.) Does the point $(1,1,-1)$ lie on the plane $\mathcal{P}$? Explain your reasoning.

Substitute $(1,1,-1)$ into the equation from (a),

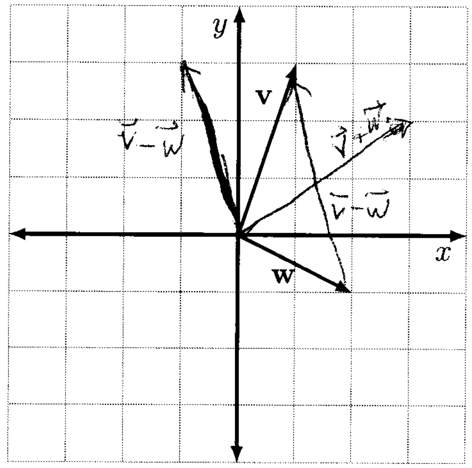
$1 + 2\cdot 1 - (-1) - 2$

$= 3 + 1 - 2 = 2$, not 0, so

$(1,1,-2)$ is not on plane P.

$\boxed{\text{No}}$

(3) (a) (5 pt.) On the grid provided please draw the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. Be sure to label both vectors clearly.



(b) (5 pt.) Assuming each grid square in part (a) above to be of unit height and width, find the angle between $\mathbf{v}$ and $\mathbf{w}$. Express your answer in radians; you do not need to simplify.

$\vec{v} = \langle 1, 3 \rangle, \quad \vec{w} = \langle 2, -1 \rangle$

$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$.

We note $\vec{v} \cdot \vec{w} = 1 \cdot 2 + 3 \cdot -1 = 2 - 3 = -1$

$\vec{v} = \sqrt{1+9} = \sqrt{10}; \quad \vec{w} = \sqrt{4+1} = \sqrt{5}$

$\cos\theta = \dfrac{-1}{\sqrt{10} \cdot \sqrt{5}} = \dfrac{-1}{5\sqrt{2}}$

$\theta = \arccos\left(\dfrac{-1}{5\sqrt{2}}\right)$

(4) Consider the surface defined by the equation $z = x^2 - y^2$ in $\mathbb{R}^3$. Find the specified traces and identify the surface. **Circle** the correct answer; you do not need to justify your reasoning. There are sample grids on the next page **p. 6** for you to use, if you find this helpful.

(a) (3 pt.) For $z = k = -1$ identify the horizontal trace:

A. a circle;

B. an ellipse;

C. a hyperbola;

D. a parabola;

E. a point.

(b) (3 pt.)For $z = k = 0$ identify the horizontal trace:

A. a circle;

B. an ellipse;

C. a hyperbola;

D. a parabola;

E. a pair of lines.

(c) (3 pt.) For $x = k = 1$ identify the vertical trace:

A. a circle;

B. an ellipse;

C. a hyperbola;

D. a parabola;

E. a pair of lines.

(d) (3 pt.) For $x = k = -1$ identify the vertical trace:

A. a circle;

B. an ellipse;

C. a hyperbola;

D. a parabola;

E. the empty set.

(e) (3 pt.) Identify the surface that is defined by the equation $z = x^2 - y^2$:

A. an ellipsoid

B. an elliptic paraboloid;

C. a hyperbolic paraboloid;

D. a cone;

E. a hyperboloid of one sheet;

F. a hyperboloid of two sheets.

(5) Consider the following equation $x^2 + y^2 + z^2 = 4$, given in rectangular coordinates in $\mathbb{R}^3$. **Circle** the correct answer to each question below; you do not need to justify your reasoning.

(a) (3 pt.) Writing this equation in cylindrical coordinates, the correct form for the equation is:

$x = r\cos\theta \quad y = r\sin\theta$

$z = z$

A. $z = \sqrt{4 - r^2}$;

B. $r^2 \cos 2\theta + z^2 = 4$;

C. $r + z = 4$;

D. $r^2 + z^2 = 4$;

E. $z^2 = 4 - r^2 \cos\theta$.

$\boxed{r^2 + z^2 = 4}$

(b) (3 pt.) Writing this equation in spherical coordinates, the correct form for the equation is:

A. $\phi = \frac{\pi}{4}$;

B. $\rho = 4$;

C. $\rho^2 \sin^2\phi \cos 2\theta - \rho^2 \cos^2\phi = 4$;

D. $\rho = 2$;

E. $\rho^2 \cos^2\phi = 4 - \rho^2 \cos\theta$.

$\rho = \sqrt{x^2+y^2+z^2}$

$\rho^2 = x^2+y^2+z^2$

$\rho^2 = 4$

$\rho = 2 \quad (\because \rho \geq 0)$

(c) (4 pt.) Identify the surface that is defined by the equation $x^2 + y^2 + z^2 = 4$ in $\mathbb{R}^3$:

A. a hyperboloid of two sheets;

B. an elliptic paraboloid;

C. a hyperbolic paraboloid;

D. a cone;

E. a sphere.

(A sphere of radius 2)

(6) Let

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sqrt{t+2})\,\mathbf{j} + \left(\frac{1}{t}\right)\mathbf{k} \text{ in } \mathbb{R}^3.$$

(a) (3 pt.) Find the domain of **r**.

Domain $\vec{r} = [-2, \infty) \cap ((-\infty, 0) \cup (0, \infty))$

$= [-2, 0) \cup (0, \infty)$

(b) (6 pt.) Find $\mathbf{r}'(t)$ wherever it is defined.

$\vec{r}'(t) = -\sin t\ \vec{i} + \frac{1}{2}(t+2)^{-\frac{1}{2}}\vec{j} - \frac{1}{t^2}\vec{k}$

$= \langle -\sin t, \frac{1}{2\sqrt{t+2}}, -\frac{1}{t^2} \rangle,$

$t \in (-2, 0) \cup (0, \infty)$

(c) (6 pt.) Find the indefinite integral $\int \mathbf{r}(t)\, dt$.

$\int \vec{r}(t)dt = \int \langle \cos t, \sqrt{t+2}, \frac{1}{t} \rangle dt$

$= \langle \sin t, \frac{(t+2)^{3/2}}{3/2}, \ln|t| \rangle + \vec{c},$

$\vec{c} \in \mathbb{R}^3$

$= \langle \sin t + c_1, \frac{2}{3}(t+2)^{3/2} + c_2, \ln|t| + c_3 \rangle,$

$c_1, c_2, c_3 \in \mathbb{R}$

[the integral must be taken over interval within the domain of $\vec{r}$]

(7) Let

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3 - t\right)\mathbf{i} + t^2\mathbf{j} \text{ in } \mathbb{R}^2.$$

(a) (6 pt.) Find the length of the curve $\mathbf{r}(t)$ from $t = 0$ to $t = 2$.

$\vec{r}(t) = \langle x(t), y(t) \rangle$, for $x(t) = \frac{t^3}{3} - t$, $y(t) = t^2$

$x'(t) = t^2 - 1$, $\quad y'(t) = 2t$

$\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(t^2-1)^2 + (2t)^2} = \sqrt{t^4 - 2t^2 + 1 + 4t^2}$

$= \sqrt{t^4 + 2t^2 + 1} = \sqrt{(t^2+1)^2} = t^2 + 1$ on $[0, 2]$

$\therefore \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^2 (t^2+1)\, dt = \left.\frac{t^3}{3} + t\right|_0^2$

$= \frac{8}{3} + 2 = \boxed{\frac{14}{3} \text{ units}}$ arclength

(b) (4 pt.) Give a formula for the unit tangent vector $\mathbf{T}(t)$ to $\mathbf{r}(t)$. You need not simplify.

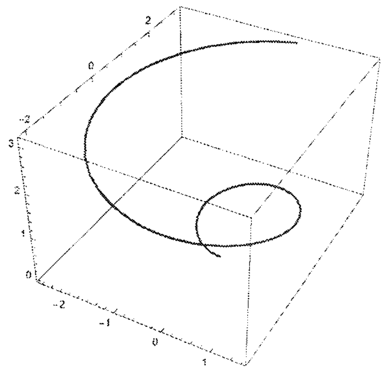
$\vec{r}'(t) = \langle t^2 - 1, 2t \rangle$

$|\vec{r}'(t)| = \sqrt{(t^2-1)^2 + (2t)^2} = t^2 + 1$

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle \frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1} \right\rangle$

(8) The graph below shows curve C traced by a particle with position vector

$$\mathbf{r}(t) = \langle -t\cos(3t), -t\sin(3t), t\rangle \text{ for time } 0 \le t \le \pi.$$



(a) (5 pt.) Find the particle's starting point, P_0, and ending point, P_π.

$P_0 = \vec{r}(0) = \langle 0, 0, 0\rangle$

$P_\pi = \vec{r}(\pi) = \langle -\pi\cos(3\pi), -\pi\sin(3\pi), \pi\rangle = \langle \pi, 0, \pi\rangle$

(b) (5 pt.) Determine the speed of the particle at time $t = \frac{\pi}{2}$.

$\vec{r}'(t) = \langle -\cos(3t) + 3t\sin(3t), -\sin(3t) - 3t\cos(3t), 1\rangle$

$\vec{r}'(\frac{\pi}{2}) = \langle -\frac{3\pi}{2}, 1, 1\rangle$

Speed at $|\vec{r}'(\frac{\pi}{2})| = \sqrt{(\frac{-3\pi}{2})^2 + (1)^2 + (1)^2}$

$= \sqrt{\frac{9\pi^2}{4} + 2}$

(c) (5 pt.) Suppose a second particle travels along the same path as the original particle, but it is moving three times faster than original particle. Find a parametric curve equation $\mathbf{c}(t)$ to describe the second particle's path from P_0 to P_π.

$\vec{c}(t) = \langle -3t\cos(9t), -3t\sin(9t), 3t\rangle,$

$0 \le t \le \frac{\pi}{3}$

Traces out same path, 3 times faster