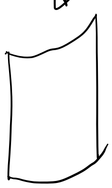
Final Practice MATH 2400-004, CALCULUS III, FALL 2019

Name:

1. Let $\vec{\mathbf{F}}(x,y,z) = \langle x, y, xe^z \rangle$ and let S be the part of the cylinder $x^2+y^2=1$ beneath the plane $z=2$ and in the first octant, with positive orientation. Calculate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$.

$\boxed{\text{Must compute directly}}$

S 

$$\vec{r}(h,\theta) = \langle \cos(\theta), \sin(\theta), h \rangle$$

$$0 \le h \le 2$$

$$0 \le \theta \le \pi/2$$

$$\vec{r}_h = \langle 0, 0, 1 \rangle$$

$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= \langle -\cos\theta, -\sin\theta, 0 \rangle$$

Wrong orientation!

$$\vec{n} = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\int_0^{\pi/2} \int_0^2 \langle \cos\theta, \sin\theta, \cos\theta e^{\cos\theta} \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle \, dh \, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \cos^2\theta + \sin^2\theta \, dh \, d\theta$$

$$= \int_0^{\pi/2} 2 \, d\theta = \boxed{\pi}$$

2. Let $\vec{\mathbf{F}}(x,y,z) = \langle e^{\sin(y)}, xz+2, \sin(y)x^2+z\rangle$ and let S be the paraboloid $y = 1-x^2-z^2$ with $y \geq 0$, with positive orientation. Calculate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$.

Almost closed!

$$\iint_S \vec{F}\cdot d\vec{S} = \underbrace{\iiint_E \operatorname{div}(\vec{F})}_{1} - \underbrace{\iint_{S'} \vec{F}\cdot d\vec{S}}_{2}$$

S' is

$\vec{r}(r,\theta) = \langle r\cos\theta, 0, r\sin\theta\rangle$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

1 $\operatorname{div}(\vec{F}) = 0 + 0 + 1$

$= 1$

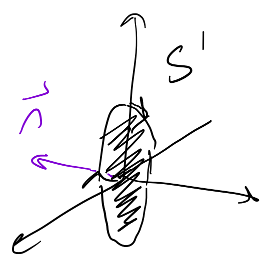
$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r \, dy\, dr\, d\theta$$

$$= 2\pi \int_0^1 r - r^3 \, dr = 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$$

2 $\vec{r}(r,\theta) = \langle r\cos\theta, 0, r\sin\theta\rangle$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$



$\vec{n} = \langle 0, -r, 0\rangle$

$$\int_0^{2\pi}\int_0^1 \langle \ldots, r^2\cos\theta\sin\theta + 2, \ldots\rangle \cdot \langle 0, -r, 0\rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r^3\cos\theta\sin\theta - 2r \, dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{\cos\theta\sin\theta}{4} - 1 \, d\theta$$

$= -2\pi$

So $\frac{\pi}{2} - (-2\pi) = \boxed{\frac{5\pi}{2}}$

3. Let $\vec{\mathbf{F}}(x,y,z) = \langle ye^{xy} - zy, xe^{xy} - xz, -xy\rangle$ and let C be the intersection of the cylinder $z^2+y^2=9$ and the paraboloid $x = y^2+z^2$, oriented clockwise. Calculate $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

$$f(x,y,z) = e^{xy} - xyz$$

$$\nabla f(x,y,z) = \langle ye^{xy} - yz, xe^{xy} - xz, -xy \rangle$$

$\vec{F}$ is conservative & C is a loop, therefore

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 0}$$

4. Let $\vec{\mathbf{F}}(x,y,z) = \langle ye^{xy} - yz, xe^{xy} - xz, 1 - xy\rangle$, and let C be the curve $\vec{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$ with $0 \leq t \leq 2\pi$. Calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

$$f(x,y,z) = e^{xy} - xyz + z$$

$\vec{F}$ conservative

$$\nabla f = \langle ye^{xy} - yz, xe^{xy} - xz, 1 - xy\rangle$$

$$\vec{r}(0) = \langle 0, 0, 0\rangle$$

$$\vec{r}(2\pi) = \langle 2\pi, 0, 2\pi\rangle$$

By FTOLI

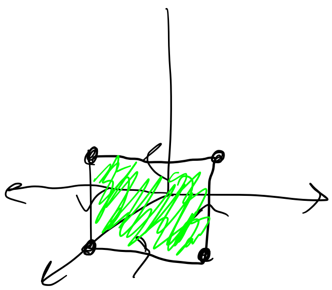
$$\int_C \vec{F} \cdot d\vec{r} = f(2\pi, 0, 2\pi) - f(0,0,0)$$

$$= (1 + 2\pi) - 1$$

$$= \boxed{2\pi}$$

5. Let $\vec{\mathbf{F}}(x,y,z) = \langle x\cos(x), xy - z, e^z + y\rangle$ and let C be the lines connecting $(1,0,0), (1,1,0), (1,1,1)$, and $(1,0,1)$, oriented counterclockwise. Calculate $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$.

$\vec{F}$ is not conservative



S is green

<u>Stokes' Theorem!!!</u>

$$\left.\begin{array}{l} \vec{r}(y,z) = \langle 1, y, z\rangle \\ 0 \le y \le 1, \quad 0 \le z \le 1 \end{array}\right\} \underline{S}$$

$$\vec{n} = \langle 1, 0, 0\rangle$$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x\cos(x) & xz - z & e^z + y \end{vmatrix} = \langle 1-1, 0, y\rangle$$

$$= \langle 0, 0, y\rangle$$

$$\int_0^1\int_0^1 \langle 0, 0, y\rangle \cdot \langle 1, 0, 0\rangle \, dy\, dz$$

$$= \boxed{0}$$

6. Let $\vec{\mathbf{F}}(x,y,z) = \langle y+\cos(x), \cos(y)e^y, z^2 - x\rangle$ and let S be the surface of the region bounded by the parametric equations

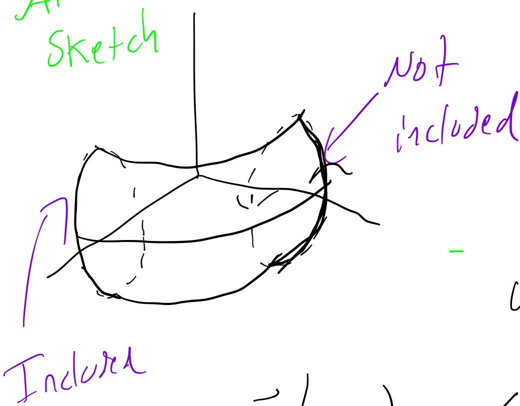
$$\vec{\mathbf{r}}(u,v) = \langle \cos(u)(2+\sin(v)), \sin(u)(2+\sin(v)), \cos(v)\rangle$$

It's a quarter of a torus

with $0 \le u \le \frac{\pi}{2}$ and $0 \le v \le 2\pi$ and the plane $y = 0$, positively oriented. Calculate $\iint_S \text{curl } \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$.

Stokes' Thm

Amateur Sketch



Boundary is circle of radius 1 centered at $(0,2,0)$.

Use Stokes' Thm to replace S with S' where S' is given by

$$\vec{r}(r,\theta) = \langle 0, 2+r\cos\theta, r\sin\theta\rangle$$

$$0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

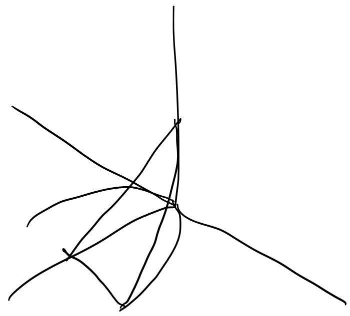
$$\text{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+\cos(x) & \cos(y)e^y & z^2 - x \end{vmatrix} = \langle 0, 1, -1\rangle$$

$$\vec{n} = \langle r, 0, 0\rangle$$

$$\int_0^{2\pi}\int_0^1 \langle 0,1,-1\rangle \cdot \langle r,0,0\rangle \, dr\, d\theta$$

$$= \boxed{0}$$

7. Evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x,y,z) = \langle e^{y^2} + xy, y^2 + z, e^{x^2} - zy \rangle$ and where S is the positively-oriented boundary of the region bounded by and including the following sides: $z = 0$, $x = 1$, $y \geq 0$, $x = y^2$, and $z - x = 1$.



$$\operatorname{div}(\vec{F}) = y + 2y - y = 2y$$

Divergence Thm!

$$\int_0^1 \int_{y^2}^1 \int_0^{1+x} 2y \, dz \, dx \, dy$$

$$= \int_0^1 \int_{y^2}^1 2y + 2yx \, dx \, dy$$

$$= \int_0^1 2yx + yx^2 \Big|_{x=y^2}^{x=1} dy$$

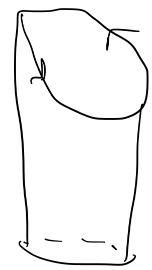
$$= \int_0^1 (2y + y) - (2y^3 + y^5) \, dy$$

$$= \int_0^1 3y - 2y^3 - y^5 \, dy$$

$$= \frac{3y^2}{2} - \frac{y^4}{2} - \frac{y^6}{6} \Big|_0^1$$

$$= \frac{3}{2} - \frac{1}{2} - \frac{1}{6} = \boxed{\frac{5}{6}}$$

8. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}}(x, y, z) = \langle \sin(x^2), yz - e^y, \sin(y^2) - x \rangle$ and C is the boundary (oriented counterclockwise) of the portion of the surface $x^2 + y^2 = 9$ (including $z = 0$) cut off at the plane $y + z = 4$.



$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(x^2) & yz - e^z & \sin(y^2) - x \end{vmatrix}$$

New surface

$$= \langle 2y\cos(y^2) - y, 1, 0 \rangle$$

$$\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 4 - r\sin\theta \rangle$$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -\sin\theta \\ -r\sin\theta & r\cos\theta & -r\cos\theta \end{vmatrix} = \langle 0, r, r \rangle$$

$$\int_0^{2\pi}\int_0^3 \langle \ldots, 1, 0 \rangle \cdot \langle 0, r, r \rangle \, dr \, d\theta$$

$$= 2\pi \int_0^3 r \, dr$$

$$= 2\pi \left(\frac{r^2}{2} \Big|_0^3 \right) = \boxed{9\pi}$$

Scalar field

9. Evaluate $\iint_S e^z\, dS$ where S is the positively-oriented surface given by $\vec{r}(u,v) = \langle \cos(u), \sin(u), \frac{uv}{2} \rangle$ with $0 \le u, v \le \pi$.

Doesn't matter

$$\vec{r}_u = \left\langle -\sin(u),\ \cos(u),\ \frac{v}{2} \right\rangle$$

$$\vec{r}_v = \left\langle 0,\ 0,\ \frac{u}{2} \right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \begin{vmatrix} i & j & k \\ -\sin(u) & \cos(u) & \frac{v}{2} \\ 0 & 0 & \frac{u}{2} \end{vmatrix} = \left| \left\langle \frac{u\cos(u)}{2},\ \frac{u\sin(u)}{2},\ 0 \right\rangle \right|$$

$$= \sqrt{\frac{u^2\cos^2 u}{4} + \frac{u^2\sin^2 u}{4}} = \frac{u}{2}$$

$$\int_0^{\pi}\int_0^{\pi} \frac{u}{2} e^{uv/2}\, dv\, du$$

$$w = \frac{uv}{2}$$

$$dw = \frac{u}{2}\, dv$$

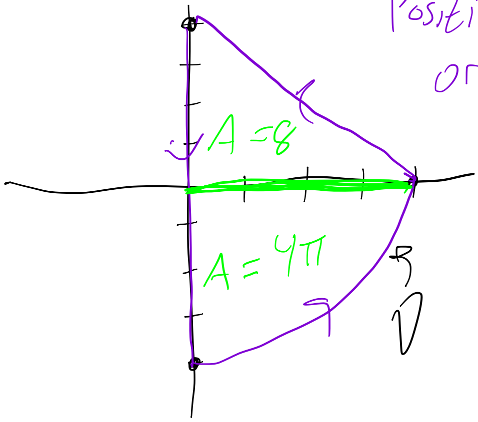
$$\int_0^{\pi}\int_0^{\frac{u\pi}{2}} e^{w}\, dw\, du$$

$$= \int_0^{\pi} e^{u\pi/2} - 1\, du$$

$$= \left(\frac{2}{\pi} e^{u\pi/2} - u\right)\Big|_0^{\pi}$$

$$= \boxed{\frac{2}{\pi} e^{\pi^2/2} - \pi - \frac{2}{\pi}}$$

10. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}}(x,y) = \langle \cos^2(x) + y^2 x, x^2 y + x - e^{y^2} \rangle$ and C is the path starting at $(0,4)$, going down the y-axis to $(0,-4)$, traveling counterclockwise along $x^2+y^2=16$ to $(4,0)$, then following $y+x=4$ back to $(0,4)$.



Positive orientation

Green's Theorem!

$$\frac{\partial P}{\partial y} = 2yx$$

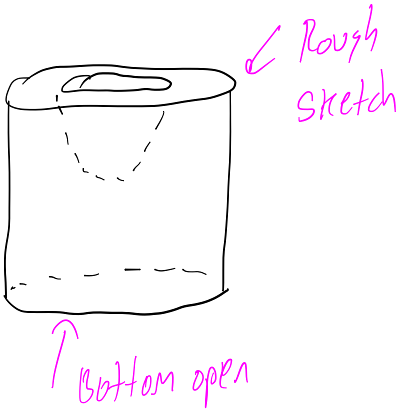
$$\frac{\partial Q}{\partial x} = 2xy + 1$$

$$\iint_D 2yx - (2yx+1)\, dA$$

$$= \iint_D -1 \, dA$$

$$= \boxed{-(8+4\pi)}$$

11. Evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x,y,z) = \langle -z, e^{x^2}, y^2 \rangle$ and S is the positively-oriented surface formed by gluing together the cylinder $x^2+y^2=9$ with $0 \le z \le 9$ with the elliptic paraboloid $5+x^2+y^2=z$ with $5 \le z \le 9$ and the surface given by the parametrization $\vec{r}(u,v) = \langle u\cos(v), u\sin(v), 9 \rangle$ where $0 \le v \le 2\pi$ and $2 \le u \le 3$.



$\mathrm{div}(\vec{F}) = 0$

S = bottom circle filled to a disc

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle$

$\vec{n} = \langle 0, 0, -r \rangle$. $\quad 0 \le r \le 3$, $0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^3 \langle \ldots, \ldots, r^2\sin^2\theta \rangle \cdot \langle 0,0,-r \rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 -r^3\sin^2\theta \, dr\, d\theta$$

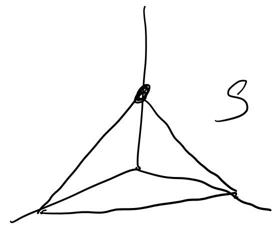
$$= -\frac{81}{4}\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{-81}{4}\left(\frac{1-\cos(2\theta)}{2}\right) d\theta$$

$$= \frac{-81}{4}\left(\frac{\theta}{2} - \frac{\sin(2\theta)}{2}\right)\Big|_0^{2\pi}$$

$$= \frac{-81\pi}{4}$$

So $\iiint \mathrm{div}(\vec{F})\, dV - \iint_S \vec{F}\cdot d\vec{S} = \boxed{\dfrac{81\pi}{4}}$

12. Evaluate $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x, y, z) = \langle xe^{y^2}, y\sin(z), 2x - ze^{y^2} \rangle$ S is the surface in the first octant formed by the planes $y = 0$, $x = 0$, and $x + y + z = 1$ oriented outwards. & $z=0$



$$\mathrm{div}(\vec{F}) = e^{y^2} + \sin(z) - e^{y^2} = \sin(z)$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sin(z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} -\cos(z) \Big|_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 1 - \cos(1-x-y)\, dy\, dx$$

$$= \int_0^1 y + \sin(1-x-y) \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 1-x + \sin(0) - \sin(1-x)\, dx$$

$$= x - \frac{x^2}{2} - \cos(1-x) \Big|_{x=0}^{x=1}$$

$$= 1 - \frac{1}{2} - \cos(0) + \cos(1)$$

$$= \boxed{\cos(1) - 1/2}$$

13. Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}}(x,y,z) = \langle ze^x, \sin(z), e^x + y\cos(z)\rangle$ and C is the curve given by $\vec{\mathbf{r}}(t) = \langle t^2\cos(t), t, t\rangle$ where $0 \le t \le 2\pi$.

$$f(x,y,z) = ze^x + y\sin(z)$$

$$\nabla f(x,y,z) = \langle ze^x, \sin(z), e^x + y\cos(z)\rangle \quad \checkmark$$

$\vec{F}$ is conservative.

$$\vec{r}(0) = \langle 0, 0, 0\rangle \qquad \vec{r}(2\pi) = \langle 4\pi^2, 2\pi, 2\pi\rangle$$

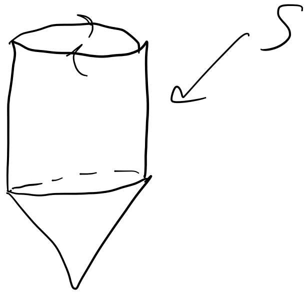
By FToLI,

$$\int_C \vec{F}\cdot d\vec{r} = f(4\pi^2, 2\pi, 2\pi) - f(0,0,0)$$

$$= 2\pi e^{4\pi^2} + 2\pi\sin(2\pi)$$

$$= \boxed{2\pi e^{4\pi^2}}$$

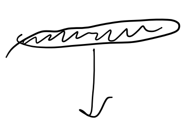
14. Evaluate $\iint_S \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x, y, z) = \langle y, -x, z\rangle$ and S is the surface, oriented outwards, made of the half-cone $z = \sqrt{x^2 + y^2}$ with $0 \le z \le 1$ and the cylinder $x^2 + y^2 = 1$ where $1 \le z \le 4$.



$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & z \end{vmatrix} = \langle 0, 0, -1-1\rangle$$

$$= \langle 0, 0, -2\rangle$$

Replace S with S'



$$\vec{r}(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 4\rangle$$

$$\vec{n} = \langle 0, 0, -r\rangle$$

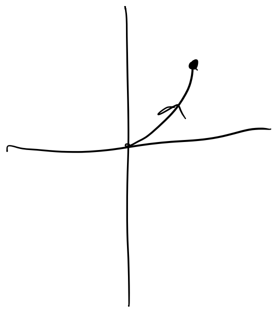
$$\int_0^{2\pi}\int_0^1 \langle 0,0,-2\rangle \cdot \langle 0,0,-r\rangle \, dr\, d\theta$$

$$= 2\pi \int_0^1 2r\, dr$$

$$= \boxed{2\pi}$$

Scalar field!

15. Evaluate $\int_C x\, ds$ where C is the curve $y = x^2$ in the xy-plane from $0 \le x \le 1$.



$$\vec{r}(t) = \langle t, t^2 \rangle$$

$$0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 1, 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1 + 4t^2}$$

$$\int_0^1 t\sqrt{1+4t^2}\, dt \qquad u = 1 + 4t^2$$

$$du = 8t\, dt$$

$$= \frac{1}{8}\int_1^5 u^{1/2}\, du$$

$$= \left(\frac{1}{8}\right)\left(\frac{2}{3}\right) u^{3/2}\Big|_1^5 = \boxed{\frac{5^{3/2} - 1}{12}}$$

16. Evaluate $\iint_S \operatorname{curl}(\vec{\mathbf{F}})\cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}} = \langle \cos(e^{y^2+z^2+x^2}), e^{\sin(xy)}, \sin(\cos(\sin(\cos(\sin(xyz)))))\rangle$ and S is the surface of the sphere $x^2+y^2+z^2=4$ above and including the plane $z=\sqrt{3}$ oriented outward.

This is a closed surface.

$$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$$

By Divergence Theorem

$$\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \boxed{0}$$