Review guide for §13.3 to §13.8

Exam date, time and room: Wednesday, December 18, 2019, 7:30 AM –10:00 AM at TBA

Exam info: `http://math.colorado.edu/math2400/2400exams.php`

For the other sections, please check out the previous 3 exam review guides.

1. **Fundamental Theorem of Line Integrals** (§13.3): If $\vec{F}$ is **conservative**, then there exists f called a **potential** function of $\vec{F}$, such that $\vec{F} = \nabla f$, and

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(\text{terminal pt}) - f(\text{initial pt}) \quad \textbf{path independent}$$

To find the potential function f of a conservative vector field $\vec{F}$, we solve $\vec{F} = \text{grad}\, f = \nabla f$,

(a) In $\mathbb{R}^2$: $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, $\vec{F} = \nabla f \iff \langle P, Q\rangle = \left\langle f_x, f_y\right\rangle \iff \begin{cases} P = f_x \\ Q = f_y \end{cases}$

(b) In $\mathbb{R}^3$: $\vec{F} = \langle P(x,y,z), Q(x,y,z).R(x,y,z)\rangle$, $\vec{F} = \nabla f \Leftrightarrow \langle P, Q, R\rangle = \left\langle f_x, f_y, f_z\right\rangle \Leftrightarrow \begin{cases} P = f_x \\ Q = f_y \\ R = f_z \end{cases}$

2. **Green's Theorem** (§12.4): If C is closed on a simple connected region D in $\mathbb{R}^2$, and $\vec{F} = \langle P,\ Q\rangle$, then

$$\oint_C \vec{F}\cdot d\vec{r} = \pm \iint_D (Q_x - P_y)\,dA$$

where ± sign is determined by the direction of C.

$A(D) = \iint_D dA = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$ — the area of $D \subset \mathbb{R}$, C is the bdry of D oriented counter-clockwise.

3. **Curl and Divergence** (§13.5):

(a) curl $\vec{F} = \nabla\times\vec{F}$: The **curl** of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} = \langle P, Q, R\rangle \in \mathbb{R}^3$ is defined by

$$\begin{aligned}
\text{curl}\,\vec{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \\
\nabla &= \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle. \\
\nabla\times\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \text{curl}\,\vec{F}
\end{aligned}$$

The curl of a vector field measures the tendency for the vector field to **swirl** around.

(b) div $\vec{F} = \nabla\cdot\vec{F}$: The **divergence** of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} = \langle P, Q, R\rangle \in \mathbb{R}^3$ is a function of 3 variables defined by div $\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ if P_x, Q_y, R_z exist.

div $\vec{F}$ — measures the **tendency** of the fluid to **diverge** from the point (x, y, z).

(c) If $\vec{F} = \langle P, Q, R\rangle \in \mathbb{R}^3$, then **div**$\vec{F} = \nabla\cdot\vec{F}$ is a **scalar** field, but **curl** $\vec{F} = \nabla\times\vec{F}$ is a **vector** field.

(d) **Theorem**: If $\vec{F} = \langle P, Q, R\rangle \in \mathbb{R}^3$, and P, Q and R have continuous 2nd-order partial derivatives, then **div (curl $\vec{F}$) = $\nabla\cdot(\nabla\times\vec{F}) = 0$**.

(e) **Theorem**: If f is scalar function of x, y and z, and has continuous 2nd-order partial derivatives, then **curl (grad f) = $\nabla\times(\nabla f) = \vec{0}$**.

(f) **Theorem**: If $\vec{F}$ is a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{F} = \vec{0}$, the $\vec{F}$ is a conservative vector field.

(g) **Conservative vector field**: If $\vec{F} = \langle P, Q, R\rangle$, $P, Q, R \in C^1(\mathbb{R}^3)$, then

$$\vec{F} \text{ is conservative} \iff \text{curl } \vec{F} = \nabla \times \vec{F} = \vec{0} \iff \text{there exists a scalar function } f \text{ such that } \nabla f = \vec{F}$$

where f is called a **potential** function of $\vec{F}$.
To find f, we solve $\nabla f = \vec{F} \Leftrightarrow f_x = P,\ f_y = Q,\ f_z = R$ for f.

(h) **Laplace operator**: $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace operator**.

$$\textbf{div (grad } f\textbf{)} = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz} \equiv \nabla^2 f$$

4. **Surface Integrals** (§13.6): S: $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ in $\mathbb{R}^3$.
$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,\mathrm{d}A$, $\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \pm \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\,\mathrm{d}A$.

5. **Stokes' Theorem** (§13.7): C is the **boundary** of S. C & S are simple, connected, and smooth. curl $\vec{F} = \nabla \times \vec{F}$.
$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \text{curl } \vec{F} \cdot \mathrm{d}\vec{S}$, where the orientations of C and S follow the **right-hand rule**.

6. **Divergence Theorem** (§13.8): div $\vec{F} = \nabla \cdot \vec{F} = P_x + Q_y + R_z$ if $\vec{F} = \langle P, Q, R\rangle \in \mathbb{R}^3$. S, E are simple & smooth.
$\oiint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_E \text{div } \vec{F}\,\mathrm{d}V$, where S oriented outward is the **boundary (closed surface)** of the solid E.

7. **Applications of line and surface integrals**

(a) **Mass and center of mass**: In the formulas below, C: $\vec{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ in $\mathbb{R}^3$. D is a general region in $\mathbb{R}^2$. S: $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ in $\mathbb{R}^3$. $\rho(x,y,z)$ or $\rho(x,y)$ is the density of the object. Once we get the mass m, for example, the mass of the wire with density $\rho(x,y,z)$, is $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = \frac{1}{m}\int_C x\rho(x,y,z)\,\mathrm{d}s$, $\bar{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,\mathrm{d}s$, $\bar{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,\mathrm{d}s$. The other center of mass is similar.

integral	**related computations**	**the meaning of the integral**
$\int_C \rho(x,y,z)\,\mathrm{d}s$	$m = \int_{t_0}^{t_1} \rho(\vec{r}(t))\,\lvert\vec{r}\,'(t)\rvert\,\mathrm{d}t$	**mass** of a wire from $\vec{r}(t_0)$ to $\vec{r}(t_1)$ along curve C in $\mathbb{R}^3$.
$\iint_D \rho(x,y)\,\mathrm{d}A$	$m = \int_a^b \int_{y=b(x)}^{y=t(x)} \rho(x,y)\,\mathrm{d}y\,\mathrm{d}x$	**mass** of a lamina D in $\mathbb{R}^2$
$\iint_S \rho(x,y,z)\,\mathrm{d}S$	$m = \iint_D \rho(\vec{r}(u,v))\,\lvert\vec{r}_u \times \vec{r}_v\rvert\mathrm{d}A$	**mass** of the surface S in $\mathbb{R}^3$ The double integral can be set up in two different orders, or by polar coordinates.
$\iiint_E \rho(x,y,z)\,\mathrm{d}V$	$m = \iint_{D_{xy}} \int_{b(x,y)}^{t(x,y)} \rho(x,y,z)\,\mathrm{d}z\,\mathrm{d}A$	**mass** of the solid E in $\mathbb{R}^3$ The integral can be set up by 3 different plane regions, or by cylindrical/spherical coordinates.

(b) **Integrals with 1 as integrand**: In the formulas below, C: $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ in $\mathbb{R}^3$. D is a general region in $\mathbb{R}^2$. S: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ in $\mathbb{R}^3$.

integral	related computations	the meaning of the integral
$\int_C \mathrm{d}s$	$L = \int_{t_0}^{t_1} \lvert\vec{r}\,'(t)\rvert \mathrm{d}t$	**arc length** from $\vec{r}(t_0)$ to $\vec{r}(t_1)$ along the curve C
$\iint_D \mathrm{d}A$	$A(D) = \int_C x\mathrm{d}y = -\int_C y\mathrm{d}x = \frac{1}{2}\int_C x\mathrm{d}y - y\mathrm{d}x$	**area** of the $D \subset \mathbb{R}^2$
$\iint_S \mathrm{d}S$	$A(S) = \iint_D \lvert\vec{r}_u \times \vec{r}_v\rvert \mathrm{d}A$	**area** of the surface $S \subset \mathbb{R}^3$
$\iiint_E \mathrm{d}V$	$V(E) = \iint_{D_{xy}} \int_{b(x,y)}^{t(x,y)} \mathrm{d}z\,\mathrm{d}A$	**volume** of the solid $E \subset \mathbb{R}^3$

(c) $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$ – **work** done by force $\vec{F}$ along the curve C. $\iint_S \vec{F}\cdot\mathrm{d}\vec{S}$ – **flux** through the surface S.

8. **Summary on the 2nd type of line and surface integrals**

(a) **How to evaluate the 2nd type of the line integral?** (§13.2, §13.3, §13.4)

i) **Direct method**: If we can parametrize the curve as C: $\vec{r} = \langle x(t), y(t), z(t)\rangle$, t from t_0 (initial time) to t_1 (terminal time), then $\int_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

ii) **FTLIs**: If $\vec{F}$ is **conservative**, then there exists a potential function f such that $\nabla f = \vec{F}$, and the integral can be evaluated by the **Fundamental Theorem of Line Integrals**:
$$\int_C \vec{F}\cdot\mathrm{d}\vec{r} = f(\text{terminal point}) - f(\text{initial point})$$

iii) By **Green's Theorem (2D)** or **Stokes' Theorem (3D)**:

A. If C is closed on a simple connected region D in $\mathbb{R}^2$, and $\vec{F} = \langle P, Q\rangle$, then
$$\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \iint_D (Q_x - P_y)\mathrm{d}A,$$
where C is the boundary of D with orientation counter-clockwise.

B. If C is closed on a simple connected region D in $\mathbb{R}^3$, and $\vec{F} = \langle P, Q\,R\rangle$, then
$$\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\mathrm{d}\vec{S},$$
where C is the boundary of S. Their orientations follow the **the right-hand rule**.

(b) **How to evaluate the 2nd type of the surface integral?** (§13.6, §13.7, §13.8)

i) **Direct method**: If we can parametrize the surface as S: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, and S is oriented with the same direction of $\vec{r}_u\times\vec{r}_v$, then
$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iint_{D_{uv}} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\mathrm{d}A.$$

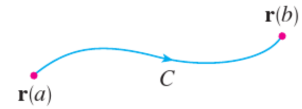
ii) **Stokes' Theorem**: If the integrand is **curl** $\vec{F} = \nabla\times\vec{F}$, or the curve C is the boundary of the surface S, $\iint_S \operatorname{curl}\vec{F}\cdot\mathrm{d}\vec{S} = \oint_C \vec{F}\cdot\mathrm{d}\vec{r}$.

iii) **Divergence Th**: If S is **closed**, $\oiint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iiint_E \operatorname{div}\vec{F}\,\mathrm{d}V$. S is the boundary of E.

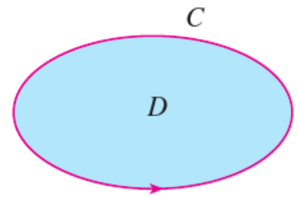
9. **Summary on Fundamental Theorem of Calculus** (§13.9):
See Page 973 on the summary of Fundamental Theorem of Calculus (FTC)

FTC $$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$$

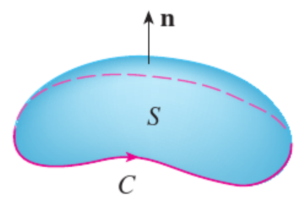
FTLIs $$\int_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$



Green's Th $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y$$



Stokes' Th $$\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = \oint_C \vec{F} \cdot \mathrm{d}\vec{r}$$



Divergence Th $$\iiint \operatorname{div} \vec{F}\,\mathrm{d}V = \oiint_S \vec{F} \cdot \mathrm{d}\vec{S}$$