Review Problems

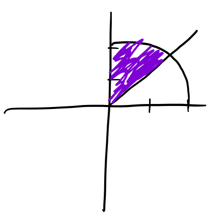
Math 2400-006, Calculus III, Spring 2019

Name:

Instructions for Doing These Problems:

1. When you first attempt the problem, **DO NOT** use any help (books, notes, etc)
2. If you cannot solve the problem during step 1, then you may use books and notes. If you solve it during this step, make a special mark next to the problem number and make sure to review that section.
3. If you cannot solve it during step 2, then I recommend either asking me or a classmate for help. Make sure to mark the problem and read the section.
4. Once you've finished all of the problems, you should focus on practice problems from the corresponding sections of your marked problems. I **highly recommend** doing odd-numbered exercises until you're almost always getting the right answer with no help.

1. Evaluate the integral $\displaystyle\int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} x^2+y^2\,dy\,dx$.

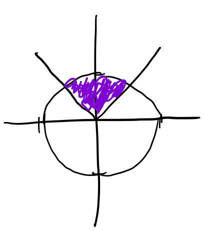


$$\int_{\pi/4}^{\pi/2}\int_0^2 r(r^2)\,dr\,d\theta$$

$$= \left(\frac{\pi}{2}-\frac{\pi}{4}\right)\left.\frac{r^4}{4}\right|_{r=0}^{r=2}$$

$$= \left(\frac{\pi}{4}\right)(4) = \boxed{\pi}$$

2. Evaluate the integral $\displaystyle\iint_D y\,dA$ where D is the region bounded by $x^2+y^2=1$ and $y=|x|$.



$$\int_{\pi/4}^{3\pi/4}\int_0^1 r(r\sin\theta)\,dr\,d\theta$$

$$= \int_{\pi/4}^{3\pi/4}\left.\frac{r^3}{3}\sin\theta\right|_{r=0}^{r=1} d\theta$$

$$= \frac{1}{3}\int_{\pi/4}^{3\pi/4}\sin\theta\,d\theta$$

$$= \frac{1}{3}(-\cos\theta)\Big|_{\theta=\pi/4}^{\theta=3\pi/4}$$

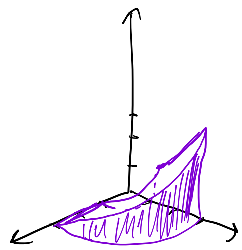
$$= \boxed{\frac{1}{3}(\sqrt{2})}$$

3. Sketch the solid whose volume is represented by the integral:

$$\int_0^{\frac{\pi}{2}} \int_1^3 r^2 \sin(\theta)\, dr\, d\theta$$

$r\,(r\sin\theta) = y$

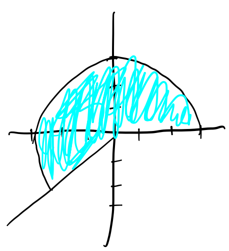
(considering r appears)



Rough sketch

4. Evaluate the sum of integrals:

$$\int_{\frac{-3}{\sqrt{2}}}^{0} \int_{-\sqrt{9-y^2}}^{y} yx^3\, dx\, dy + \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} yx^3\, dx\, dy$$



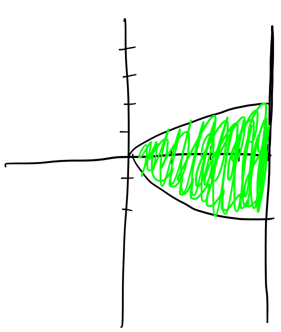
$$= \int_0^{5\pi/4} \int_0^3 r\,(r\sin\theta)(r^3\cos^3\theta)\, dr\, d\theta$$

$$= \int_0^{5\pi/4} \frac{r^6}{6} \sin\theta \cos^3\theta \Big|_{r=0}^{r=3} d\theta$$

$$= \frac{243}{2} \int_0^{5\pi/4} \sin\theta \cos^3\theta\, d\theta$$

$$= \frac{243}{2} \left(-\frac{\cos^4\theta}{4} \right) \Big|_{\theta=0}^{\theta=\frac{5\pi}{4}} = \boxed{\frac{243}{8}\left(\left(\frac{\sqrt{2}}{2}\right)^4 - 1 \right)}$$

5. Calculate the mass of the lamina with density $\rho(x, y) = x^2$ and shape bounded by $x = y^2$ and $x = 4$.



$$\int_{-2}^{2} \int_{y^2}^{4} x^2\, dx\, dy$$

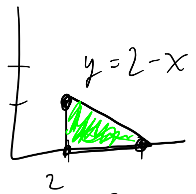
$$= \int_{-2}^{2} \frac{x^3}{3} \Big|_{x=y^2}^{x=4} dy$$

$$= \int_{-2}^{2} \frac{64}{3} - \frac{y^6}{3}\, dy$$

$$\frac{64}{3} y - \frac{y^7}{21} \Big|_{-2}^{2}$$

$$\boxed{= \frac{128}{3} - \frac{128}{21} + \frac{64}{3} - \frac{128}{21}}$$

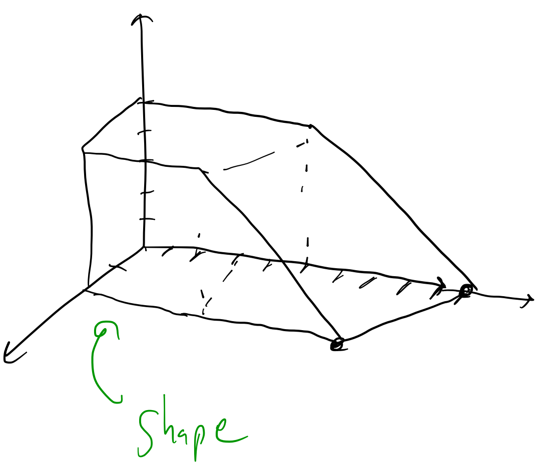
6. Calculate the x-coordinate of the center of mass of the lamina with density $\rho(x,y) = \frac{y}{x}$ and shape bounded by the vertices $(1,1),(2,0),(1,0)$.



$$M = \int_1^2 \int_0^{2-x} \frac{y}{x}\, dy\, dx$$

$$\boxed{\bar{x} = \frac{\int_1^2 \int_0^{2-x} y \, dy\, dx}{\int_1^2 \int_0^{2-x} \frac{y}{x}\, dy\, dx}}$$

7. Set up the integral for the mass of the object with density $\rho(x,y,z) = x + y$ and shape in the first octant bounded by $z = 5$, $x = 2$, and $y = 10 - x - z$.



$$\boxed{M = \int_0^5 \int_0^2 \int_0^{10-x-z} x + y \; dy\, dx\, dz}$$

8. Find the surface area of the surface defined by the parametrization $\vec{r}(u,v) = \langle u - v, 3 + v + u, u \rangle$ where $0 \le u \le 2$ and $-1 \le v \le 1$.

$\vec{r}_u = \langle 1, 1, 1 \rangle$

$\vec{r}_v = \langle -1, 1, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \langle -1, -1, 2 \rangle$$

$$\int_{-1}^{1} \int_0^2 \sqrt{1+1+4}\; du\, dv$$

$$= (2)(2)\sqrt{6}$$

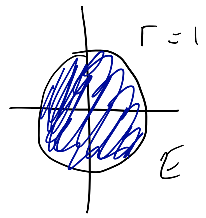
$$= \boxed{4\sqrt{6}}$$

9. Find the surface area of the surface defined by $f(x, y) = x^2 - y^2$ where $x^2 + y^2 \leq 1$.

$x = x$

$y = y$

$z = x^2 - y^2$



$$\begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} = \langle -2x, 2y, 1 \rangle$$

$$\iint_E \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$= \boxed{\int_0^{2\pi}\int_0^1 r\sqrt{1 + 4r^2}\, dr\, d\theta}$$

10. Find the surface area of the elliptic paraboloid $x = y^2 + z^2$ within the sphere $x^2 + y^2 + z^2 = 6$.

≥ 0

$x = y^2 + z^2$

$y = y$

$z = z$

$$\begin{vmatrix} i & j & k \\ 2y & 1 & 0 \\ 2z & 0 & 1 \end{vmatrix} = \langle 1, -2y, -2z \rangle$$

$\sqrt{1 + 4y^2 + 4z^2}$

$x^2 + x = 6$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

So $x = 2$

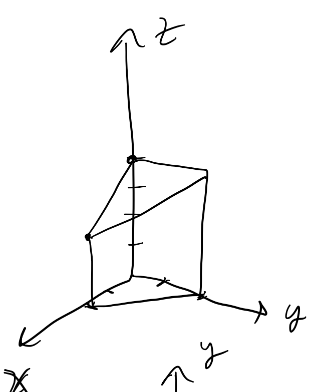
$4 + y^2 + z^2 = 6$

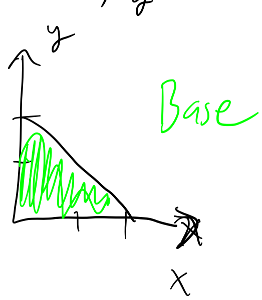
$y^2 + z^2 = 2$

$$\boxed{\int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr\, d\theta}$$

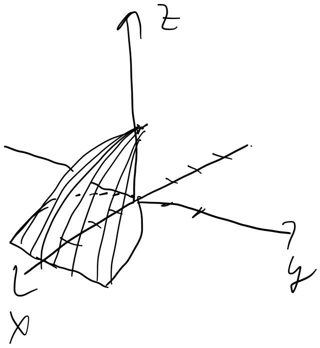
11. Find the volume of the solid in the 1st octant bounded between the planes $z = 4 - x$ and $y = 2 - x$.

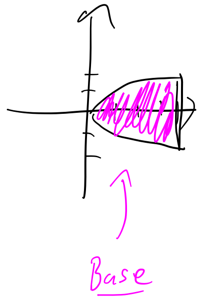
$$\boxed{\int_0^{2}\int_0^{2-x}\int_0^{4-x} dz\, dy\, dx}$$





12. Set up, but do not evaluate, $\iiint_E e^y \, dV$ where E is the region bounded by $x = y^2$, $z = 16 - x^2$, and $z = 0$.

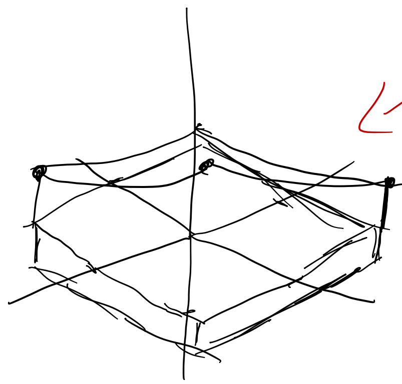




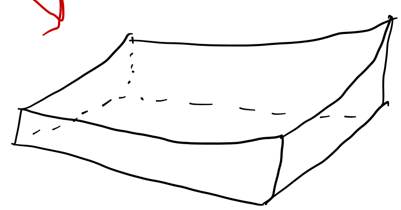
$$\int_{-2}^{2} \int_{y^2}^{4} \int_{0}^{16-x^2} e^y \, dz \, dx \, dy$$

13. Sketch the solid whose volume is given by

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{e^{x+y}} 1 \, dz \, dx \, dy$$



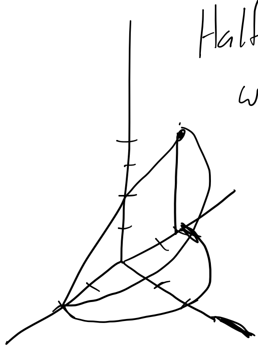
Very rough Sketches



14. Let $f(x, y, z)$ be an arbitrary continuous function. Switch the order of integration of

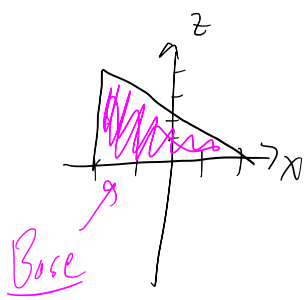
$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{2-x} f(x, y, z) \, dz \, dy \, dx$$

to $dy \, dx \, dz$.



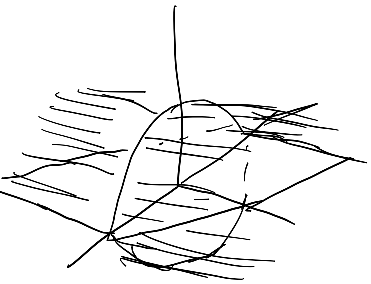
Half-cylinder with plane slicing it

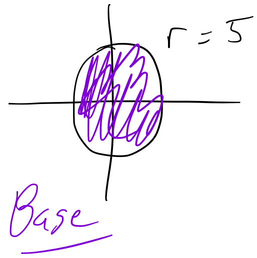
$x = z + 2$



$$\boxed{\int_{0}^{4} \int_{-2}^{2-z} \int_{0}^{\sqrt{4-x^2}} f(x,y,z) \, dy \, dx \, dz}$$

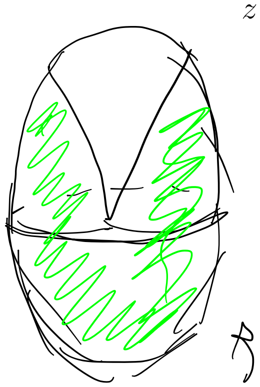
15. Evaluate the integral $\iiint_E z + 2\,dV$ where E is the region inside the cylinder $x^2 + z^2 = 25$ and bounded by the planes $y = x + 5$ and $y = x - 5$.





$$\boxed{\int_0^{2\pi}\int_0^{5}\int_{r\cos\theta-5}^{r\cos\theta+5} r(r\sin\theta+2)\,dy\,dr\,d\theta}$$

16. Evaluate the integral $\iiint_E x^2 + y^2\,dV$ where E is the region beneath the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 36$.

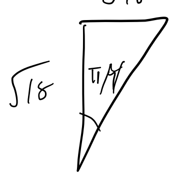


Imagine I can draw a sphere...

$$2(x^2+y^2) = 36$$

$$x^2+y^2 = 18$$

$\sqrt{18}$



$$\boxed{\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^{6} (\rho^2\sin\phi)(\rho^2\sin^2\phi)\,d\rho\,d\phi\,d\theta}$$

17. Sketch the region whose volume is represented by the integral

$$\int_0^{\pi}\int_{-1}^{2}\int_0^{\sqrt{1+z^2}} r\,dr\,dz\,d\theta.$$

cylindrical, so forget r in integrand

half

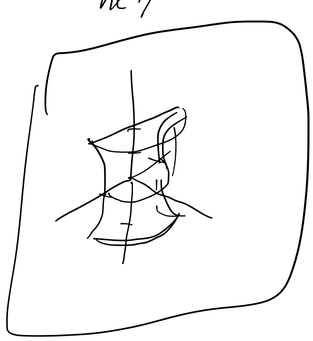
Controls height

$r = \sqrt{1+z^2}$

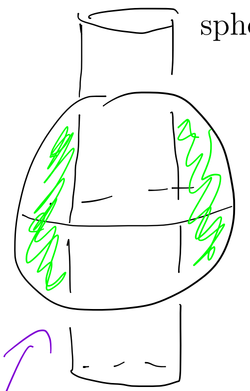
$r^2 = 1 + z^2$

$x^2 + y^2 - z^2 = 1$

Hyperboloid of 1-Sheet



18. Find the volume of the region outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

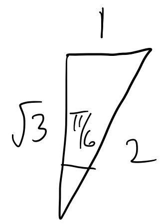


Intersect at $z = \pm\sqrt{3}$

$x^2 + y^2 = 1$

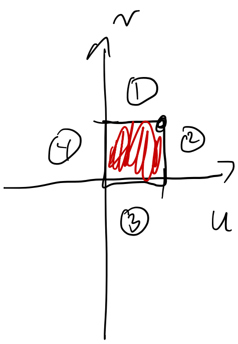
$\rho^2 \sin^2\phi = 1$

$\rho = \frac{1}{\sin\phi}$

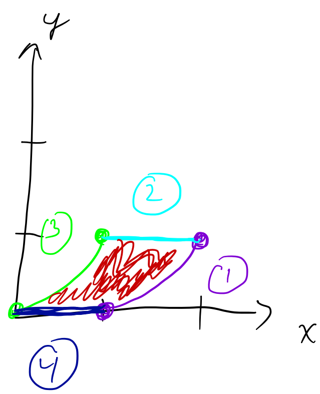


$$\boxed{\int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\frac{1}{\sin\phi}}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

19. Sketch the region $T(R)$ where R is the rectangle with vertices $(0, 0), (1, 0), (0, 1), (1, 1)$ and T is the transformation $T(u, v) = \langle u + v, \sqrt{u} \rangle$



1 $v = 1$	2 $u = 1$	3 $v = 0$	4 $u = 0$
$x = u + 1$	$x = v + 1$	$x = u$	$x = v$
$y^2 = u$	$y = 1$	$y^2 = u$	$y = 0$
$x = y^2 + 1$	$0 \le v \le 1$	$x = y^2$	$0 \le v \le 1$
$0 \le y \le 1$		$0 \le y \le 1$	



20. Evaluate the integral $\iint_D \frac{x-2y}{x^2+y^2+2xy+1} \, dA$ where D is the region bounded by $y = 1 - x$, $x + y = 2$, $x = 2y$ and $x - 2y = \sqrt{x+y}$ using the transformation $T(u, v) = \left\langle \frac{u+2v}{3}, \frac{v-u}{3} \right\rangle$

$3x = u + 2v$

$3y = -u + v$

$v = x + y$

$3x = u + 2v$

$-6y = +2u - 2v$

$u = x - 2y$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{vmatrix} = \left| \frac{1}{9} + \frac{2}{9} \right| = \frac{1}{3}$$

$v \begin{cases} y + x = 1 \\ x + y = 2 \end{cases}$

$u \begin{cases} x - 2y = 0 \\ x - 2y = \sqrt{x+y} = \sqrt{v} \end{cases}$

$$\frac{1}{3} \int_1^2 \int_0^{\sqrt{v}} \frac{u}{1+v^2} \, du \, dv$$

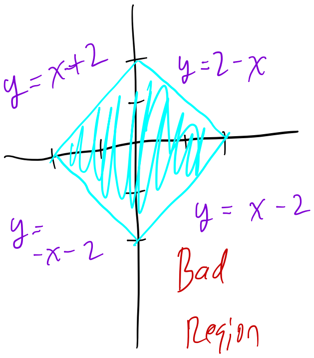
$$= \frac{1}{6} \int_1^2 \frac{u^2}{1+v^2} \Big|_{u=0}^{u=\sqrt{v}} dv$$

$$= \frac{1}{6} \int_1^2 \frac{v}{1+v^2} \, dv$$

$$= \frac{1}{12} \ln(1+v^2) \Big|_{v=1}^{v=2}$$

$$= \boxed{\frac{1}{12} \left(\ln(5) - \ln(2) \right)}$$

21. Evaluate the integral $\iint_D xe^{x^2-y^2} + ye^{x^2-y^2}\, dA$ where D is the region bounded by $|x| + |y| \leq 2$.



Let $u = x - y$

$v = x + y$

$x - y = -2$

$x - y = 2$

$x + y = 2$

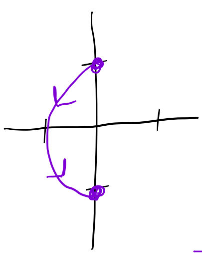
$x + y = -2$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}\right| = 2$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

$$\boxed{\frac{1}{2}\int_{-2}^{2}\int_{-2}^{2} ve^{uv}\, du\, dv}$$

22. Evaluate $\int_C xy\, ds$ where C is the section of the circle $x^2 + y^2 = 1$ starting at $(0, -1)$ and going to $(0, 1)$. Which direction? Let's say counterclockwise!



$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

$\pi/2 \leq t \leq \frac{3\pi}{2}$

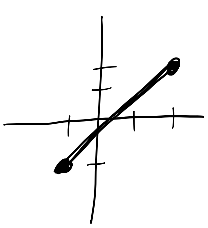
$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

$|\vec{r}\,'(t)| = 1$

$$\int_{\pi/2}^{3\pi/2} \cos(t)\sin(t)\, dt$$

$$= \frac{\sin^2(t)}{2}\Bigg|_{t=\pi/2}^{3\pi/2} = \boxed{0}$$

23. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle -y, x \rangle$ and C is the line segment connecting $(-1, -1)$ to $(2, 2)$. Looks like $y = x$



Let's use $\vec{r}(t) = \langle t, t \rangle$

$\vec{r}\,'(t) = \langle 1, 1 \rangle$

$$\int_{-1}^{2} \langle -t, t \rangle \cdot \langle 1, 1 \rangle\, dt$$

$$= \int_{-1}^{2} 0\, dt = \boxed{0}$$

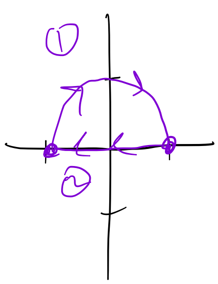
24. Evaluate $\int_C y^2\, ds$ where C is given by $\mathbf{r}(t) = \langle 2\cos(t), 2, 2\sin(t)\rangle$ where $0 \le t \le 2\pi$.

$\vec{r}\,'(t) = \langle -2\sin(t), 0, 2\cos(t)\rangle$

$|\vec{r}\,'(t)| = 2$

$\int_0^{2\pi} 8\, dt = \boxed{16\pi}$

25. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle y, -x\rangle$ and C is the path that starts at $(-1, 0)$, goes to $(1, 0)$ clockwise, then travels back to $(-1, 0)$ along the x-axis.



1 $\vec{r}(t) = \langle -\cos(t), \sin(t)\rangle$

$0 \le t \le \pi$

$\vec{r}\,'(t) = \langle \sin(t), \cos(t)\rangle$

$\int_0^{\pi} \langle \sin(t), \cos(t)\rangle \cdot \langle \sin(t), \cos(t)\rangle\, dt$

$= \int_0^{\pi} dt$

$= \pi$

2 $\vec{r}(t) = \langle -t, 0\rangle$

$-1 \le t \le 1$

$\vec{r}\,'(t) = \langle -1, 0\rangle$

$\int_{-1}^{1} \langle 0, t\rangle \cdot \langle -1, 0\rangle\, dt$

$= 0$

So $\boxed{\pi}$