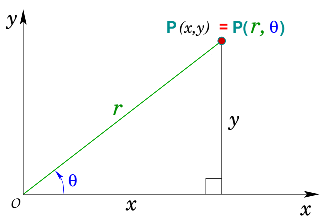
Review guide for mid-term exam 3

Exam date and time: Monday, November 18, 2019, 5:15–6:45PM

Exam info: `http://math.colorado.edu/math2400/2400exams.php`

1. **Double Integrals in Polar Coordinates** (§12.4)



$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$.

The area element $dA = dx\,dy = r\,dr\,d\theta$.

$$\iint_D f(x,y)\,dA = \iint_{\widetilde{D}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where D is the region in xy−coordinates, and $\widetilde{D}$ is the corresponding region in polar coordinates.

2. **Surface Area** (§12.6): If a smooth parametric surface S is given by the equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle, \quad (u,v) \in D$$

$$A(S) = \iint_D \left|\vec{r}_u \times \vec{r}_v\right| dA, \qquad \text{where } \vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle,\ \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$$

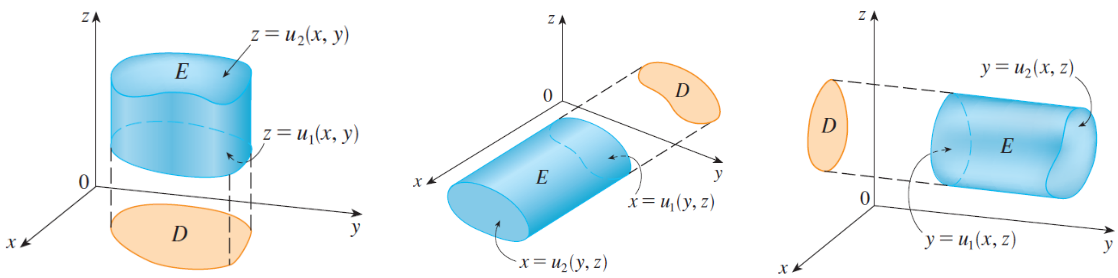
$\vec{n} = \vec{r}_u \times \vec{r}_v$ is normal to the tangent plane of S at (u,v).

A special case: The equation of the surface S is given by $z = f(x,y)$, $(x,y) \in D$, $f_x, f_y \in C(D)$, then

$$A(S) = \iint_D \sqrt{(f_x(x,y))^2 + \left(f_y(x,y)\right)^2 + 1}\ dA = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA = \iint_D |\vec{n}|\ dA$$

$\vec{n} = \left\langle -f_x,\ -f_y,\ 1\right\rangle$ is normal to the tangent plane of S at (x,y). D = projection of S onto xy-plane.

3. **Triple Integrals** (§12.7): **3 types of orders of integration:**



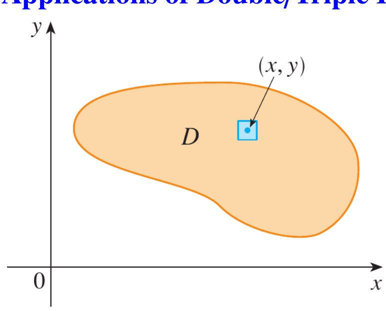
Type 1: $$\iiint_E f(x,y,z)\,dV = \iint_{D_{xy}} \int_{u_1(x,y)}^{u_2(x,y) \leftarrow \text{surf } z = u_2(x,y) \text{ in “+” } z \text{ direction}} f(x,y,z)\,dz\,dA$$

Type 2: $$\iiint_E f(x,y,z)\,dV = \iint_{D_{yz}} \int_{u_1(y,z)}^{u_2(y,z) \leftarrow \text{surf } x = u_2(y,z) \text{ in “+” } x \text{ direction}} f(x,y,z)\,dx\,dA$$

Type 3: $$\iiint_E f(x,y,z)\,dV = \iint_{D_{xz}} \int_{u_1(x,z)}^{u_2(x,z) \leftarrow \text{surf } y = u_2(x,z) \text{ in “+” } y \text{ direction}} f(x,y,z)\,dy\,dA$$

D_{xy}, D_{yz}, D_{xz}, are the **projections** of the solid onto xy, yz, xz-planes, respectively.

4. **Applications of Double/Triple Integrals** (§12.5 &§12.7):

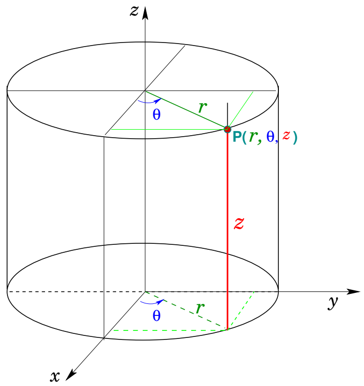


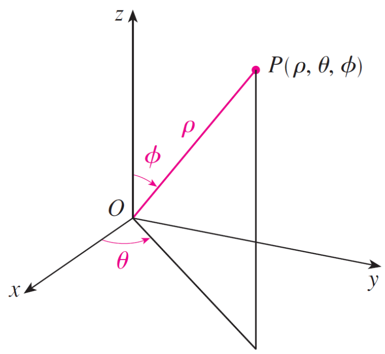
The mass m, center of mass $(\bar{x}, \bar{y})$ of a lamina with density $\rho(x,y)$ are $m = \iint_D \rho(x,y)\,dA$,

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA, \quad \bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

- The **volume of the solid** E: $\displaystyle\iiint_E \mathbf{d}V = \iint_{D_{xy}} (\text{top} - \text{bot})\mathbf{d}A$ (if E is type 1 plane region).
- The **mass of a solid** E with density $\rho(x,y,z)$ is $\displaystyle \boldsymbol{m} = \iiint_E \rho(x,y,z)\,\mathbf{d}V$.
- The **center of mass** $(\bar{x},\bar{y},\bar{z})$ of the solid:
$\displaystyle \bar{x} = \frac{1}{m}\iiint_E x\,\rho(x,y,z)\,\mathbf{d}V,\ \bar{y} = \frac{1}{m}\iiint_E y\,\rho(x,y,z)\,\mathbf{d}V,\ \bar{z} = \frac{1}{m}\iiint_E z\,\rho(x,y,z)\,\mathbf{d}V.$

5. **Triple Integrals in Cylindrical and Spherical Coordinates** (§12.8):





- In cylindrical coordinates
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$
$\mathbf{d}V = r\,\mathbf{d}z\,\mathbf{d}r\,\mathbf{d}\theta$.
- In spherical coordinates
$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$
$\mathbf{d}V = \rho^2\sin\phi\,\mathbf{d}\rho\,\mathbf{d}\theta\,\mathbf{d}\phi$.

6. **Change of Variables in Multiple Integrals** (§12.9):
$T\colon (u,v)\to(x,y)$ is given by $x = x(u,v)$, $y = y(u,v)$. T images S into R. $T^{-1}\colon R\to S$.

$$\iint_R f(x,y)\,\mathbf{d}A = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathbf{d}A'$$

$\frac{\partial(x,y)}{\partial(u,v)}$ **is called the Jacobian of** T. $\left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{\partial(x,y)}{\partial(u,v)}$. $\mathbf{d}A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathbf{d}A'$.
For triple integral, $T\colon (u,v,w)\to(x,y,z)$:

$$\iiint_R f(x,y,z)\,\mathbf{d}V = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathbf{d}V'$$

7. **Vector Fields** (§13.1):
$2D$: $\vec{F}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$. $3D$: $\vec{F}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$.
f is a **scalar** function of 2 variables, **grad** $f = \nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = \langle f_x,\ f_y\rangle$.
f is a **scalar** function of 3 variables, **grad** $f = \nabla f = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k} = \langle f_x,\ f_y,\ f_z\rangle$.

8. **Line Integrals** (§13.2):
   (a) Type 1: $\displaystyle\int_C f(\vec{r}(t))\,\mathbf{d}s = \int_{t_1}^{t_2} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,\mathbf{d}t$ — the **mass** of a wire with density $f(\vec{x})$.
   (b) Type 2: $\displaystyle\int_C \vec{F}\cdot\mathbf{d}\vec{r} = \int_{t_1}^{t_2} f(\vec{r}(t))\cdot\vec{r}\,'(t)\,\mathbf{d}t$ — the **work** done by a force $\vec{F}$.
   (c) **Center of mass**: $\displaystyle\bar{x} = \frac{1}{m}\int_C x\rho(x,y,z)\,\mathrm{d}s,\ \bar{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,\mathrm{d}s,\ \bar{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,\mathrm{d}s,$
   where $\displaystyle m = \int_C \rho(x,y,z)\,\mathrm{d}s$ is the mass of the wire with density $\rho(x,y,z)$.