Review guide for mid-term exam 2

Exam date and time: Monday, October 21, 2019, 5:15–6:45PM

Exam info: `http://math.colorado.edu/math2400/2400exams.php`

1. **Functions of several variables** (§11.1, §11.2)

   (a) **Domain, independent/dependent variables, level curves/contour map**

   (b) **Limit:** The **limit of $f(x,y)$ as (x,y) approaches (a,b)** is L: $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

   i. If the limit exists, then $f(x,y)$ must approach the same limit **no matter how (x,y) approaches (a,b)**. **Note**: The existence of limit does **NOT** indicate the existence of $f(a,b)$.

   ii. If the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ exists, then as $(x,y)\to(a,b)$ along **any path C**, $f(x,y)$ approaches the **same limit** L.

   iii. If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along a path C_1, and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ doesn't exist.
   But "$L_1 = L2$" does **not** indicate the existence or non-existence of the limit!

   iv. The **Squeeze Theorem**: If $g(x,y)\le f(x,y)\le h(x,y)$ for all (x,y) in the domain, and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = \lim\limits_{(x,y)\to(a,b)} h(x,y) = L$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.
   Note: The theorem can not be used to show the non-existence of the limit.

   v. We may also use the polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) combining with the Squeeze Theorem to find the limit.

   (c) **Continuity:** A function f of two variables is called **continuous at** (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. It implies 3 things, which you should check when you check the continuity.
   (i) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists; (ii) $f(a,b)$ exists; (iii) the limit and $f(a,b)$ are equal.

2. **Partial Derivatives, The Chain Rule** (§11.3 & §11.5). Pay attention to the **dependent/independent** variables!

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y),\ f_y(x,y) = \frac{\partial}{\partial y} f(x,y),\ f_{xx} = \frac{\partial}{\partial x}(f_x),\ f_{xy} = \frac{\partial}{\partial y}(f_x),\ f_{yx} = \frac{\partial}{\partial x}(f_y),\ f_{yy} = \frac{\partial}{\partial y}(f_y).$$

Case I: $z = f(x,y),\ x = x(t),\quad y = y(t),\qquad \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Case II: $z = f(x,y),\ x = x(s,t),\ y = y(s,t),\qquad \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s},\ \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

General Case: $u = f(x_1,\cdots,x_n),\ x_i = x_i(t_1,\cdots,t_n),\ \frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$

Implicit Differentiation: $y = y(x)$ is defined by $F(x,y) = 0$, $\quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$

(How to derive them by Chain Rule?) $z = z(x,y)$ is defined by $F(x,y,z) = 0$ $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}},\quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

3. **Tangent Plane, Normal Line and Linear Approximation** (§11.4).
   Let $\vec{n} = \langle A,B,C\rangle$ be the normal direction of the tangent plane through $P_0(x_0,y_0,z_0)\in S$.
   Plane Eqn: $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$. **Normal Line:** $\frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C}$.
   Linerization: $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.
   Differential: $dz = d(f(x,y)) = f_x\,dx + f_y\,dy$ if $z = f(x,y)$. $dz = f_x\,dx + f_y\,dy + f_z\,dz$ if $z = f(x,y,z)$.

   Surface S is given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$: The uv-coordinates (u_0,v_0) corresponds to the Cartesian point $P_0(x_0,y_0,z_0)$. Then the **normal direction** of the tangent plane (also the direction vector of the normal line) is $\vec{n} = \vec{r}_u(u_0,v_0)\times\vec{r}_v(u_0,v_0)$.

   Surface S is given by $z = f(x,y)$ ($\vec{r}(x,y) = \langle x,y,f(x,y)\rangle$)**:** The **normal direction** of the tangent plane (also the direction vector of the normal line) is $\vec{n} = \left\langle -f_x(x_0,y_0),\ -f_y(x_0,\ y_0), 1\right\rangle$.

   Surface S is given by $F(x,y,z) = k$ (k – constant): The **normal direction** of the tangent plane is $\vec{n} = \left\langle F_x(x_0,y_0,z_0),\ F_y(x_0,y_0,z_0),\ F_z(x_0,y_0,z_0)\right\rangle$.

4. **The Gradient Vector and Directional Derivative** (§11.6)

   (a) The **gradient** of f: $\begin{cases} \nabla \mathbf{f}(\mathbf{x_0}, \mathbf{y_0}) = \left\langle \mathbf{f_x}(\mathbf{x_0}, \mathbf{y_0}),\ \mathbf{f_y}(\mathbf{x_0}, \mathbf{y_0}) \right\rangle, & \text{(if } f \text{ is a function of } x \text{ and } y) \\ \nabla \mathbf{f}(\mathbf{x_0}, \mathbf{y_0}, \mathbf{z_0}) = \left\langle \mathbf{f_x}(\mathbf{x_0}, \mathbf{y_0}, \mathbf{z_0}),\ \mathbf{f_y}(\mathbf{x_0}, \mathbf{y_0}, \mathbf{z_0}),\ \mathbf{f_z}(\mathbf{x_0}, \mathbf{y_0}, \mathbf{z_0}) \right\rangle, & \end{cases}$

   (b) The **directional derivative** of f at P_0 in the direction of a **unit** vector $\vec{u}$ ($|\vec{u}| = 1$) is

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u} \quad = f_x(x, y)\, a + f_y(x, y)\, b \quad (f \text{ is a function of } x \text{ and } y)$$
$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u} = f_x(x, y, z)\, a + f_y(x, y, z)\, b + f_z(x, y, z)\, c$$

   If we let $\vec{x} = \langle x, y \rangle$ (2D) or $\vec{x} = \langle x, y, z \rangle$ (3D), then $D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$, $D^2_{\vec{u}} f(\vec{x}) = \nabla \left(D_{\vec{u}} f(\vec{x}) \right) \cdot \vec{u}$.

   (c) The **max rate of change** of f is $|\nabla f|$, and it occurs in the direction ∇f.

   (d) The gradient direction is **orthogonal** or **perpendicular** to the level curve in the contour map.

5. **Maximum and Minimum Values** (§11.7)
   If f has a local max/min at (a, b), and f_x, f_y exist at (a, b), $\nabla f(a, b) = \langle 0, 0 \rangle \Leftrightarrow f_x(a, b) = f_y(a, b) = 0$. If (a, b) is a point such that $\nabla f(a, b) = \vec{0} = \langle 0, 0 \rangle$, it is called a **critical point** of f.
   $\nabla f(a, b) = \vec{0}$ is a necessary, but not a sufficient condition for f to have a local max/min at (a, b).

   Second Derivative Test: $z = f(x, y)$, $f_{xx}, f_{xy}, f_{yy} \in C(\mathfrak{D})$, $\mathfrak{D}$ is a disk with center (a, b),
   $\nabla f(a, b) = \mathbf{0}$, or $f_x(a, b) = f_y(a, b) = 0$. $D = D(a, b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$. Then

   (a) If $D(a, b) > 0$, and $f_{xx}(a, b) > 0$, $f(a, b)$ is **a local minimum**.
   (b) If $D(a, b) > 0$, and $f_{xx}(a, b) < 0$, $f(a, b)$ is **a local maximum**.
   (c) If $D(a, b) < 0$ $f(a, b)$ is **not a local max/min**. In this case, (a, b) is called a **saddle point**.
   (d) If $D(a, b) = 0$, or $D(a, b) > 0$ but $f_{xx}(a, b) = 0$, it's inconclusive.

   Procedure for finding extreme values of f in $\mathfrak{D}$:

   (a) Find the values of f at all the critical points. (b) Find the extreme values of f on the boundary.
   (c) Compare the values in steps (a) & (b). The largest = abs. max; The smallest = abs. min of f.

6. **Method of Lagrange Multipliers** (§11.8) (let f and g be functions of x, y. similarly for 3 variables.)
   To find the max/min values of $f(x, y)$ subject to constraint $g(x, y) = k$, where k is a constant:

   (a) Find all values of (x, y) and λ such that $\nabla f = \lambda \nabla g$, $g(x, y) = k$.
   (b) Evaluate f at all points (x, y) that result from (a). The largest/smallest = the abs max/min of f.

   For 2 or more constraints, replace the eqns in (a) with $\nabla f = \lambda \nabla g + \mu \nabla h, g = k_1, h = k_2$.

7. **Double Integrals** (§12.1) **The volume of the solid under $z = f(x, y)$ and above $R = [a, b] \times [c, d] \subset \mathbb{R}^2$:**

$$V = \underbrace{\iint_R f(x, y) dA}_{\text{double integral}} = \underbrace{\lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A}_{\text{Double Riemann Sum}}$$

   area element: $dA = dx\, dy = dy\, dx$
   $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, $\Delta A = \Delta x\, \Delta y$.
   sample point $(x_i^*, y_j^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

8. **Iterated Integrals – Fubini's Th** (§12.2) $\displaystyle \iint_R f(x, y)\, \mathrm{d}A = \int_a^b \int_c^d f(x, y)\, \mathrm{d}y\, \mathrm{d}x = \int_c^d \int_a^b f(x, y)\, \mathrm{d}x\, \mathrm{d}y$

9. **Double Integrals over D** (§12.3) $\displaystyle \iint_D f(x, y)\, \mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, \mathrm{d}y\, \mathrm{d}x = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$

$$D = \left\{ (x, y) \middle| a \le x \le b,\ g_1(x) \le y \le g_2(x) \right\} = \left\{ (x, y) \middle| c \le y \le d,\ g_1(y) \le x \le g_2(y) \right\}$$

   In rectangular coordinates, the area element $\mathrm{d}A = \mathrm{d}x\, \mathrm{d}y = \mathrm{d}y\, \mathrm{d}x$.