Review Problems

Math 2400 Calculus III

Name: Solutions

Instructions for Doing These Problems:

1. When you first attempt the problem, **DO NOT** use any help (books, notes, etc)
2. If you cannot solve the problem during step 1, then you may use books and notes. If you solve it during this step, make a special mark next to the problem number and make sure to review that section.
3. If you cannot solve it during step 2, then I recommend either asking me or a classmate for help. Make sure to mark the problem and read the section.
4. Once you've finished all of the problems, you should focus on practice problems from the corresponding sections of your marked problems. I **highly recommend** doing odd-numbered exercises until you're almost always getting the right answer with no help.

1. Find the distance between the points $(1, 2, 3)$ and $(-1, -2, 0)$.

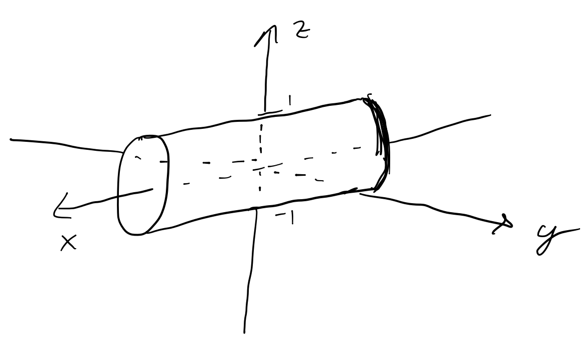
$$d = \sqrt{(1-(-1))^2 + (2-(-2))^2 + (3-0)^2} = \sqrt{4+16+9} = \boxed{\sqrt{29}}$$

2. Find the distance between the center of the sphere $(x-3)^2 + (y+1)^2 + z^2 = 9$ and the point $(0, 9, -3)$.

Center $= (3, -1, 0)$

$$d = \sqrt{(0-3)^2 + (9-(-1))^2 + (-3-0)^2} = \sqrt{9+100+9} = \boxed{\sqrt{118}}$$

3. Sketch the graph of the equation $y^2 + z^2 = 1$ in $\mathbb{R}^3$.



4. Find the dot product of the vectors $\langle -1, 1, -1\rangle$ and $\langle 2, 10, -3\rangle$.

$$\langle -1, 1, -1\rangle \cdot \langle 2, 10, -3\rangle = -2 + 10 + 3 = \boxed{11}$$

5. Determine whether the vectors $\langle 2, -1, 1\rangle$ and $\langle 3, 3, -3\rangle$ are orthogonal.

$$\langle 2, -1, 1\rangle \cdot \langle 3, 3, -3\rangle = 6 - 3 - 3 = 0$$

$$\boxed{\text{Orthogonal}}$$

6. Find the angle between the vectors $\langle 3, 0, -2\rangle$ and $\langle -1, -1, -1\rangle$.

$$|\langle 3, 0, -2\rangle| \, |\langle -1, -1, -1\rangle| \cos\theta = \langle 3, 0, -2\rangle \cdot \langle -1, -1, -1\rangle$$

$$(\sqrt{9+4})(\sqrt{3}) \cos\theta = -3 + 2$$

$$\boxed{\theta = \arccos\left(\frac{-1}{\sqrt{39}}\right)}$$

7. Compute the scalar projection of $\langle 2, 0, 4\rangle$ on $\langle 4, 0, 0\rangle$.

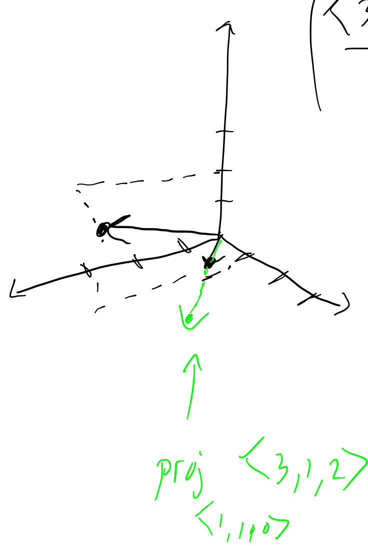
$$\frac{\langle 2, 0, 4\rangle \cdot \langle 4, 0, 0\rangle}{\sqrt{16+0+0}} = \frac{8}{4} = \boxed{2}$$

8. Sketch the vector projection of $\langle 3, 1, 2\rangle$ on $\langle 1, 1, 0\rangle$ along with the vectors $\langle 3, 1, 2\rangle$ and $\langle 1, 1, 0\rangle$.

$$\left(\frac{\langle 3, 1, 2\rangle \cdot \langle 1, 1, 0\rangle}{\sqrt{2}}\right)\left(\frac{\langle 1, 1, 0\rangle}{\sqrt{2}}\right)$$

$$= \frac{4}{2} \langle 1, 1, 0\rangle$$

$$= \langle 2, 2, 0\rangle$$



9. Find a vector perpendicular to both $\langle 0, -2, 5\rangle$ and $\langle 1, 0, 1\rangle$.

$$\begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & 0 & 1 \end{vmatrix} = \boxed{\langle -2, 5, 2\rangle}$$

10. Determine whether the vectors $\langle 12, 4, 15\rangle$ and $\langle 3, 1, 4\rangle$ are parallel.

$$\begin{vmatrix} i & j & k \\ 12 & 4 & 15 \\ 3 & 1 & 4 \end{vmatrix} = \langle \overset{\neq 0}{16 - 15}, \ldots\rangle \qquad \boxed{\text{So not parallel}}$$

11. Calculate the area of the parallelogram formed by the vectors $\langle 7, 3, -2\rangle$ and $\langle 0, 19, -1\rangle$.

$$\begin{vmatrix} i & j & k \\ 7 & 3 & -2 \\ 0 & 19 & -1 \end{vmatrix} = \langle -3 + 38, 7, 133\rangle = \langle 35, 7, 133\rangle$$

$$\boxed{A = \sqrt{35^2 + 7^2 + 133^2}}$$

12. Calculate the volume of the parallelepiped formed by the vectors $\langle 2, -2, -1\rangle$, $\langle -1, -1, 1\rangle$, and $\langle 3, 2, 5\rangle$.

9.4

Equation 7 in book let's us calculate

$$|\langle 2, -2, -1\rangle \cdot (\langle -1, -1, 1\rangle \times \langle 3, 2, 5\rangle)|$$

$$\left|\begin{vmatrix} 2 & -2 & -1 \\ -1 & -1 & 1 \\ 3 & 2 & 5 \end{vmatrix}\right| = |-10 - 6 + 2 - 3 - 10 - 4|$$

$$= |-31| = \boxed{31}$$

13. Determine whether or not the vectors $\langle -1, 0, 3\rangle$, $\langle 4, 1, 2\rangle$, and $\langle -3, -1, 2\rangle$ lie on the same plane.

$$\begin{vmatrix} -1 & 0 & 3 \\ 4 & 1 & 2 \\ -3 & -1 & 2 \end{vmatrix} = -2 + 0 - 12 + 9 - 0 - 2$$

$$\neq 0$$

$$\boxed{\text{Not coplanar}}$$

14. Write the equation of the line $\frac{x-2}{3} = \frac{y}{4} = z + 2$ in parametric form.

$t = \frac{x-2}{3}$ $\quad t = z+2$

$t = y/4$

$$\boxed{\vec{r}(t) = \langle 3t+2, 4t, t-2 \rangle}$$

15. Write the equation of the line $\langle t+1, -3t+4, 7t-2 \rangle$ as symmetric equations.

$$\boxed{\frac{x-1}{1} = \frac{4-y}{3} = \frac{z+2}{7}}$$

16. Write the equation of the line that goes through the point $(0, 0, 3)$ and has direction vector $\langle 4, -2, -1 \rangle$.

$$\boxed{\vec{r}(t) = \langle 4t, -2t, 3-t \rangle}$$

17. Write the equation of the line that goes through the points $(-5, 1, 0)$ and $(0, -2, -5)$.

$\vec{d} = \langle -5, 1, 0 \rangle - \langle 0, -2, -5 \rangle = \langle -5, 3, 5 \rangle$

$$\boxed{\vec{r}(t) = \langle -5t, 3t-2, 5t-5 \rangle}$$

18. Determine whether the points $(1, 1, 1)$, $(2, 3, -4)$ and $(-1, -3, 4)$ lie on the same line.

$\vec{d}_1 = \langle 1, 1, 1 \rangle - \langle 2, 3, -4 \rangle = \langle -1, -2, 5 \rangle$

$\vec{d}_2 = \langle 1, 1, 1 \rangle - \langle -1, -3, 4 \rangle = \langle 2, 4, -3 \rangle$

$\vec{d}_1 \neq K\vec{d}_2$ $\quad K$, constant $\Rightarrow$ $\boxed{\text{Not on same line}}$

19. Find the equation of the plane with normal vector $\langle 6, 2, 5 \rangle$ and point on the plane $(0, 0, -1)$.

$\langle 6, 2, 5 \rangle \cdot \langle x, y, z+1 \rangle = 0$

$$\boxed{6x + 2y + 5z + 5 = 0}$$

20. Find the equation of the intersection of the planes $x+y+z=1$ and $-x-2z=4$.

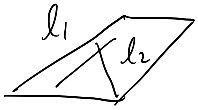
$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 0 & -2 \end{vmatrix} = \langle -2, 1, 1 \rangle$$

$$\begin{aligned} x+y+z &= 1 \\ -x-2z &= 4 \\ \hline y-z &= 5 \end{aligned}$$

Point: $(-6, 6, 1)$

$$\boxed{\vec{r}(t) = \langle -2t-6,\ t+6,\ t+1 \rangle}$$

21. Find the equation of the plane containing the lines $\langle t, 9-t, 2t\rangle$ and $\frac{x}{2} = y = z+4$.



$\vec{d}_1 = \langle 1, -1, 2 \rangle$

$\vec{d}_2 = \langle 2, 1, 1 \rangle$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \langle -3, 3, 3 \rangle$$

Point: $(0, 9, 0)$

$$\langle x, y-9, z \rangle \cdot \langle -3, 3, 3 \rangle = 0$$

$$\boxed{-3x + 3(y-9) + 3z = 0}$$

22. Find the angle between the planes $2x - y = 6$ and $3y + 2z = -2$.

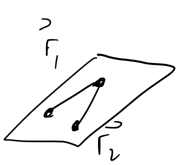
$\vec{n}_1 = \langle 2, -1, 0 \rangle$

$\vec{n}_2 = \langle 0, 3, 2 \rangle$

$$\langle 2, -1, 0 \rangle \cdot \langle 0, 3, 2 \rangle = (\sqrt{5})(\sqrt{13}) \cos\theta$$

$$\boxed{\theta = \arccos\left(\frac{-3}{\sqrt{65}}\right)}$$

23. Find the equation of the plane containing the points $(0, 0, -3)$, $(1, 2, 0)$, and $(7, 1, 1)$.



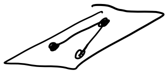
$\vec{r}_1 = \langle 7, 1, 1 \rangle - \langle 0, 0, -3 \rangle = \langle 7, 1, 4 \rangle$

$\vec{r}_2 = \langle 7, 1, 1 \rangle - \langle 1, 2, 0 \rangle = \langle 6, -1, 1 \rangle$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 7 & 1 & 4 \\ 6 & -1 & 1 \end{vmatrix} = \langle 1+4, 24-7, -7-6 \rangle = \langle 5, 17, -13 \rangle$$

$$\langle 5, 17, -13 \rangle \cdot \langle x-0, y-0, z+3 \rangle = 0$$

$$\boxed{5x + 17y - 13(z+3) = 0}$$



24. Find the equation of a plane parallel to the plane containing the points $(0, 8, 9)$, $(3, 12, 1)$, $(19, 19, 0)$.

$\vec{r}_1 = \langle 0, 8, 9\rangle - \langle 3, 12, 1\rangle = \langle -3, -4, 8\rangle$

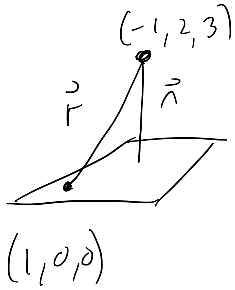
$\vec{r}_2 = \langle 19, 19, 0\rangle - \langle 3, 12, 1\rangle = \langle 16, 7, -1\rangle$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -3 & -4 & 8 \\ 16 & 7 & -1 \end{vmatrix} = \langle 4-56,\ 128-3,\ -21+64\rangle = \langle -52, 125, 43\rangle$$

$\langle -52, 125, 43\rangle \cdot \langle x-0, y-8, z-7\rangle = 0$

$$\boxed{-52x + 125(y-8) + 43(z-9) = 0}$$

25. Find the distance between the point $(-1, 2, 3)$ and the plane $x - y + z = 1$.

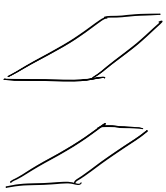


$\vec{n} = \langle 1, -1, 1\rangle$

$\vec{r} = \langle -1, 2, 3\rangle - \langle 1, 0, 0\rangle = \langle -2, 2, 3\rangle$

$$\mathrm{comp}_{\vec{n}}\, \vec{r} = \frac{\langle 1,-1,1\rangle \cdot \langle -2,2,3\rangle}{\sqrt{1^2+(-1)^2+1^2}} = \frac{-4+3}{\sqrt{3}} \qquad \boxed{\frac{1}{\sqrt{3}}}$$

26. Find the distance between the planes $x - y = 3$ and $x - y = -1$.

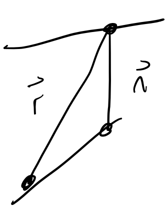


$$\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\frac{3}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \boxed{\frac{4}{\sqrt{2}}}$$

27. Find the distance between the skew lines $\langle -t, t-2, 3+2t\rangle$ and $\langle t, 3t-1, t-2\rangle$.



$\vec{d}_1 = \langle -1, 1, 2\rangle \qquad \vec{d}_2 = \langle 1, 3, 1\rangle$

$l_1(0) = \langle 0, -2, 3\rangle$

$l_2(0) = \langle 0, -1, -2\rangle$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = \langle -5, 3, -4\rangle$$

$\vec{r} = \langle 0, -2, 3\rangle - \langle 0, -1, -2\rangle$

$= \langle 0, -1, 5\rangle$

$$|\mathrm{comp}_{\vec{n}}\, \vec{r}| = \frac{|\langle -5, 3, -4\rangle \cdot \langle 0, -1, 5\rangle|}{\sqrt{50}} = \boxed{\frac{23}{\sqrt{50}}}$$

28. Name the surface given by the equation $x - y^2 + z^2 = 0$ and say where it's centered.

$x = y^2 - z^2$

Not really concerned with "center" here

$\boxed{\text{Hyperbolic Paraboloid}}$

29. Name the surface given by the equation $2x^2 + y^2 + z^2 = 3$ and say where it's centered.

$\boxed{\text{Ellipsoid}}$ $\boxed{\text{Center doesn't make sense}}$

30. Name the surface given by the equation $-3z^2 - 3y^2 = -x^2$ and say where it's centered.

$\boxed{\text{Cone}}$ $\boxed{\text{Centered on } x\text{-axis}}$

31. Name the surface given by the equation $x^2 - 2y + z^2 = 0$ and say where it's centered.

$\boxed{\text{Elliptic Paraboloid}}$ $\boxed{\text{Centered on } y\text{-axis}}$

32. Name the surface given by the equation $x^2 - y^2 - z^2 = 5$ and say where it's centered.

$\boxed{\text{Hyperboloid of Two Sheets}}$ $\boxed{\text{Centered on } x\text{-axis}}$

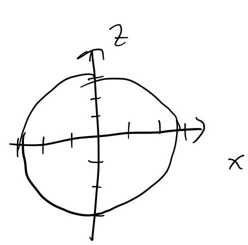
33. Name the surface given by the equation $x^2 - y^2 = 1 - z^2$ and say where it's centered.

$\boxed{\text{Hyperboloid of One Sheet}}$ $\boxed{\text{Centered on } y\text{-axis}}$

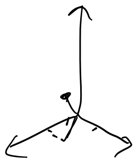
34. Sketch the trace of the surface $x^2 - 4y^2 + z^2 = 5$ when $y = 1$.

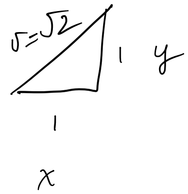
$x^2 - 4 + z^2 = 5$

$x^2 + z^2 = 9$



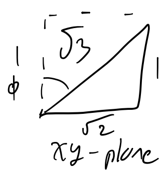
35. Convert the point $(1, 1, 1)$ from rectangular to cylindrical coordinates.





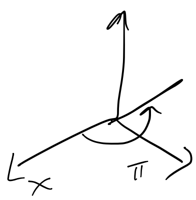
$$\boxed{\left(\sqrt{2}, \frac{\pi}{4}, 1\right)}$$

36. Convert the point $(1, 1, 1)$ from rectangular to spherical coordinates.



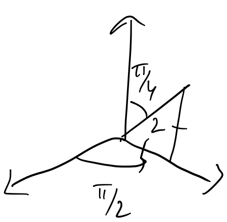
$$\boxed{\left(\sqrt{3}, \frac{\pi}{4}, \arccos\left(\frac{1}{\sqrt{3}}\right)\right)}$$

37. Convert the point $(2, \pi, -2)$ from cylindrical to rectangular coordinates.



$$\boxed{(-2, 0, -2)}$$

38. Convert the point $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$ from spherical to rectangular coordinates.



$$\boxed{(0, \sqrt{2}, \sqrt{2})}$$

39. Convert the equation $x^2 - y^2 - z^2 = 2$ into spherical coordinates.

$$\boxed{(\rho \sin\phi \cos\theta)^2 - (\rho \sin\phi \sin\theta)^2 - (\rho \cos\phi)^2 = 2}$$

40. Convert the equation $x^2 + 2y^2 + 2z^2 = 1$ into cylindrical coordinates.

$$\boxed{r^2 \cos^2\theta + 2r^2 \sin^2\theta + 2z^2 = 1}$$

41. Find the equation of the space curve in the intersection of the cylinder $x^2+y^2=4$ and the plane $z-x-y=1$

$$\boxed{\begin{aligned} x &= 2\cos\theta \\ y &= 2\sin\theta \qquad\qquad 0 \le \theta \le 2\pi \\ z &= 1 + 2\cos\theta + 2\sin\theta \end{aligned}}$$

42. Calculate $\int_1^4 \langle t^2, e^t, \cos(t)\rangle\, dt$.

$$\left\langle \frac{t^3}{3}, e^t, \sin(t) \right\rangle \Big|_{t=1}^{t=4} = \boxed{\left\langle 21, e^4 - e^1, \sin(4) - \sin(1) \right\rangle}$$

43. Calculate the derivative of $\underset{\vec{r}(t)}{\langle e^t \sin(t^2), \ln(t), t^2+5t\rangle}$.

$$\boxed{\vec{r}\,'(t) = \left\langle e^t \sin(t^2) + 2te^t\cos(t^2), \frac{1}{t}, 2t+5 \right\rangle}$$

44. Find the unit tangent vector to the curve $\langle e^t, t^2+5t, -\cos(t)\rangle$ when $t=0$.

$\vec{r}\,'(t) = \langle e^t, 2t+5, \sin(t)\rangle \qquad r'(0) = \langle 1, 5, 0\rangle$

$$\boxed{\left\langle \frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}, 0 \right\rangle}$$

45. Find the equation of the tangent line(s) to the curve $\langle t^2, t^3, t\rangle$ at the point $(4, 8, 2)$.

$t^3 = 8$

$t = 2$

$r'(t) = \langle 2t, 3t^2, 1\rangle$

$r'(2) = \langle 4, 12, 1\rangle$

$$\boxed{l(t) = \langle 4+4t, 8+12t, 2+t\rangle}$$

$s=0$

$t=0 \qquad \langle -1, s, s^2+2\rangle$

46. Find the angle of intersection of the curves $\langle t-1, t^2, 2\rangle$ and $\langle -1, t, t^2+2\rangle$.

$t-1=-1$

$t=0$

$P=\langle -1,0,2\rangle$

$\vec{r}_1'(t) = \langle 1, 2t, 0\rangle \Rightarrow \vec{r}_1'(0) = \langle 1,0,0\rangle$

$\vec{r}_2'(s) = \langle 0, 1, 2s\rangle \Rightarrow \vec{r}_2'(0) = \langle 0,1,0\rangle$

$\cos\theta = 0$

$\boxed{\theta = \pi/2}$

47. Set up the integral to determine the arc length of the curve $\langle e^t, t^2, t^7 - 5t^2\rangle$ from $t = 4$ to $t = 8$.

$r'(t) = \langle e^t, 2t, 7t^6 - 10t\rangle$

$$\boxed{\int_4^8 \sqrt{e^{2t} + 4t^2 + (7t^6-10t)^2}\, dt}$$

48. Reparametrize the curve $\langle \cos(t), \sin(t), 5t\rangle$ by arc length.

$r'(t) = \langle -\sin(t), \cos(t), 5\rangle$

$s(t) = \int_0^t \sqrt{\cos^2 t + \sin^2 t + 25}\, du$

$= \int_0^t \sqrt{26}\, du$

$= u\sqrt{26}\Big|_0^t = t\sqrt{26}$

$t(s) = \dfrac{s}{\sqrt{26}}$

$$\boxed{\left\langle \cos\left(\frac{s}{\sqrt{26}}\right), \sin\left(\frac{s}{\sqrt{26}}\right), 5\left(\frac{s}{\sqrt{26}}\right)\right\rangle}$$

49. Parametrize a cylinder of radius 3 centered on the y-axis.

$x = 3\cos\theta \qquad 0 \le \theta \le 2\pi$

$y = y \qquad y \in \mathbb{R}$

$z = 3\sin\theta$

50. Parametrize a sphere of radius 2 centered at the origin.

$x = 2\cos\theta \sin\phi \qquad 0 \le \theta \le 2\pi$

$y = 2\sin\theta\sin\phi \qquad 0 \le \phi \le \pi$

$z = 2\cos\phi$

51. Parametrize the plane $2x - y + 3z = -2$.

$$x = x$$
$$y = 2x + 3z + 2 \qquad x, z \in \mathbb{R}$$
$$z = z$$

52. Parametrize the cone $4x^2 + 2y^2 = z^2$.

$$x = \frac{r}{2}\cos\theta \qquad 0 \le \theta \le 2\pi$$
$$y = \frac{r}{\sqrt{2}}\sin\theta \qquad r \in \mathbb{R}$$
$$z = r$$

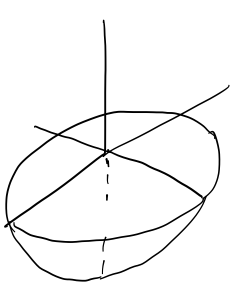
53. Parametrize the ellipsoid $3x^2 + 5y^2 + z^2 = 1$.

$$x = \frac{1}{\sqrt{3}}\cos\theta\sin\phi \qquad 0 \le \theta \le 2\pi$$
$$y = \frac{1}{\sqrt{5}}\sin\phi\sin\theta \qquad 0 \le \phi \le \pi$$
$$z = \cos\phi$$

54. Parametrize the elliptic paraboloid $y = 2z^2 + x^2$.

$$x = r\cos\theta \qquad 0 \le \theta \le 2\pi$$
$$y = r^2 \qquad r \in \mathbb{R}$$
$$z = \frac{r}{\sqrt{2}}\sin\theta$$

55. Parametrize the bottom hemisphere of a sphere with radius 10 centered at the origin.



$$x = 10\cos\theta\sin\phi \qquad 0 \le \theta \le 2\pi$$
$$y = 10\sin\theta\sin\phi \qquad \frac{\pi}{2} \le \phi \le \pi$$
$$z = 10\cos\phi$$