Review guide for mid-term exam 1 – Fall 2019

Exam date and time: Monday, September 23, 2019, 5:15–6:45PM

1. **Vectors and Their Operations**: $\vec{a} \pm \vec{b}$, $c\vec{a}$, $\vec{a}\cdot\vec{b}$, $\vec{a}\times\vec{b}$, $\vec{a}\cdot(\vec{b}\times\vec{c})$.
   Let $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle \in V_3$, and $c \in \mathbb{R}$ be a scalar.

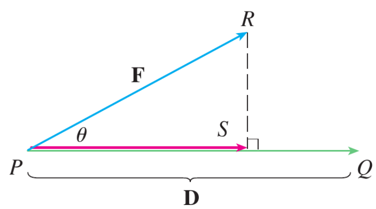
   (a) If $P(x, y, z) \in \mathbb{R}^3$, the vector $\vec{v} = \overrightarrow{OP} = \langle x, y, z\rangle$ is the position vector of the point P.
   $|\vec{v}| = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

   (b) $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$, $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$, $c\vec{a} = \langle ca_1, ca_2, ca_3\rangle$,
   $\vec{a}\cdot\vec{b} = \langle a_1b_1, a_2b_2, a_3b_3\rangle = |\vec{a}|\,|\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

   $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \langle a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1\rangle.$$

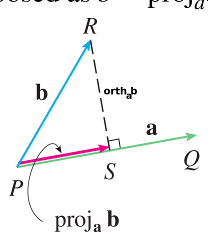
2. **More on Vectors**:

   (a) $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. $\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$.

   $\vec{F} = \overrightarrow{PQ}$ is the force moves the object from P to Q with with **displacement vector** $\vec{D} = \overrightarrow{PQ}$. The work done by $\vec{F}$ is $W = |\vec{F}|\,|\vec{D}|\cos\theta = \vec{F}\cdot\vec{D}$.

   (b) The addition of any two vectors follows the triangle and parallelogram laws. A vector $\vec{b}$ can be decomposed as $\vec{b} = \text{proj}_{\vec{a}}\vec{b} + \text{orth}_{\vec{a}}\vec{b}$.

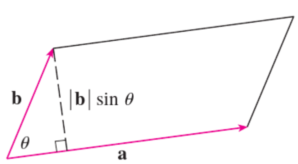
   scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

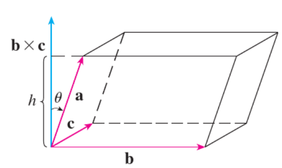
   vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$

   orthogonal projection of $\vec{b}$ onto $\vec{a}$: $\text{orth}_{\vec{a}}\vec{b} = \vec{b} - \text{proj}_{\vec{a}}\vec{b}$

   (c) $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\vec{n}$, where n is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, and whose direction is determined by the right-hand rule.

   (d) **The area of a parallelogram and the volume of a parallelepiped**:

   area $= |\vec{a}|\,(|\vec{b}|\sin\theta) = |\vec{a}\times\vec{b}|$

   $$\text{vol} = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right| = \text{Abs}\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right)$$

   i. The area of the triangle with 3 vertices A, B and C is $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$ — half area of the parallelogram.

   ii. The volume of the tetrahedron with vertices A, B, C and D is $\frac{1}{6}$ of the volume of the parallelepiped spanned by $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$.

   (e) If θ is the angle between $\vec{a}$ and $\vec{b}$, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$, $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$.

   (f) $\vec{a}\perp\vec{b} \iff \vec{a}\cdot\vec{b} = 0$. ($\vec{0}\perp$ any vector.)

   (g) $\vec{a}\parallel\vec{b} \iff \vec{a}\times\vec{b} = \vec{0} \iff \vec{a} = c\vec{b}$ or $\vec{b} = c\vec{a} \iff \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ (if $\vec{b}\neq\vec{0}$). ($\vec{0}\parallel$ any vector.)

   (h) $\vec{a}\times\vec{b}\perp\vec{a}$, $\vec{a}\times\vec{b}\perp\vec{b}$. $\vec{a}\times\vec{b}$ is orthogonal to the plane determined by $\vec{a}$ and $\vec{b}$ if $\vec{a}\nparallel\vec{b}$.

3. **Lines and Planes in A Three Dimensional Space:**

(a) **Equation of a line** L through $P_0(x_0, y_0, z_0)$ with direction $\vec{v} = \langle a,\ b,\ c\rangle$.

(1) **Vector equation:** $\vec{r} = \vec{r}_0 + t\ \vec{v}$, where $\vec{r}_0 = \overrightarrow{OP_0}$ is the position vector of P_0, $t \in \mathbb{R}$.

(2) **Parametric equations:** $x = x_0 + a\,t,\ \ y = y_0 + b\,t,\ \ z = z_0 + c\,t$, parameter $t \in \mathbb{R}$.

(3) **Symmetric equations:** $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

(b) **Equation of a line segment** from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$. Let $\vec{r}_0 = \overrightarrow{OP_0}$, $\vec{r}_1 = \overrightarrow{OP_1}$, $\vec{r} = \overrightarrow{OP}$.

$$\overrightarrow{P_0P} \parallel \overrightarrow{P_0P_1} \Leftrightarrow \overrightarrow{P_0P} = t\ \overrightarrow{P_0P_1} \Leftrightarrow \vec{r}(t) - \vec{r}_0 = t\ (\vec{r}_1 - \vec{r}_0) \Leftrightarrow \vec{r}(t) = (1-t)\vec{r}_0 + t\ \vec{r}_1,\ \ 0 \le t \le 1.$$

(c) **Equation of a plane** π through a point $P_0(x_0, y_0, z_0)$ with a normal direction $\vec{n} = \langle a,\ b,\ c\rangle$.

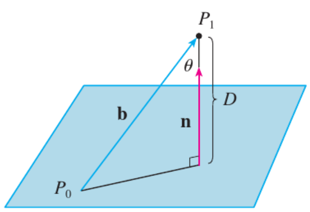
(1) **Vector equation of the plane:** $\vec{n}\cdot(\vec{r} - \vec{r}_0) = 0$, or $\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$ where $\vec{r}_0 = \overrightarrow{OP_0}$.

(2) **Scalar equation of the plane:** $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

(3) **Linear equation of the plane:** $ax + by + cz + d = 0$, where $d = -(ax_0 + by_0 + cz_0)$

(d) **Distances:** – Formulas can be derived by **projection** of vectors.

(1) The distance between a point $P_1(x_1, y_1, z_1)$ and a plane π: $ax + by + cz + d = 0$.



$$D = \text{dist}(P_1, \pi_2) = \left|\text{comp}_{\vec{n}}\vec{b}\right| = \left|\text{proj}_{\vec{n}}\vec{b}\right| = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}$$
$$= \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (ax_0 + by_0 + cz_0 + d = 0)$$

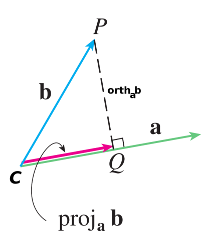
(2) The distance between two parallel planes: π_1: $ax + by + cz + d_1 = 0$; π_2: $ax + by + cz + d_2 = 0$.

$$D = \text{dist}(\pi_1, \pi_2) = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} \overset{\text{or}}{=} \text{dist}(P, \pi_2), \quad (P \text{ is any point on } \pi_1)$$

(3) The distance between parallel line (L) and plane (π): $D = \text{dist}(L,\ \pi) = \text{dist}(P,\ \pi)$, where P can be any point on L.

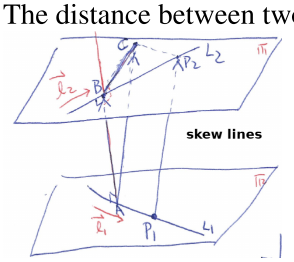
(4) The distance between parallel lines ($L_1 \parallel L_2$) $D = \text{dist}(L_1, L_2) = \text{dist}(P, L_2)$, where P is any point on L_1.

(5) The distance between a point $P(x_0, y_0, z_0)$ and a line L: $\frac{x-c_1}{a_1} = \frac{y-c_2}{a_2} = \frac{z-c_3}{a_3}$.



$C(c_1,\ c_2,\ c_3) \in L$ is on L. $\vec{a} = \langle a_1,\ a_2,\ a_3\rangle$ is the direction vector of L. Let $\vec{b} = \overrightarrow{CP}$. Then
$D = |PQ| = \left|\text{orth}_{\vec{a}}\vec{b}\right| = \left|\vec{b} - \text{proj}_{\vec{a}}\vec{b}\right|$

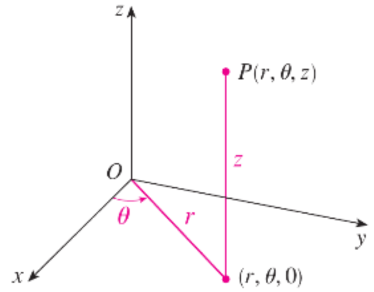
(6) The distance between two skew lines:

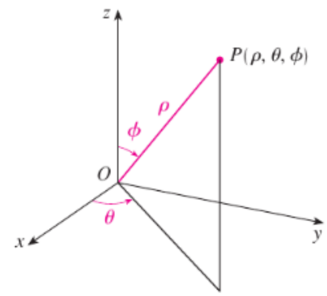


$$D = \text{dist}(L_1, L_2) = \left|\text{Comp}_{\vec{n}}\overrightarrow{P_1P_2}\right| = \frac{\left|\vec{n}\cdot\overrightarrow{P_1P_2}\right|}{|\vec{n}|},$$

where $\vec{n} = \vec{l}_1 \times \vec{l}_2$, and $\vec{l}_1$, $\vec{l}_2$ are the direction vectors of L_1 and L_2. $P_1 \in L_1$, $P_2 \in L_2$.

4. **Cylindrical and Spherical Coordinates:**





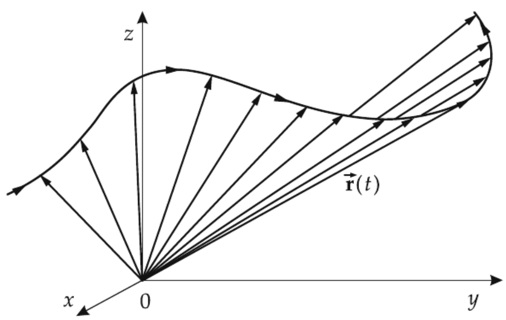
Cylindrical coord (r,θ,z): $\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{cases}$ Spherical coord: (ρ,θ,ϕ): $\begin{cases} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{cases}$

(a) Convert points or equations in Cartesian (or rectangular) coordinates to that in cylindrical or spherical coordinates, and verse visa.

(b) Space curves, surfaces, solids in cylindrical or spherical coordinates.

5. **Space Curves and Surfaces:**

(a) A space curve C can be described by a vector function $\vec{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$, or parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, where t is the parameter.



The domain of the vector function is the intersection of the domains of the components.
$\vec{r}(t)$ is continuous if $\lim_{x\to a}\vec{r}(t) = \vec{r}(a)$.
$\vec{r}\,'(t) = \langle x'(t),\ y'(t),\ z'(t)\rangle$.
$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left\langle \int_a^b x(t)\,\mathrm{d}t,\ \int_a^b y(t)\,\mathrm{d}t,\ \int_a^b z(t)\,\mathrm{d}t\right\rangle$.
If a particle position at time t is described by $\vec{r}(t)$, $\vec{r}\,'(t)$ is the velocity of the particle at t, and $\vec{r}\,''(t)$ is the acceleration of the particle.

(b) Equation of the tangent line:
If C: $\vec{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ is a space curve, and $P_0(x_0,\ y_0,\ z_0) \in C$, and P_0 corresponds to $t = t_0$, then the equation of the tangent line through P_0 is

$$\vec{r}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0) \textbf{ or } x = x_0 + at,\ \ y = y_0 + bt,\ \ z = z_0 + ct, \textbf{ or } \frac{x - x_0}{a} = \frac{x - y_0}{b} = \frac{z - z_0}{c}$$

where $\vec{r}(t_0) = \overrightarrow{OP_0} = \langle x_0,\ y_0,\ z_0\rangle$, $\vec{r}\,'(t_0) = \langle a,\ b,\ c\rangle$.
Equation of a normal plane through P_0 is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

(c) The arc length of a space curve C: $\vec{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ from $t = a$ to $t = b$ is
$L = \int_C \mathrm{d}s = \int_a^b |\vec{r}\,'(t)|\,\mathrm{d}t = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t$, (similarly for arc length in $2D$.)

(d) In 3D, a space surface can be described by a vector function $\vec{r}(u,v)$ with two parameters:
$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$
where $(u,v) \in D$. D is the domain of $\vec{r}$.
Parametric equations of surface S: $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$.

(e) Quadric surfaces: cylinder, ellipsoid, elliptic paraboloid, hyperbolic paraboloid, cone, hyperboloid with one or two sheet(s), etc. See Page 679 for the plots of some basic quadric surfaces.

(f) Parametrize a space curve or a surface:
A space curve C: $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ usually has **one** parameter (t).
A space surface S: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. usually has **two** parameters (u, v).