LECTURE 1: DIFFERENTIAL FORMS

1. 1-FORMS ON $\mathbb{R}^n$

In calculus, you may have seen the differential or exterior derivative $df$ of a function $f(x, y, z)$ defined to be

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$ 

The expression $df$ is called a 1-form. But what does this really mean?

**Definition:** A smooth 1-form $\phi$ on $\mathbb{R}^n$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^n$, i.e.,

$$\phi : T \mathbb{R}^n \to \mathbb{R}$$

with the properties that

1. $\phi$ is linear on the tangent space $T_x \mathbb{R}^n$ for each $x \in \mathbb{R}^n$.
2. For any smooth vector field $v = v(x)$, the function $\phi(v) : \mathbb{R}^n \to \mathbb{R}$ is smooth.

Given a 1-form $\phi$, for each $x \in \mathbb{R}^n$ the map

$$\phi_x : T_x \mathbb{R}^n \to \mathbb{R}$$

is an element of the dual space $(T_x \mathbb{R}^n)^*$. When we extend this notion to all of $\mathbb{R}^n$, we see that the space of 1-forms on $\mathbb{R}^n$ is dual to the space of vector fields on $\mathbb{R}^n$.

In particular, the 1-forms $dx^1, \ldots, dx^n$ are defined by the property that for any vector $v = (v^1, \ldots, v^n) \in T_x \mathbb{R}^n$,

$$dx^i(v) = v^i.$$

The $dx^i$'s form a basis for the 1-forms on $\mathbb{R}^n$, so any other 1-form $\phi$ may be expressed in the form

$$\phi = \sum_{i=1}^{n} f_i(x) \, dx^i.$$

If a vector field $v$ on $\mathbb{R}^n$ has the form

$$v(x) = (v^1(x), \ldots, v^n(x)),$$
then at any point $x \in \mathbb{R}^n$, 
$$
\phi_x(v) = \sum_{i=1}^{n} f_i(x) v^i(x).
$$

2. $p$-FORMS ON $\mathbb{R}^n$

The $1$-forms on $\mathbb{R}^n$ are part of an algebra, called the \textit{algebra of differential forms} on $\mathbb{R}^n$. The multiplication in this algebra is called \textit{wedge product}, and it is skew-symmetric:

$$
dx^i \wedge dx^j = -dx^j \wedge dx^i.
$$

One consequence of this is that $dx^i \wedge dx^i = 0$.

If each summand of a differential form $\phi$ contains $p$ $dx^i$'s, the form is called a $p$-\textit{form}. Functions are considered to be 0-forms, and any form on $\mathbb{R}^n$ of degree $p > n$ must be zero due to the skew-symmetry.

A basis for the $p$-forms on $\mathbb{R}^n$ is given by the set 
$$
\{dx^{i_1} \wedge \cdots \wedge dx^{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}.
$$

Any $p$-form $\phi$ may be expressed in the form

$$
\phi = \sum_{|I| = p} f_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}
$$

where $I$ ranges over all multi-indices $I = (i_1, \ldots, i_p)$ of length $p$.

Just as 1-forms act on vector fields to give real-valued functions, so $p$-forms act on $p$-tuples of vector fields to give real-valued functions. For instance, if $\phi, \psi$ are 1-forms and $v, w$ are vector fields, then 

$$
(\phi \wedge \psi)(v, w) = \phi(v)\psi(w) - \phi(w)\psi(v).
$$

In general, if $\phi_1, \ldots, \phi_p$ are 1-forms and $v_1, \ldots, v_p$ are vector fields, then 

$$
(\phi_1 \wedge \cdots \wedge \phi_p)(v_1, \ldots, v_p) = \sum_{\sigma \in S_p} \text{sgn}(\sigma) \phi_1(v_{\sigma(1)}) \phi_2(v_{\sigma(2)}) \cdots \phi_n(v_{\sigma(n)}).
$$

3. THE EXTerior DERivative

The \textit{exterior derivative} is an operation that takes $p$-forms to $(p + 1)$-forms. We will first define it for functions and then extend this definition to higher degree forms.

**Definition:** If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then the exterior derivative of $f$ is the 1-form $df$ with the property that for any $x \in \mathbb{R}^n$, $v \in T_x \mathbb{R}^n$,

$$
df_x(v) = v(f),
$$

i.e., $df_x(v)$ is the directional derivative of $f$ at $x$ in the direction of $v$. 
It is not difficult to show that
\[ df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i. \]

The exterior derivative also obeys the Leibniz rule
\[ d(fg) = g df + f dg \]
and the chain rule
\[ d(h(f)) = h'(f) df. \]

We extend this definition to \( p \)-forms as follows:

**Definition:** Given a \( p \)-form \( \phi = \sum_{|I|=p} f_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \), the exterior derivative \( d\phi \) is the \((p+1)\)-form
\[ d\phi = \sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \]

If \( \phi \) is a \( p \)-form and \( \psi \) is a \( q \)-form, then the Leibniz rule takes the form
\[ d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi. \]

**Very Important Theorem:** \( d^2 = 0 \). i.e., for any differential form \( \phi \),
\[ d(d\phi) = 0. \]

**Proof:** First suppose that \( f \) is a function, i.e., a \( 0 \)-form. Then
\[
d(df) = d\left( \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \right)
= \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i
= \sum_{i<j} \left( \frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^i \wedge dx^j
= 0
\]
because mixed partials commute.

Next, note that \( dx^i \) really does mean \( d(x^i) \), where \( x^i \) is the \( i \)th coordinate function. So by the argument above, \( d(dx^i) = 0 \). Now suppose that
\[ \phi = \sum_{|I|=p} f_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \]
Then by the Leibniz rule,
\[
d(d\phi) = d\left( \sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \right)
\]
\[
= \sum_{|I|=p} \left[ d(df_I) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} - df_I \wedge d(dx^{i_1}) \wedge \cdots \wedge dx^{i_p} + \ldots \right]
\]
\[
= 0. \quad \square
\]

**Definition:** A \( p \)-form \( \phi \) is **closed** if \( d\phi = 0 \). \( \phi \) is **exact** if there exists a \((p - 1)\)-form \( \eta \) such that \( \phi = d\eta \).

By the Very Important Theorem, every exact form is closed. The converse is only partially true: every closed form is **locally** exact. This means that given a closed \( p \)-form \( \phi \) on an open set \( U \subset \mathbb{R}^n \), any point \( x \in U \) has a neighborhood on which there exists a \((p - 1)\)-form \( \eta \) with \( d\eta = \phi \).

### 4. Differential Forms on Manifolds

Given a smooth manifold \( M \), a **smooth 1-form** \( \phi \) on \( M \) is a real-valued function on the set of all tangent vectors to \( M \) such that

1. \( \phi \) is linear on the tangent space \( T_xM \) for each \( x \in M \).
2. For any smooth vector field \( v \) on \( M \), the function \( \phi(v) : M \to \mathbb{R} \) is smooth.

So for each \( x \in M \), the map
\[
\phi_x : T_xM \to \mathbb{R}
\]
is an element of the dual space \((T_xM)^*\).

Wedge products and exterior derivatives are defined similarly as for \( \mathbb{R}^n \). If \( f : M \to \mathbb{R} \) is a differentiable function, then we define the exterior derivative of \( f \) to be the 1-form \( df \) with the property that for any \( x \in M \), \( v \in T_xM \),
\[
df_x(v) = v(f).
\]

A local basis for the space of 1-forms on \( M \) can be described as before in terms of any local coordinate chart \((x^1, \ldots, x^n)\) on \( M \), and it is possible to show that the coordinate-based notions of wedge product and exterior derivative are in fact independent of the choice of local coordinates and so are well-defined.

More generally, suppose that \( M_1, M_2 \) are smooth manifolds and that \( F : M_1 \to M_2 \) is a differentiable map. For any \( x \in M_1 \), the differential \( dF \) (also denoted \( F_* \)) : \( T_xM_1 \to T_{F(x)}M_2 \) may be thought of as a **vector-valued** 1-form, because it is a linear map from \( T_xM_1 \) to the vector space \( T_{F(x)}M_2 \). There is an analogous map in the opposite direction for differential forms, called the **pullback** and denoted \( F^* \). It is defined as follows.
Definition: If $F : M_1 \to M_2$ is a differentiable map, then

1. If $f : M_2 \to \mathbb{R}$ is a differentiable function, then $F^*f : M_1 \to \mathbb{R}$ is the function
   
   
   $$(F^*f)(x) = (f \circ F)(x).$$

2. If $\phi$ is a $p$-form on $M_2$, then $F^*\phi$ is the $p$-form on $M_1$ defined as follows:
   if $v_1, \ldots, v_p \in T_xM_1$, then
   
   $$(F^*\phi)(v_1, \ldots, v_p) = \phi(F_*(v_1), \ldots, F_*(v_p)).$$

In terms of local coordinates $(x^1, \ldots, x^n)$ on $M_1$ and $(y^1, \ldots, y^m)$ on $M_2$, suppose that the map $F$ is described by

$$y^i = y^i(x^1, \ldots, x^n), \quad 1 \leq i \leq m.$$ 

Then the differential $dF$ at each point $x \in M_1$ may be represented in this coordinate system by the matrix

$$\left[ \frac{\partial y^i}{\partial x^j} \right].$$

The $dx^j$’s are forms on $M_1$, the $dy^i$’s are forms on $M_2$, and the pullback map $F^*$ acts on the $dy^i$’s by

$$F^*(dy^i) = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j} dx^j.$$ 

The pullback map behaves as nicely as one could hope with respect to the various operations on differential forms, as described in the following theorem.

**Theorem:** Let $F : M_1 \to M_2$ be a differentiable map, and let $\phi, \eta$ be differential forms on $M_2$. Then

1. $F^*(\phi + \eta) = F^*\phi + F^*\eta$.
2. $F^*(\phi \wedge \eta) = F^*\phi \wedge F^*\eta$.
3. $F^*(d\phi) = d(F^*\phi)$.

5. The Lie derivative

The final operation that we will define on differential forms is the Lie derivative. This is a generalization of the notion of directional derivative of a function.
Suppose that $v(x)$ is a vector field on a manifold $M$, and let $\varphi : M \times (-\varepsilon, \varepsilon) \to M$ be the flow of $v$. This is the unique map that satisfies the conditions
\[
\frac{\partial \varphi}{\partial t}(x, t) = v(\varphi(x, t)) \quad \text{and} \quad \varphi(x, 0) = x.
\]
In other words, $\varphi_t(x) = \varphi(x, t)$ is the point reached at time $t$ by flowing along the vector field $v(x)$ starting from the point $x$ at time 0.

Recall that if $f : M \to \mathbb{R}$ is a smooth function, then the directional derivative of $f$ at $x$ in the direction of $v$ is
\[
v(f) = \lim_{t \to 0} \frac{f(\varphi_t(x)) - f(x)}{t} = \lim_{t \to 0} \frac{(\varphi_t^\star(f) - f)(x)}{t}.
\]
Similarly, given a differential form $\omega$ we define the Lie derivative of $\omega$ along the vector field $v(x)$ to be
\[
\mathcal{L}_v \omega = \lim_{t \to 0} \frac{\varphi_t^\star \omega - \omega}{t}.
\]
Fortunately there is a practical way to compute the Lie derivative. First we need the notion of the left-hook of a differential form with a vector field. Given a $p$-form $\omega$ and a vector field $v$, the left-hook $v \lhd \omega$ of $\omega$ with $v$ (also called the interior product of $\omega$ with $v$) is the $(p - 1)$-form defined by the property that for any $w_1, \ldots, w_{p-1} \in T_x \mathbb{R}^n$,
\[
(v \lhd \omega)(w_1, \ldots, w_{p-1}) = \omega(v, w_1, \ldots, w_{p-1}).
\]
For instance,
\[
\frac{\partial}{\partial x} \lhd (dx \wedge dy + dz \wedge dx) = dy - dz.
\]
Now according to Cartan’s formula, the Lie derivative of $\omega$ along the vector field $v$ is
\[
\mathcal{L}_v \omega = v \lhd d\omega + d(v \lhd \omega).
\]

Exercises

1. Classical vector analysis avoids the use of differential forms on $\mathbb{R}^3$ by converting 1-forms and 2-forms into vector fields by means of the following one-to-one correspondences. ($\varepsilon_1, \varepsilon_2, \varepsilon_3$ will denote the standard basis $\varepsilon_1 = [1, 0, 0], \varepsilon_2 = [0, 1, 0], \varepsilon_3 = [0, 0, 1]$.)
\[
\begin{aligned}
f_1 \, dx^1 + f_2 \, dx^2 + f_3 \, dx^3 & \leftrightarrow f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3 \\
f_1 \, dx^2 \wedge dx^3 + f_2 \, dx^3 \wedge dx^1 + f_3 \, dx^1 \wedge dx^2 & \leftrightarrow f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3 
\end{aligned}
\]

Vector analysis uses three basic operations based on partial differentiation:

1. **Gradient** of a function \( f \):
   \[
   \text{grad}(f) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} \varepsilon_i
   \]

2. **Curl** of a vector field \( v = \sum_{i=1}^{3} v^i(x) \varepsilon_i \):
   \[
   \text{curl}(v) = \left( \frac{\partial v^3}{\partial x^1} - \frac{\partial v^1}{\partial x^3} \right) \varepsilon_1 + \left( \frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1} \right) \varepsilon_2 + \left( \frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2} \right) \varepsilon_3
   \]

3. **Divergence** of a vector field \( v = \sum_{i=1}^{3} v^i(x) \varepsilon_i \):
   \[
   \text{div}(v) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i}
   \]

Prove that all three operations may be expressed in terms of exterior derivatives as follows:

1. \( df \leftrightarrow \text{grad}(f) \)
2. If \( \phi \) is a 1-form and \( \phi \leftrightarrow v \), then \( d\phi \leftrightarrow \text{curl}(v) \).
3. If \( \eta \) is a 2-form and \( \eta \leftrightarrow v \), then \( d\eta \leftrightarrow \text{div}(v) \, dx^1 \wedge dx^2 \wedge dx^3 \).

Show that the identities
   \[
   \text{curl(\text{grad}(f))} = 0 \]
   \[
   \text{div(\text{curl}(v))} = 0
   \]
   follow from the fact that \( d^2 = 0 \).

2. Let \( f \) and \( g \) be real-valued functions on \( \mathbb{R}^2 \). Prove that
   \[
   df \wedge dg = \left| \begin{array}{cc}
   \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
   \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
   \end{array} \right| \, dx \wedge dy.
   \]
   (You may recognize this from the change-of-variables formula for double integrals.)

3. Suppose that \( \phi, \psi \) are 1-forms on \( \mathbb{R}^n \). Prove the Leibniz rule
   \[
   d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi.
   \]
4. Prove the statement above that if $F : M_1 \rightarrow M_2$ is described in terms of local coordinates by

$$y^i = y^i(x^1, \ldots, x^n), \quad 1 \leq i \leq m$$

then

$$F^*(dy^i) = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j} dx^j.$$ 

5. Let $(r, \theta)$ be coordinates on $\mathbb{R}^2$ and $(x, y, z)$ coordinates on $\mathbb{R}^3$. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$F(r, \theta) = (\cos \theta, \sin \theta, r).$$

Describe the differential $dF$ in terms of these coordinates and compute the pullbacks $F^*(dx), F^*(dy), F^*(dz)$. 