Discontinuities in the identical satisfaction of equations

by

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Abstract. For a metric space (A, d), and a set Σ of equations, some quantities are introduced that measure the size of discontinuities that must occur in operations satisfying Σ (identically) on A. We are able to evaluate these quantities in a few easy cases.

Contents

0 Introduction.

This paper is part of a continuing investigation—see the author's papers [32] (1986), [34] (2000), and [35] (2006)—into the *compatibility* relation (see (2) below) between a topological space A and a set Σ of equations, which we will briefly review in §0.1.

The main results so far are Theorems 4, 5, 6, 7 and 28, in §2.3, §2.4, §2.5, §2.6 and §3.8. They showcase a new quantity that measures the incompatibility—see §0.1—of a metric space A with an equational theory Σ. This quantity will be denoted $\mu(A, \Sigma)$—see §0.3. In the first four of the aforementioned theorems we are able to calculate some non-trivial μ-values, with $A = S^1$, the one-dimensional sphere, and Σ taken, for example, as the well-known ternary majority laws. Then Theorem 28 deals with lattice theory on a Y-shaped space. It invokes some more sophisticated measures, μ_n and $\mu_n^\star$, which measures jumps in n-fold iterates of the operations.

These calculations indicate the feasibility of studying this μ, and show that the quantity may have some independent interest. At the same time, we seek a broader applicability of the concept and associated methods. Thus we will for a while keep the project open, both for the ultimate form of this paper, and for future writings.

At the start of §0.3 we contrast this μ with an earlier measure [36] of incompatibility known as λ.

0.1 Compatibility—context and background.

In this context, Σ typically denotes a set (finite or infinite) of equations[1], which are understood as universally quantified. We usually expect that Σ has a specified similarity type. This means that we are given a set T and whole numbers $n_t \geq 0$ $(t \in T)$, that for each $t \in T$ there is an operation symbol[2] F_t of arity $n(t)$, and that the operation symbols of Σ are included among these F_t.

Given a set A and for each $t \in T$ a function $\overline{F_t} : A^{n(t)} \longrightarrow A$ (called an operation), we say that the operations $\overline{F_t}$ *satisfy* Σ and write

$$(A, \overline{F}_t)_{t \in T} \;\models\; \Sigma, \tag{1}$$

if for each equation $\sigma \approx \tau$ in Σ, both σ and τ evaluate to the same function when the operations $\overline{F_t}$ are substituted for the symbols F_t appearing in σ and τ. Given a *topological space* A and a set of equations Σ, we write

$$A \;\models\; \Sigma, \tag{2}$$

and say that A and Σ are *compatible*, iff there exist *continuous* operations $\overline{F_t}$ on A satisfying Σ.

While the definitions are simple, the relation (2) remains mysterious. The algebraic topologists long knew that the n-dimensional sphere S^n is compatible with H-space theory ($x \cdot e \;\approx\; x \;\approx\; e \cdot x$) if and only if $n = 1, 3$ or 7. For $A = \mathbb{R}$, the relation (2) is algorithmically undecidable for Σ [35]; i.e. there is no algorithm that inputs an arbitrary finite Σ and outputs the truth value of (2) for $A = \mathbb{R}$. In any case, (2) appears to hold only sporadically, and with no readily discernable pattern.

The mathematical literature contains many scattered examples of the truth or falsity of specific instances of (2). The author's earlier papers [32], [34], [35], [36] collectively refer to most of what is known, and in fact many of the earlier examples are recapitulated throughout the long article [36]. We will therefore not attempt to write a list of examples for this introduction.

[1] A (formal) equation is an ordered pair of terms (σ, τ), more frequently written $\sigma \approx \tau$. As such it makes no assertion, but merely presents two terms for consideration. The actual mathematical assertion is made by the satisfaction relation $\models$.

[2] In examples we may sometimes give the operation symbols familiar names like $+$ or $\wedge$, or use F, G, etc. without a subscript. All of these variations may be thought of as colloquial expressions for the more formal F_t.

0.2 Metric approximation to compatibility.

If the topological space A is metrizable, and if a metric d is selected for A, then in addition to the modeling relation $\models$, one may also study some real-valued measures of approximate satisfaction. In our previous paper [36] we wrote

$$A \models_\varepsilon \Sigma \tag{3}$$

(for real $\varepsilon > 0$) to mean that there exist continuous operations $\overline{F}_t$ on A such that, for each equation $\sigma \approx \tau$ in Σ, the terms σ and τ evaluate to functions $\overline{\sigma}$ and $\overline{\tau}$ that are within ε of each other. We then defined $\lambda_A(\Sigma)$ to be *the smallest non-negative*[3] *real such that* $A \models_\varepsilon \Sigma$ *for every* $\varepsilon > \lambda_A(\Sigma)$.

As one might imagine, the precise value of $\lambda_A(\Sigma)$ depends strongly on the metric d chosen to represent the topology of A; moreover, its value can increase if Σ is augmented by the inclusion of some of its own logical consequences. The earlier article [36] illustrates these points with detailed estimations of $\lambda_A(\Sigma)$ for many different A and Σ.

0.3 Measuring continuity-failure in models of Σ.

In looking to have $A \models \Sigma$ for a space A and a set Σ of equations, we demand both the continuity of the operations $\overline{F}_t$ of $(A, \overline{F}_t)_{t \in T}$, and the exact satisfaction of Σ by these operation. The outlook reviewed in §0.2 was to relax the need for exact satisfaction, and to see how close we can come with approximate satisfaction.

In this paper we examine a different—opposite, really—way of relaxing our requirements. Namely, we require exact satisfaction while measuring how far our operations must deviate from continuity.

Let B be a topological space, and (A, d) a metric space. Let us consider a function $\overline{F} \colon B \longrightarrow A$. If $\overline{F}$ is continuous at $b \in B$, then for each $\varepsilon > 0$ there is a neighborhood U of b with $\overline{F}[U] \subseteq B(\overline{F}(b), \varepsilon/2)$ (the open ball about $\overline{F}(b)$ with radius $\varepsilon/2$). Consequently, if $\overline{F}$ is continuous at $b \in B$, then for each $\varepsilon > 0$ there is a neighborhood U of b such that $\overline{F}[U]$ has diameter $< \varepsilon$, in other words

$$\inf \{ \text{diameter}\, \overline{F}[U] \,:\, U \text{ open},\ b \in U \} \;=\; 0. \tag{4}$$

[3] If there is any real number satisfying this condition, then there is a smallest one, by completeness. If there is no such real number, then we let $\lambda_A(\Sigma) = \infty$.

If $\overline{F}$ is not continuous at b we may still define the quantity

$$\chi(\overline{F}, b) \;=\; \inf\{\, \text{diameter}\, \overline{F}[U] \;:\; U \text{ open},\; b \in U \,\}, \tag{5}$$

which we will call the *jump of* $\overline{F}$ *at* b, as a real number (or ∞), and take this quantity as a measure of failure of continuity at b. The following lemma is almost immediate.

Lemma 1 *$\overline{F}\colon B \longrightarrow A$ is continuous at $b \in B$ iff $\chi(\overline{F}, b) = 0$.*

Proof. We already saw that if $\overline{F}$ is continuous at b, then $\chi(\overline{F}, b) = 0$. For the converse, let us be given $\chi(\overline{F}, b) = 0$ and prove continuity at b. Suppose we are given $\varepsilon > 0$. The infimum appearing in (4) and (5) is zero, which means in particular that there is an open $U \subseteq B$ with $b \in U$ and such that $\overline{F}[U]$ has diameter $< \varepsilon$. In other words, $d(f(u), f(b)) < \varepsilon$ for all $u \in U$. Thus $\overline{F}$ is continuous at b. ∎

Then $\chi(\overline{F})$, the jump of $\overline{F}$ is defined to be the supremum of $\chi(\overline{F}, b)$, for b ranging over B. (This supremum is either a non-negative real number or ∞. Its value obviously depends on the choice of metric on A.) Clearly $\chi(\overline{F}) = 0$ if and only if $\overline{F}$ is continuous.

It will be of interest to have an analog of $\chi(\overline{F}, b)$ for uniform continuity. If $\overline{F}\colon B \longrightarrow A$ is uniformly continuous, then for each real $\varepsilon > 0$ there exists real $\delta > 0$ such that if U is any δ-ball in B, then $\overline{F}[U]$ has diameter $\leq \varepsilon$. We define

$$\chi_u(\overline{F}) \;=\; \inf_{\delta > 0} \sup_{b \in B} \text{ diameter } \overline{F}[B_\delta(b)], \tag{6}$$

where the subscript u stands for "uniform," and where $B_\delta(b)$ stands for the δ-ball in B centered at b. This quantity may be called the *uniform jump of* $\overline{F}$ *on* B. By analogy with Lemma 1, we have that $\overline{F}$ is uniformly continuous on B iff $\chi_u(\overline{F}) = 0$ (formal statement and proof omitted). The following lemma is a slight extension of the well-known equivalence, for compact spaces, of continuity with uniform continuity. (It may also be well known.)

Lemma 2 *$\chi(\overline{F}) \leq \chi_u(\overline{F})$, with equality holding for compact spaces.*

Proof. Removing the b-supremum from (6) we immediately have, for each $b \in B$,

$$\begin{aligned} \chi_u(\overline{F}) \;&\geq\; \inf_{\delta>0} \text{diameter } \overline{F}[B_\delta(b)] \\ &=\; \inf\{\, \text{diameter}\, \overline{F}[U] \,:\, U \text{ open},\; b \in U \,\} \;=\; \chi(\overline{F}, b). \end{aligned}$$

It follows immediately that

$$\chi_u(\overline{F}) \;\geq\; \sup_{b\in B} \chi(\overline{F}, b) \;=\; \chi(\overline{F}).$$

We now consider the opposite inequality $\chi_u(\overline{F}) \leq \chi(\overline{F}) = \sup_{b\in B} \chi(\overline{F}, b)$. If the supremum on the right is infinite, the result is immediate. So we will assume that it is finite. Let us consider arbitrary real $K > \sup_{b\in B} \chi(\overline{F}, b)$; it will suffice to prove that $K \geq \chi_u(\overline{F})$. We are given that $K > \chi(\overline{F}, b)$ for every $b \in B$. Thus by the definition (5), for every $b \in B$ there exists an open set U_b with $b \in U$ and such that diameter $\overline{F}[U_b] < K$.

The sets U_b form an open cover of A; by compactness there is a Lebesgue number for this covering. In other words, there exists $\delta > 0$ such that each δ-ball $B_\delta(b)$ is a subset of $U_{b'}$ for some $b' \in B$. Therefore diameter $\overline{F}[B_\delta(b)]$ $< K$ for each $b \in B$. Therefore, for this one value of δ,

$$\sup_{b\in B} \text{ diameter } \overline{F}[B_\delta(b)] \;\leq\; K.$$

So finally we have

$$\chi_u(\overline{F}) \;=\; \inf_{\delta>0} \sup_{b\in B} \text{ diameter } \overline{F}[B_\delta(b)] \;\leq\; K,$$

the desired inequality. ∎

Let (A, d) be a metric space, and $\mathbf{A} = (A, \overline{F}_t)_{t\in T}$ an algebra based on A. We define

$$\chi(\mathbf{A}, d) \;=\; \sup_{t\in T} \chi(\overline{F}_t).$$

When the metric d is clear from the context, we may write $\chi(\mathbf{A})$ for $\chi(\mathbf{A}, d)$. It should be clear that $\chi(\mathbf{A}, d) = 0$ if and only if $\mathbf{A}$ is a topological algebra.

Finally, for (A, d) a metric space, and Σ a set of equations of similarity type $\langle n_t \,:\, t \in T\rangle$, we define

$$\mu(A, d, \Sigma) \;=\; \inf\{\, \chi(\mathbf{A}, d) \,:\, \mathbf{A} = (A, \overline{F}_t)_{t\in T} \models \Sigma \,\}; \tag{7}$$

in other words, it is the infimum taken over all algebras built on A that satisfy Σ. When the metric d is clear from the context, we may write $\mu(A, \Sigma)$ for $\mu(A, d, \Sigma)$.

All these notions have uniform versions, denoted with a subscript u, based on χ_u in place of χ. Most of this paper deals with compact metric spaces, on which the two concepts coincide. In some cases (for example see §§2.3–2.5) it turns out to be easier to prove an estimate for χ_u than for χ. One should bear in mind that, even when A has been given a metric, the value of $\chi_u(\mathbf{A})$ still depends on the metric that is chosen to represent the topology on the finite powers A^{n_t}. By Lemma 2, however, there is no dependence in the compact case. We will mostly work in the compact case.

In this paper we shall apply (7) only in situations where (i) A is infinite, (ii) Σ is finite or countable, and (iii) Σ defines a consistent equational theory (i.e. it has a model of more than one element). In these circumstanes, there is at least one model of Σ based on A; in other words, the infimum appearing in (7) is over a non-empty set. If one or more of (i–iii) should fail, then it is possible for the infimum of (7) to be over the empty set. In that case, we would naturally define $\mu(A, d, \Sigma)$ to be ∞.

This quantity $\mu(A, d, \Sigma)$ will be the main object of study in this paper. Like the previously studied $\lambda_A(\Sigma)$—see §0.2— $\mu(A, d, \Sigma)$ measures deviation from $A \models \Sigma$, as follows: in every model of Σ based on A, some operation has a discontinuity at least as large as $\mu(A, d, \Sigma)$. (ONLY APPROXIMATELY TRUE)

It should be apparent that if $A \models \Sigma$, then $\mu(A, d, \Sigma) = 0$. The converse is false, as we shall see. Under the right circumstances, there are some connections between the values of $\lambda_A(\Sigma)$ and $\mu(A, \Sigma)$. See e.g. Corollaries 20 and 21 of §3.3.

1 Elementary remarks about $\mu(A, d, \Sigma)$.

I do not yet have a clear idea of the full scope of §1. Maybe it won't really be necessary, but for the moment I will file remarks here as I think of them.

1.1 $\Sigma \subseteq \Sigma'$.

It is almost obvious that if $\Sigma \subseteq \Sigma'$ then $\mu(A, d, \Sigma) \leq \mu(A, d, \Sigma')$. (This is simply because, in evaluating the infimum in (7), the set of algebras for Σ'

is a subset of the set of algebras for Σ.)

It is also immediate from (7) that if Σ' is any collection of logical consequences of Σ, then $\mu(A, d, \Sigma) \geq \mu(A, d, \Sigma')$. Thus, if Σ' includes all of Σ together with any subset of the consequences of Σ, then $\mu(A, d, \Sigma) = \mu(A, d, \Sigma')$.

It thus follows that, unlike $\lambda_A(\Sigma)$ (see §0.2), $\mu(A, d, \Sigma)$ is a logical invariant of Σ.

1.2 Topological products.

To come.

1.3 Products of theories.

To come.

2 Some sample values of $\mu(A, d, \Sigma)$.

2.1 An injective binary operation: $\mu(A, d, \Sigma) = 0$.

Consider Σ consisting of the two equations

$$F_0(G(x_0, x_1)) \approx x_0, \qquad F_1(G(x_0, x_1)) \approx x_1. \tag{8}$$

They imply, among other things, that in any topological model $\overline{G}$ must be a one-one continuous binary operation. Euclidean spaces of non-zero finite dimension do not have such operations, hence are not compatible with Σ. In §2.1 we will be concerned with $A = [0, 1]$. Although this A is not compatible with Σ, we shall show that $\mu(A, \delta, \Sigma) = 0$ (where d is the ordinary Euclidean metric).

To begin we let $\overline{P}$ be a continuous function mapping $A = [0, 1]$ onto A^2. (Such an area-filling curves was devised by G. Peano in 1890—see [33, pp. 116–7], or many other sources.) By the Axiom of Choice, $\overline{P}$ has a (discontinuous) inverse $\overline{H}$; in other words we have

$$A^2 \xrightarrow{\overline{H}} A \xrightarrow{\overline{P}} A^2$$

with $\overline{P} \circ \overline{H}$ the identity function on A^2. Let $\lambda = \chi(\overline{H})$. We remarked above that $\overline{H}$ cannot be continuous; hence $0 < \lambda \leq 1$.

Now to establish our claim that $\mu(A, \Sigma) = 0$, we must prove that the infimum appearing in (7) is zero. It will suffice, given $\varepsilon > 0$, to exhibit an algebra $\mathbf{A}$ based on A with $\chi(\mathbf{A}) \leq \varepsilon$, and such that $A \models \Sigma$.

Our algebra $\mathbf{A}$ is as follows. Its binary operation $\overline{G}$ is defined via

$$\overline{G}(a_0, a_1) \;=\; \varepsilon\, \overline{H}(a_0, a_1).$$

The unary operations $\overline{F}_0$ and $\overline{F}_1$ are defined to be the two components of the function $\overline{F}\colon A \longrightarrow A^2$ that is defined via

$$\overline{F}(a) \;=\; \overline{P}(1 \wedge (a/\varepsilon)),$$

where $\wedge$ denotes the smaller of two real numbers. Clearly $\overline{F}$ is well-defined and continuous; hence the same is true of $\overline{F}_0$ and $\overline{F}_1$.

To verify Σ for the operations, we first calculate

$$\begin{aligned} \overline{F}(\overline{G}(a_0, a_1)) &= \overline{P}(1 \wedge [\varepsilon \overline{H}(a_0, a_1)/\varepsilon]) \\ &= \overline{P}(1 \wedge [\overline{H}(a_0, a_1)]) \;=\; \overline{P}(\overline{H}(a_0, a_1)) \;=\; (a_0, a_1), \end{aligned}$$

with the final equation true by our choice of $\overline{H}$. We now have $\overline{F} \circ \overline{G} = 1$, which is tantamount to the equations Σ.

As for $\chi(\mathbf{A})$, we first note that $\chi(\overline{G}) \leq \varepsilon\lambda \leq \varepsilon$. By continuity, $\chi(\overline{F}_0) = \chi(\overline{F}_1) = 0$. Thus $\chi(\mathbf{A}) \leq \varepsilon$. This completes the description of this example.

2.2 Σ non-Abelian and simple; $A = S^1$

Following [36, §3.2.3], we define a set Σ of equations to be *Abelian* iff it is interpretable (in the sense of [14]) in the equational theory of Abelian groups. Equivalently, Σ is Abelian if and only if it has a model based on $\mathbb{Z}$ with operations of the form

$$\overline{F}(x_1, \cdots, x_n) \;=\; m_1 x_1 + \cdots + m_n x_n, \tag{9}$$

where each $m_i \in \mathbb{Z}$.

It was proved in Theorem 41 on page 234 of [34] that Σ is Abelian iff Σ is compatible with S^1, and then in §3.2.3 of [36] that Σ is Abelian iff $\lambda_{S^1}(\Sigma) = 0$. In fact, if S^1 is given its natural metric as a circle embedded in the Euclidean space $\mathbb{R}^2$, scaled to diameter 1, then we have

$$\lambda_{S^1}(\Sigma) \;=\; \begin{cases} 0 & \text{if } \Sigma \text{ is Abelian} \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

The first assertion of (10) obviously holds for $\mu(S^1, \Sigma)$—namely that $\mu(S^1, \Sigma) = 0$ if Σ is Abelian—but the corresponding second assertion is false.

By a *simple term* in the language defined by the operation symbols F_t of §0.1, we mean[4] a term that contains at most one F_t, and moreover contains at most one instance of that F_t. In other words, according to the usual recursive definition of terms, a simple term is either a variable or created at the first stage beyond the inclusion of variables. An equation $\sigma \approx \tau$ is *simple* iff both σ and τ are simple terms.

We cannot prove anything like (10) for μ in place of λ. However, for some simple non-Abelian theories Σ, we can prove the surprisingly exact result that $\mu(S^1, \Sigma) = 2/3$. See §§2.3–2.5.

2.3 $\Sigma =$ multiplication with zero and one; $A = S^1$.

Throughout §2.3 we let Σ be the theory of a single binary operation with a zero and a one. (Specifically a left zero and a left one.) Specifically, Σ is given by these equations:

$$F(0, x) \approx 0; \qquad F(1, x) \approx x. \tag{11}$$

Let us represent the one-sphere S^1 as a circle of circumference 2 (hence radius $1/\pi$). We then give it the metric of arc length: $d(P, Q)$ is the length of the shorter of the two circular arcs joining A and B. In this metric, the space has diameter 1. By an *arc* in this space we mean the smaller of two circular arcs joining two points, considered as a closed subset of S^1. We shall prove that, in this metric, $\mu(S^1, \Sigma) = 2/3$.

Lemma 3 *If F is a finite subset of S^1 with diameter$(F) < 2/3$, then there is an arc A of S^1, of length $=$ diameter(F), such that $F \subseteq A$.*

Proof. Let λ denote diameter(F). Choose $P, Q \in F$ such that $d(P, Q) = \lambda$, and define A to be the arc $\overline{PQ}$. Clearly A has length λ. To prove that $F \subseteq A$, we consider three intervals of length λ. The first is A itself; the second is B which meets A only in the point P; the third is C which meets A only at the point Q. Since $\lambda < 2/3$, the intervals B and C are disjoint. Now every member of $F$is within λ of P, hence belongs either to A or to B. Similarly

[4] This terminology was used, perhaps for the first time, in García and Taylor [14], and then again by Taylor in [35].

every member of F belongs either to A or to C. Now suppose that $R \in F$ but $\notin A$. Then R must belong to both B and C, which is a contradiction; this contradiction completes the proof that $F \subseteq A$. ∎

Remarks. The proof can easily be extended to infinite F, although we will not need this refinement. The number 2/3 is sharp for this lemma, as follows. Consider F whose members are three points equally spaced at distance 2/3 about the circle S^1. Clearly no arc contains F, but diameter$(F) = 2/3$.

Theorem 4 $\mu(S^1, \Sigma) = 2/3$. *(With Σ as defined in (11).)*

Proof. **Part 1.** $\mu(S^1, \Sigma) \leq 2/3$. We must exhibit an algebra $\mathbf{A} = \langle S^1, \overline{F}, \overline{0}, \overline{1}\rangle$ (with $\overline{F}$ binary) satisfying equations (11), and with $\chi(\overline{F}) \leq 2/3$. For convenience, let us take R to stand for $1/\pi$, the radius of our circle S^1. We now define the operations of $\mathbf{A}$ as follows: $\overline{1} = R$, $\overline{0} = -R$, and $\overline{F}$ is given by these formulas:

$$\overline{F}(R, z) \;=\; z; \qquad \overline{F}(-R, z) \;=\; -R; \tag{12}$$

$$\overline{F}(w \neq \pm R,\, Re^{i\theta}) \;=\; \begin{cases} R\,e^{i\pi/3} & \text{if } 0 \leq \theta < 2\pi/3, \\ -R & \text{if } 2\pi/3 \leq \theta < 4\pi/3, \\ R\,e^{5i\pi/3} & \text{if } 4\pi/3 \leq \theta < 6\pi/3. \end{cases} \tag{13}$$

The satisfaction of (11) is immediate from (12) of the definition.

To evaluate $\chi(\overline{F})$, we first consider $\chi(\overline{F}, (w, z))$, where $w \neq R$. It is obvious from (12–13) that (w, z) has a neighborhood U such that $\overline{F}[U] \subseteq R\{-1, e^{i\pi/3}, e^{5i\pi/3}\}$. This latter set has diameter 2/3, and so we may turn our attention to the case of $w = R$.

We begin the case of $w = R$ by remarking that $\overline{F}$ has a 3-fold symmetry, as follows: if $w \neq -R$, then $\overline{F}(w, e^{2\pi i/3}z) = e^{2\pi i/3}\overline{F}(w, z)$. Since multiplication by any unimodular complex scalar is a rotation, and does not change diameters, it will be enough to focus our attention on $\overline{F}(R, Re^{i\theta})$ where $0 \leq \theta < 2\pi i/3$. We first assume that $\theta \neq 0$. We may then consider a neighborhood $U_0 \times U_1$ of $(R, Re^{i\theta})$, where $-R \notin U_0$, and U_1 is a small arc about $Re^{i\theta}$ that lies interior to the arc $\overline{R\,Re^{2\pi i/3}}$. Then $\overline{F}[U_0 \times U_1]$ is $U_1 \cup \{Re^{\pi i/3}\}$. The reader may easily check that this set has diameter $< 2/3$.

It finally remains to consider $w = R$ and $\theta = 0$, which is to say, to evaluate $\chi(\overline{F}, (R, R))$. Things go exactly as before, except that a neighborhood $U_0 \times U_1$

of (R, R) will contain some points of the form $(v, Re^{i\theta})$ where $v \neq \pm R$ and $4\pi/3 < \theta < 2\pi$. Thus $\overline{F}[U_0 \times U_1]$ will be $U_1 \cup \{Re^{\pi i/3},\ Re^{5\pi i/3}\}$.

Part 2. $\mu(S^1, \Sigma) \geq 2/3$. For a proof by contradiction, let us assume that $\mu(S^1, \Sigma) < 2/3$. By (7), there is an algebra $\mathbf{A} = \langle S^1, \overline{F}, \overline{0}, \overline{1}\rangle$ such that $\chi(\overline{F}) < 2/3$ and such that $\mathbf{A} \models \Sigma$. Let us give $(S^1)^2$ the sum metric

$$d((a,b),\,(c,d)) \;=\; d(a,c) \,+\, d(b,d). \tag{14}$$

By Lemma 2, we also have $\chi_u(\overline{F}) < 2/3$. Referring to (6) (the definition of χ_u), we see that there exists $\delta > 0$ such that $d((a,b),(c,d)) < \delta$ implies $d(\overline{F}(a,b), \overline{F}(c,d)) < 2/3$. Let

$$t_0,\ t_1,\ \cdots\ t_{N-1},\ t_N \;=\; t_0$$

be points of S^1 such that

(a) The points t_i are evenly spaced around the circle, with $d(t_i, t_{i+1}) < \delta/2$ for all appropriate i. We will refer to the portion of S^1 between t_i and t_{i+1} as a *segment* of the circle.

(b) This sequence of points continues around the circle in the same direction, and goes around the circle exactly once.

(c) For convenience, we make sure that $\overline{0} = t_0$ and $\overline{1} = t_K$ for some K.

For the remainder of the proof we consider the restriction of $\overline{F}$ to the finite set $\{(t_i, t_j) \,:\, 0 \leq i \leq K,\ 0 \leq j < N\}$. From (a) and (14) and our choice of δ, we immediately have

$$\text{diameter}\,\{\overline{F}(t_i,t_j), \overline{F}(t_i,t_{j+1}), \overline{F}(t_{i+1},t_j), \overline{F}(t_{i+1},t_{j+1})\,\} \;<\; 2/3. \tag{15}$$

for $0 \leq i < K,\ 0 \leq j < N$. We will finish the proof by showing that the metric arrangement (15) and Equations (11) together lead to a contradiction.

Applying Lemma 3 to (15) we see that for $0 \leq i < K,\ 0 \leq j < N$ there is an arc A_{ij} of length $< 2/3$ such that

$$\overline{F}(t_i,t_j),\ \overline{F}(t_i,t_{j+1}),\ \overline{F}(t_{i+1},t_j),\ \overline{F}(t_{i+1},t_{j+1}) \;\in\; A_{ij}. \tag{16}$$

For $i = 0, \ldots, K$, let us take a continuous function $\overline{\gamma}_i\!: S^1 \longrightarrow S^1$, in such a way that the following five conditions are met:

(i) $\overline{\gamma}_i(t_j) = \overline{F}(t_i, t_j)$ and $\overline{\gamma}_i(t_{j+1}) = \overline{F}(t_i, t_{j+1})$;

(ii) For $0 \leq i < K$, $\overline{\gamma}_i$ maps the arc $\overline{t_j t_{j+1}}$ into the arc A_{ij}.

(iii) For $0 < i \leq K$, $\overline{\gamma}_i$ maps the arc $\overline{t_j t_{j+1}}$ into the arc $A_{(i-1)j}$.

(iv) $\overline{\gamma}_0$ is the constantly $\overline{0}$ function.

(v) $\overline{\gamma}_K$ is the identity function.

(Condition (i) can be met directly. For conditions (ii) and (iii), we use (16) to see that the endpoints t_j and t_{j+1} both map into the arc $A_{ij} \cap A_{(i-1)j}$; hence the arc between them can be mapped into $A_{ij} \cap A_{(i-1)j}$ (here we use the fact that a non-empty intersection of two arcs of diameter < 1 is itself an arc). For condition (iv), we recall that $\overline{0}$ is t_0; therefore the equations (11), together with condition (i), tell us that all the values $\overline{\gamma}_0(t_j)$ are $\overline{0}$. Therefore we easily satisfy conditions (ii) and (iii) by making γ_0 constantly equal to $\overline{0}$—which yields also (iv). Condition (v) is satisfied similarly.)

Now, for a contradiction, we will prove that $\overline{\gamma}_0$ is homotopic to $\overline{\gamma}_K$, in contradiction to conditions (iv) and (v). Using the transitivity of homotopy, it will be enough to prove that $\overline{\gamma}_i$ is homotopic to $\overline{\gamma}_{i+1}$ for $0 \leq i < K$. So we fix a value of i in this range, and proceed to define the required homotopy.

For $0 \leq j < N$, we define a continuous S^1-valued function $\overline{G}_j$, whose domain is the arc $\overline{t_i t_{i+1}}$ and which satisfies

(vi) $\overline{G}_j(t_i) = \overline{F}(t_i, t_j)$ and $\overline{G}_j(t_{i+1}) = \overline{F}(t_{i+1}, t_j)$.

(vii) $\overline{G}_j$ maps the arc $\overline{t_i t_{i+1}}$ into the arc A_{ij}.

(viii) $\overline{G}_{j+1}$ maps the arc $\overline{t_i t_{i+1}}$ into the arc A_{ij}.

(Again, these conditions are all possible by (16).)

We now consider the set $B_{ij} = \overline{t_i t_{i+1}} \times \overline{t_j t_{j+1}} \subseteq S^1 \times S^1$. We define a function $\overline{\phi}_{ij}$ from the boundary of B_{ij} to the arc A_{ij}, as follows:

$$\overline{\phi}_{ij}(s, t_j) = \overline{G}_j(s) \tag{17}$$
$$\overline{\phi}_{ij}(s, t_{j+1}) = \overline{G}_{j+1}(s) \tag{18}$$
$$\overline{\phi}_{ij}(t_i, t) = \overline{\gamma}_i(t) \tag{19}$$
$$\overline{\phi}_{ij}(t_{i+1}, t) = \overline{\gamma}_{i+1}(t) \tag{20}$$

(The reader may check, from what has come before, that Range $(\overline{\phi}_{ij}) \subseteq A_{ij}$, and that $\overline{\phi}_{ij}$ is well-defined, and hence continuous, at the corners of B_{ij}.)

Any continuous function from the boundary of a plane disk to the real line extends to a continuous function defined on the full disk. (This is Tietze's Extension Theorem.) Thus there exists a continuous function $\overline{\Phi}_{ij} : B_{ij} \longrightarrow A_{ij}$ that restricts to $\overline{\phi}_{ij}$ on the boundary.

We will now show that

$$\overline{\Phi}_i \;=\; \bigcup_{j=0}^{N-1} \overline{\Phi}_{ij}$$

is the desired homotopy between $\overline{\gamma}_i$ and $\overline{\gamma}_{i+1}$. Clearly its domain is $\bigcup_{j=0}^{N-1} B_{ij} = \overline{t_i t_{i+1}} \times S^1$, and by (17–18), for each j the component functions $\overline{\phi}_{ij}$ and $\overline{\phi}_{i(j+1)}$ agree where they overlap. Thus $\overline{\Phi}_i$ is a continuous function defined on $\overline{t_i t_{i+1}} \times S^1$. Finally, from (19–20) it follows that, for all $t \in S^1$, we have $\overline{\Phi}_i(t_i, t) = \gamma_i(t)$ and $\overline{\Phi}_i(t_{i+1}, t) = \gamma_{i+1}(t)$. Thus $\overline{\Phi}_i$ is the desired homotopy. As mentioned above, transitivity yields a homotopy between the identity and a constant function. This contradiction to known results completes the proof of the theorem. ▮

Remark on the proof. In fact, what we have done here is—for a contradiction—to begin with a solution $\overline{F}$ to the equations Σ, such that $\overline{F}$ is discontinuous, but by no more than 2/3. We have then focused on a finite subset of $\overline{F}$ (comprising the function-values $\overline{F}(t_i, t_j)$). Finally we have interpolated a continuous function $\overline{G}$ through these values that also satisfies Σ. Since no such $\overline{G}$ exists, we have our contradiction. In §2.4 and §2.5 we will see this method to be widely applicable. See also §2.5.1.

2.4 $\Sigma =$ commutative idempotent binary; $A = S^1$

In §2.4 we consider the following Σ, which defines commutative idempotent binary operations:

$$F(x, y) \;\approx\; F(y, x); \qquad F(x, x) \;\approx\; x. \tag{21}$$

In §3.4.1 of [36] we remarked that this Σ is non-Abelian, hence not compatible with S^1. Since it is also simple, its μ-value is amenable to estimation. By a method similar to that of §2.3 (again using Lemma 3) we will prove

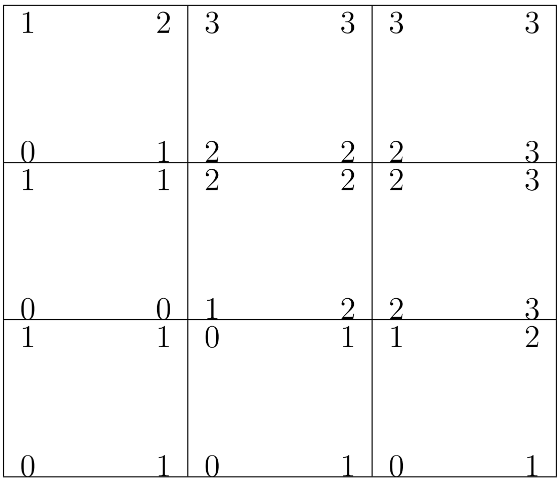
Theorem 5 $\mu(S^1, \Sigma) \;=\; 2/3$. *(With Σ as defined in (21).)*

Proof.

Part 1. $\mu(S^1, \Sigma) \le 2/3$. We must exhibit an algebra $\mathbf{A} = \langle S^1, \overline{F}\rangle$ (with $\overline{F}$ binary) satisfying equations (21), and with $\chi(\overline{F}) \le 2/3$. To avoid fractions, we will represent elements of our circle as real numbers modulo 3, and will assume that these numbers parametrize the distance. In this reframing, the circle has diameter $3/2$, and so we expect to prove that $\chi(\overline{F}) \le 1$. We define the operation $\overline{F}$ of $\mathbf{A}$ as follows:

$$\overline{F}(s,t) \;=\; \overline{F}(t,s) \;=\; \begin{cases} s \vee t & \text{if } 0 \le s,t \le 1 \\ s \vee t & \text{if } 1 \le s,t \le 3 \\ 1\,+\,s\,+\,t & \text{if } 0 \le s \le 1 \text{ and } 2 \le t \le 3 \\ 2 + t & \text{if } 0 \le s \le 1 \text{ and } 1 \le t \le 2, \end{cases} \tag{22}$$

where of course the addition is taken modulo 3. It is obvious that this $\overline{F}$ satisfies the Σ in (21). In order to estimate $\chi(\overline{F})$ we consider the following diagram:



(23)

This illustration depicts $[0,3] \times [0,3]$, divided into nine squares of dimensions 1×1. If we consider the diagram modulo 3 in each direction, then we have our version of the torus $S^1 \times S^1$. As the reader may check—case by case—the four values shown in each small square are the corner $\overline{F}$-values for that square, as supplied by our definition (22). Moreover, on each small edge, the $\overline{F}$-values (considered not modulo 3, but as reals in $[0, 3]$) vary linearly between the indicated corner values.

It is now not hard to observe—again, case by case—that no jump is greater than 1 in the limit. The most serious case occurs at the upper-right corner, call it P, of the upper-left small square: the values at P are 0, 1, 2, 3 = 0. Given $\varepsilon > 0$, there is a neighborhood U of P such that $\overline{F}[U] \subseteq [-\varepsilon, \varepsilon] \cup [1 - \varepsilon, 1 + \varepsilon] \cup [2 - \varepsilon, 2 + \varepsilon]$. This last set has diameter $1 + 2\varepsilon$. Then $\chi(\overline{F}, P)$ is the relevant infimum, which clearly is 1. We have now established the required properties of $\mathbf{A} = \langle S^1, \overline{F} \rangle$, and hence the proof of Part 1 is complete.[5]

Part 2. $\mu(S^1, \Sigma) \geq 2/3$ For a proof by contradiction, we assume that $\mu(S^1, \Sigma) < 2/3$. By (7), there is an algebra $\mathbf{A} = \langle S^1, \overline{F} \rangle$ such that $\chi(\overline{F}) < 2/3$ and such that $\mathbf{A} \models \Sigma$. As in the proof of Theorem 4, we give $(S^1)^2$ the sum metric. As before, there exists $\delta > 0$ such that $d((a, b), (c, d)) < \delta$ implies $d(\overline{F}(a, b), \overline{F}(c, d)) < 2/3$. Let

$$t_0,\ t_1,\ \cdots\ t_{N-1},\ t_N\ =\ t_0$$

be points of S^1 satisfying (a–c) in the proof of Theorem 4. As before, we have[6]

$$\text{diameter}\ \{\overline{F}(t_i, t_j), \overline{F}(t_i, t_{j+1}), \overline{F}(t_{i+1}, t_j), \overline{F}(t_{i+1}, t_{j+1})\ \}\ <\ 2/3. \tag{24}$$

for $0 \leq i, j < N$. The nearness relations (24) will make it possible to define a continuous binary operation $\overline{G}$ that interpolates the N^2 discrete function values $\overline{F}(t_i, t_j)$ $(0 \leq i, j < N)$. Using the fact that these values obey (21) we will be able to make sure that the interpolated operation $\overline{G}$ also obeys (21). Thus we will have $\langle S^1, \overline{G} \rangle \models \Sigma$ with $\overline{G}$ continuous, in contradiction to the known fact [36, §3.4.1] that $S^1 \not\models \Sigma$; this contradiction will complete the proof of the theorem.

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[5] Actually, formulas (22) are not especially relevant or important to the proof. The important thing is the sudoku-like puzzle of finding Diagram (23): the values shown must illustrate idempotence, commutativity and small jumps. From there one can easily contrive a function like our $\overline{F}$.

[6] The "+1" appearing in subscripts in (24–25), and elsewhere, is of course to be understood modulo N.

Applying Lemma 3 to (24) we see that for $0 \leq i, j < N$ there is an arc A_{ij} of length $< 2/3$ such that

$$\overline{F}(t_i, t_j),\, \overline{F}(t_i, t_{j+1}),\, \overline{F}(t_{i+1}, t_j),\, \overline{F}(t_{i+1}, t_{j+1}) \;\in\; A_{ij}. \tag{25}$$

From (25) we easily derive, for $0 \leq i, j < N$, that

$$\overline{F}(t_i, t_j),\, \overline{F}(t_i, t_{j+1}) \;\in\; A_{ij} \,\cap\, A_{(i-1)j} \tag{26}$$

$$\overline{F}(t_i, t_j),\, \overline{F}(t_{i+1}, t_j) \;\in\; A_{ij} \,\cap\, A_{i(j-1)}. \tag{27}$$

From (25) and Equations (21) we also have

$$t_i \;\in\; A_{ij} \cap A_{i(j-1)} \cap A_{(i-1)j} \cap A_{(i-1)(j-1)} \tag{28}$$

for all i. By symmetry (21), we have $\overline{F}(t_i, t_j) \;=\; \overline{F}(t_j, t_i)$ for all i and j; hence we may further require that

$$A_{ij} \;=\; A_{ji} \tag{29}$$

for all appropriate i and j. Moreover, since each A_{ij} is an arc of length $\leq 2/3$, the right-hand sides of (26), (27) and (28) are themselves arcs of S^1.

Turning to the definition of $\overline{G}$, we begin with what may be called the coordinate circles, (s, t_j) and (t_i, t) for $s, t \in S^1$ and $0 \leq i, j < N$. From (26) and (27) it is clear that for these domain values we may now define a continuous binary operation $\overline{G}$ satisfying the following conditions for all $s, t \in S^1$ and $0 \leq i, j < N$:

(i) $\overline{G}(t_i, t_j) \;=\; \overline{F}(t_i, t_j)$;

(ii) $\overline{G}(s, t_j) \in \; A_{ij} \,\cap\, A_{i(j-1)}$, for s in the arc $\overline{t_i\, t_{i+1}}$;

(iii) $\overline{G}(t_i, t) \in \; A_{ij} \,\cap\, A_{(i-1)j}$, for t in the arc $\overline{t_j\, t_{j+1}}$.

By the symmetry that we already have, e.g. (29), we may further require

(iv) $\overline{G}(s, t_j) \;=\; \overline{G}(t_j, s)$

for all appropriate j and s.

For each i, j we have defined $\overline{G}$ on the boundary of the rectangle $\overline{t_i\, t_{i+1}} \times \overline{t_j\, t_{j+1}}$, which consists of the following four arcs:

$$\{t_i\} \times \overline{t_j\, t_{j+1}}\,, \quad \overline{t_i\, t_{i+1}} \times \{t_{j+1}\}\,, \quad \{t_{i+1}\} \times \overline{t_j\, t_{j+1}}\,, \quad \overline{t_i\, t_{i+1}} \times \{t_j\}\,. \tag{30}$$

By (ii) and (iii), our partial operation $\overline{G}$ maps each of these four arcs into A_{ij}. In other words $\overline{G}$ maps the boundary of the rectangle $\overline{t_i\, t_{i+1}} \times \overline{t_j\, t_{j+1}}$ into the topological interval A_{ij}. As is well known, $\overline{G}$ may be extended to a continuous function on the entire rectangle:

$$\overline{G}_{ij} : \overline{t_i\, t_{i+1}} \times \overline{t_j\, t_{j+1}} \longrightarrow A_{ij}.$$

Let us take such a $\overline{G}_{ij}$ for every i and j with $i \leq j$. Then for $i > j$ we will define $\overline{G}_{ij}$ by the formula

$$\overline{G}_{ij}(s,t) \;=\; \overline{G}_{ji}(t,s). \tag{31}$$

It is obvious from (iv) that the $\overline{G}_{ij}$ defined by (31) also extends our given $\overline{G}$ as defined on the boundary of $\overline{t_i\, t_{i+1}} \times \overline{t_j\, t_{j+1}}$.

It should now be clear that $\bigcup_{0 \leq i,j < N} \overline{G}_{ij}$ is a continuous binary operation on S^1 that extends our partial operation $\overline{G}$. We will denote this full operation also by $\overline{G}$. For (21), we need to check its idempotence and its symmetry. For this, we need to make two further stipulations in the definition of $\overline{G}_{ii}$ (for $0 \leq i < N$). Since $\overline{F}$ satisfies (21), we have $\overline{F}(t_i, t_i) \;=\; t_i$ for all i. By (i) we have $\overline{G}(t_i, t_i) \;=\; t_i$ for all i. Let us first extend $\overline{G}_{ii}(s,s)$ to have the value s for each $s \in \overline{t_i\, t_{i+1}}$. The diagonal $\{(s,s) : s \in \overline{t_i\, t_{i+1}}\,\}$ divides $\overline{t_i\, t_{i+1}} \times \overline{t_i\, t_{i+1}}$ into two triangles, and $\overline{G}_{ii}$ has been defined on the boundary of each of these triangles. Then $\overline{G}_{ii}$ may be extended to one triangle (as before), and reflected to the other triangle by the formula $\overline{G}_{ii}(t,s) \;=\; \overline{G}_{ii}(t,s)$. This completes a definition of $\overline{G}_{ii}$ on the full rectangle $\overline{t_i\, t_{i+1}} \times \overline{t_i\, t_{i+1}}$.

It is now obvious that $\overline{G}$ satisfies (22) if the variables are assigned values in any rectangle $\overline{t_i\, t_{i+1}} \times \overline{t_i\, t_{i+1}}$. For values outside such a rectangle, idempotence is moot, and (31) suffices to prove symmetry. We have thus constructed a continuous commutative idempotent operation on S^1, in contradiction to known results. This contradiction completes the proof of the theorem. ∎

2.5 $\Sigma =$ ternary majority laws; $A = S^1$

In §2.5 we follow the general path of §2.3 and §2.4, but this time we consider a non-Abelian simple theory about a *ternary* operation symbol F. In this section we let Σ denote the following equations, known sometimes as the *majority equations*:

$$F(x,x,z) \;\approx\; F(x,z,x) \;\approx\; F(z,x,x) \;\approx\; x. \tag{32}$$

For the sake of completeness we also consider the *symmetric majority equations:*

$$\Sigma' \;=\; \Sigma \cup \{F(x,y,z) \approx F(x,z,y) \approx F(y,z,x)\}. \tag{33}$$

Again using Lemma 3, we will prove

Theorem 6 $\mu(S^1, \Sigma) \;=\; \mu(S^1, \Sigma') \;=\; 2/3$. *(With Σ, Σ' as defined in (32), (33), resp.)*

Sketch of proof. It will of course be enough to prove that $2/3 \leq \mu(S^1, \Sigma) \leq \mu(S^1, \Sigma') \leq 2/3$.

Part 1. $\mu(S^1, \Sigma') \leq 2/3$. We must exhibit an algebra $\mathbf{A} = \langle S^1, \overline{F}\rangle$ (with $\overline{F}$ ternary) satisfying equations (33), and with $\chi(\overline{F}) \leq 2/3$. For convenience, as in the proof of Theorem 4, we represent S^1 as a circle of radius $R = 1/\pi$, with metric determined by arc length around the circle. Thus in this representation S^1 has diameter 1.

On S^1 we shall construct a ternary operation $\overline{F}$ satisfying three properties, which guarantee (33) and which allow us to make the desired estimate of $\chi(\overline{F})$:

(i) $\overline{F}$ satisfies $F(x,y,z) \approx F(x,z,y) \approx F(y,z,x)$.

(ii) $\overline{F}(a,b,c) \in \{a,b,c\}$ for all a,b,c.

(iii) If $d(a,a') < 2/3$, then $\overline{F}(a,a',b) \in \overline{a\,a'}$.

Here is the definition of $\overline{F}$. Given $a,b,c \in S^1$, we examine the three distances $d(a,b)$, $d(b,c)$ and $d(c,a)$.

Definition of $\overline{F}$, clause (1). If all three distances are $< 2/3$, then one of these distances is the sum of the other two. For example $d(a,c) = d(a,b) + d(b,c)$. In that case b is said to be between a and c, and we define $\overline{F}(a,b,c)$ to be b. The same formula, *mutatis mutandis*, yields a (resp. c) between the other two, in which case $\overline{F}(a,b,c)$ is a (resp. c).

Definition of $\overline{F}$, clause (2). If exactly two of the three distances are $< 2/3$, then a, b and c must be distinct, as the reader may verify. For example we might have $0 < d(a,b), d(b,c) < 2/3$ and $d(a,c) > 2/3$. In this case, a and c must lie on opposite sides of b, for otherwise $d(a,c)$ would be too small.

In this case, we define $\overline{F}(a,b,c)$ to be b. We extend the definition, *mutatis mutandis*, to the other two possible arrangements.

Definition of $\overline{F}$, clause (3). If exactly one of the three distances is $< 2/3$, say $d(a,b) < 2/3$, then we define $\overline{F}(a,b,c)$ to be either a or b, chosen at random. We extend the definition, *mutatis mutandis*, to the other two possible arrangements.

Definition of $\overline{F}$, clause (4). Finally, if none of the three distances is $< 2/3$, then all three must be equal to $2/3$. In this case we let $\overline{F}(a,b,c)$ be a or b or c, chosen at random.

We now turn to the verification of (i), (ii) and (iii) for our operation $\overline{F}$. Condition (i) is immediate, since in all cases the definition concerns e.g. a set of distances; it does matter in which order the three variables enter the triple (a,b,c). Condition (ii) is immediate from the construction of $\overline{F}$.

As for Condition (iii), let us consider the definition of $\overline{F}(a,a',b)$, where $d(a,a') < 2/3$. If $\overline{F}(a,a',b)$ falls into clause (1) of the definition, then we may discern two cases: (a) b is between a and a', and (b) it is not. In case (a), $\overline{F}(a,a',b)$ is b, which lies in the interval $\overline{a\,a'}$. In case (b), $\overline{F}(a,a',b)$ is either a or a', and both of these lie in the arc $\overline{a\,a'}$.

Verifying Condition (iii) for clause (2) of the definition, if $d(a,a') < 2/3$ then we cannot have a and a' on opposite sides of b (for then all three interals would be small). Thus either a and b are on opposite sides of a', or a' and b are on opposite sides of a. Thus we have $\overline{F}(a,a',b)$ equal to a or a', and hence in the arc $\overline{a\,a'}$.

The verification of Condition (iii) for clause (3) of the definition is immediate. Clause (4) cannot occur in the calculation of $\overline{F}(a,a',b)$. Hence we have considered all clauses for the evaluation of $\overline{F}(a,a',b)$; hence Condition (iii) is verified.

Having established conditions (i–iii), we turn now to our previous claim that these conditions imply the desired properties for $\overline{F}$. As for equations (33), Condition (i) is symmetry itself, and condition (iii) immediately yields the majority laws (32). All that remains for Part 1 of the proof is to estimate $\chi(\overline{F},(a,b,c))$ for $(a,b,c) \in (S^1)^3$. Our estimate will be based solely on conditions (i–iii). We consider two possibilities for the triple (a,b,c).

Case 1: $d(a,b) = d(b,c) = d(c,a) = 2/3$. In this case, $\{a,\ b,\ c\}$ is an equilateral triangle of diameter $2/3$. For a neighborhood of (a,b,c) in $(S^1)^3$, we may consider a set $U \times V \times W$, where U (resp. V, W) is a neighborhood of a (resp. b, c). From Condition (ii), we easily see that

$$\overline{F}[U \times V \times W] \subseteq U \cup V \cup W.$$

By making the neighborhoods U, V and W small, we obviously have diameter $\overline{F}[U \times V \times W] < 2/3 + \varepsilon$ for any $\varepsilon > 0$. Thus $\chi(\overline{F}, (a, b, c)) \leq 2/3$.

Case 2: either $d(a, b) \neq 2/3$ or $d(b, c) \neq 2/3$ or $d(c, a) \neq 2/3$. Then obviously one of these three distances must be $< 2/3$. Since $\overline{F}$ is symmetric, we assume without loss of generality that $d(a, b) < 2/3$. Choose real δ with $0 < 2\delta < (2/3 - d(a, b))$, and let U (resp. V) be the δ-ball about a (resp. b) with radius δ. If $u \in U$ and $v \in V$, then $d(u, v) < 2/3$. Hence, for any w, we have $\overline{F}(u, v, w) \in \overline{u\,v}$, by (iii). In other words, we have

$$\overline{F}\,[U \times V \times S^1] \subseteq U \cup V \cup \overline{a\,b}. \tag{34}$$

This last is a set of diameter $d(a, b) + 2\delta$; by our choice of δ this diameter $< 2/3$. In other words, we have now shown that $\chi(\overline{F}, (a, b, c)) < 2/3$.

Combining Cases 1 and 2, we see that $\chi(\overline{F}) \leq 2/3$, and hence that $\mu(S^1, \Sigma) \leq 2/3$. This finishes Part 1 of the proof.

Part 2. $\mu(S^1, \Sigma) \geq 2/3$. For a proof by contradiction, we assume that $\mu(S^1, \Sigma) < 2/3$. By (7), there is an algebra $\mathbf{A} = \langle S^1, \overline{F} \rangle$ such that $\chi(\overline{F}) < 2/3$ and such that $\mathbf{A} \models \Sigma$. As in the proof of Theorems 4 and 5, we give $(S^1)^3$ the sum metric (in this case, the sum of distances over three coordinates). As before, there exists $\delta > 0$ such that $d((a, b, c), (d, e, f)) < \delta$ implies $d(\overline{F}(a, b, c), \overline{F}(d, e, f)) < 2/3$. Let

$$t_0,\ t_1,\ \cdots\ t_{N-1},\ t_N = t_0$$

be points of S^1 satisfying (a–c) in the proof of Theorem 4. As before, we have[7]

$$\text{diameter } \{\overline{F}(t_u, t_v, t_w) : u = i, i{+}1;\ v = j, j{+}1;\ w = k, k{+}1\} < 2/3. \tag{35}$$

for $0 \leq i, j, k < N$. The nearness relations (35) will make it possible to define a continuous ternary operation $\overline{G}$ that interpolates the N^3 discrete function values $\overline{F}(t_i, t_j, t_k)$ $(0 \leq i, j, k < N)$. Using the fact that these values obey (32) we will be able to make sure that the interpolated operation $\overline{G}$ also obeys

[7] The "+1" appearing in (35–36), and elsewhere, is again modulo N.

(32). Thus we will have $\langle S^1, \overline{G}\rangle \models \Sigma$ with $\overline{G}$ continuous, in contradiction to the known fact [31] that $S^1 \not\models \Sigma$; this contradiction will complete the proof of the theorem.

Applying Lemma 3 to (35) we see that for $0 \le i, j, k < N$ there is an arc A_{ijk} of length $< 2/3$ such that

$$\{\, \overline{F}(t_u, t_v, t_w) \,:\, u = i, i{+}1, \;\; v = j, j{+}1, \;\; w = k, k{+}1 \,\} \;\subseteq\; A_{ijk}. \qquad (36)$$

Now the proof continues much like Part 2 of the proof of Theorem 5; we omit the details. The function values $\overline{F}(t_i, t_j, t_k)$ will be interpolated to a continuous operation $\overline{G}$. The interpolation is done first along grid lines $\{(t_i, t_j, u)\}$, $\{(t_i, t, t_k)\}$ and $\{(s, t_j, t_k)\}$, where s, t and u range over S^1. It is then extended to the grid surfaces $\{(t_i, t, u)\}$, $\{(s, t_j, u)\}$ and $\{(s, t, t_k)\}$, and finally to the entire 3-dimensional figure $(S^1)^3$. As before, it is carried out one cell at a time in the given subdivision, and as before (36) ensures that a continuous extension always exists, one cell at a time.

To accommodate Equations (32) we need first notice, for example, that $\overline{G}(t_i, t_j, t_j) = \overline{F}(t_i, t_j, t_j) = t_j$, and likewise $\overline{G}(t_i, t_{j+1}, t_{j+1}) = \overline{F}(t_i, t_{j+1}, t_{j+1}) = t_{j+1}$. Therefore for t ranging over the arc $\overline{t_j\, t_{j+1}}$ it is possible to define $\overline{G}$ in such a way that $\overline{G}(t_i, t, t) = t$, which is a start on proving (32) for $\overline{G}$. This can then be incorporated into the determination of the two-dimensional interpolation $\overline{G}(t_i, t, w)$, by interpolating over two triangles, as we did in the proof of Theorem 5. At the three-dimensional level we must divide a cube into two triangular prisms. We omit further details. ∎

2.5.1 Comment on the proofs of Theorems 4–6.

Theorems 4, 5 and 6, in §2.3, §2.4 and §2.5, each conclude that $\mu(S^1, \Sigma) = 2/3$ for a certain theory Σ. The proofs for $\mu(S^1, \Sigma) \ge 2/3$ are essentially identical: each involves interpolating a discontinuous operation over a fine grid, and producing a continuous operation. (The proof of Theorem 4 is not directly phrased this way, but it could easily be rewritten to this form.) We are confident that this method would extend to many more simple non-Abelian theories Σ, perhaps all of them. (Perhaps one would need to invoke [35] to satisfy Σ continuously at the cellular level.)

This common argument relies essentially on Lemma 3, which allows each cell to be mapped into an interval, which is topologically very feasible. We believe it will be possible to find analogs to Lemma 3 for higher dimensions (e.g. for S^n); in that case the method may extend to the study of non-Abelian simple theories on S^n.

On the other hand the three proofs for $\mu(S^1, \Sigma) \leq 2/3$ seem to have arisen ad hoc, on a completely case-by-case basis. To remind the reader: each of these proofs involved the construction of an algebra $(S^1, \overline{F})$ satisfying Σ and with $\chi(\overline{F}) \leq 2/3$. At this time there seems to be little scope for extension of these methods to another set Σ of equations, or to spaces of higher dimension.

2.6 $\Sigma =$ commutative idempotent binary; $A = S^2$.

Here we begin to explore whether the method of Theorems §§4–6 will extend to other spaces. We first note that the two-dimensional sphere S^2 is incompatible with the spaces that appear in those theorems ([34]; see also [36, §3.2 and §3.2.1]). In fact we will sketch a proof of

Theorem 7 *If Σ is the theory either of a binary operation with zero and one (§2.3), or of a symmetric idempotent operation (§2.4), or of a ternary majority operation (§2.5), then $\mu(S^2, \Sigma) \geq 2/3$.*

Before sketching the proof, we will state an analog of Lemma 3. In our previous applications of Lemma 3, the essential part of the conclusion is that F lies in some convex subset of S^1, i.e. an arc. Let us suppose that S^2 is given the great-circle metric, with diameter scaled to 1. For a subset $A \subseteq S^2$, we say that A is *convex* iff for each two points $P, Q \in A$, we have $d(P, Q) < 1$ and $\overline{PQ} \subseteq A$.

Lemma 8 *If F is a finite subset of S^2 with diameter$(F) < 2/3$, then there is a convex subset A of S^2 such that $F \subseteq A$.* ∎

The proof of Lemma 8 is like that of Lemma 3, and omitted for now. Notice again that 2/3 is best possible for this conclusion: three equally spaced points on a great circle form a set F of diameter 2/3 that does not lie in a convex set.

We will use two facts about convex subsets: the first is that the intersection of any family of convex subsets is convex. The second is the property of being an *absolute extensor* (**AE**). A metrizable space A is defined to be an AE in the family of metrizable spaces iff it satisfies the following property:

if B is a closed subspace of a metrizable space F, and if $g : B \longrightarrow A$ is a continuous function, then there exists a continuous function $\phi : F \longrightarrow A$ such that $\phi \restriction B = g$. (See e.g. [15, pages 34–35].) Each convex subset of S^2 is homeomorphic to a convex subset (in the ordinary sense) of the plane, and hence is an AE, by [15, pages 84–87]. (See also [6] for the general theory of AE's (and absolute retracts).)

Sketch of proof of Theorem 7. We will restrict our attention to the case where Σ is the theory of a symmetric idempotent operation (§2.4). For a proof by contradiction, we assume that $\mu(S^1, \Sigma) < 2/3$. By (7), there is an algebra $\mathbf{A} = \langle S^2, \overline{F} \rangle$ such that $\chi(\overline{F}) < 2/3$ and such that $\mathbf{A} \models \Sigma$. As in the proofs of Theorems 4 and 5, we give $(S^2)^2$ the sum metric. As before, there exists $\delta > 0$ such that $d((a, b), (c, d)) < \delta$ implies $d(\overline{F}(a, b), \overline{F}(c, d)) < 2/3$.

Now let us assume that $(S^2)^2$ has been triangulated in such a way that each 4-simplex has diameter $< \delta$. Moreover the triangulation must be symmetric in the following sense. Let ι be the involution of $(S^2)^2$ given by $\iota(a, b) = (b, a)$, where $a, b \in S^2$. Our symmetry condition is that if σ is a simplex of the triangulation, then so is $\iota[\sigma]$. Our final condition is that the diagonal of $(S^2)^2$ — namely $\{(a, a) : a \in S^2\}$ — must be a subcomplex of this triangulation. Such a triangulation is clearly possible.

We now proceed to define a continuous binary operation $\overline{G}$ on S^2, which is symmetric and idempotent. This will contradict the known fact [34, Theorem 1] that no such $\overline{G}$ exists; this contradiction will complete the proof of Theorem 7.

We define $\overline{G}$ on simplices of successively higher dimension. For a 0-simplex (point) P we simply define $\overline{G}(P) = \overline{F}(P)$; then obviously $\overline{G}$ is symmetric and idempotent at the level of 0-simplices.

For each 4-simplex σ, Lemma 8 yields a convex subset A_σ of S^2 such that $\overline{F}[\overline{\sigma}] \subseteq A_\sigma$. (Here $\overline{\sigma}$ denotes the closure of σ, which is σ together with all its subsimplices.) From the symmetry of $\overline{F}$, we may further take the sets A_σ so that $A_{\iota(\sigma)} = A_\sigma$ for all σ. We will use the A_σ's in defining $\overline{G}$ over simplices of dimensions 1, 2, 3 and 4. For a simplex τ of any dimension ≤ 4, we define

$$A_\tau \ = \ \bigcap \{A_\sigma : \dim(\sigma) = 4;\ \overline{\tau} \subseteq \overline{\sigma}\}. \tag{37}$$

It is not hard to check that A_τ is a nonempty convex subset of S^2 such that

$$\overline{F}[\overline{\rho}] \subseteq A_\rho; \quad \text{if } \rho \leq \tau, \text{ then } A_\rho \subseteq A_\tau. \tag{38}$$

We will now show inductively that for $n = 1, 2, 3, 4$, it is possible to define $\overline{G}$ on the n-skeleton of our triangulation in such a way that, $\overline{G}[\overline{\tau}] \subseteq A_\tau$ for each n-simplex τ. We prove this for $n = 3$; the other cases are similar. If τ is a 3-simplex, then $\overline{G}$ has already been defined on all 2-simplices in the boundary of τ. For each boundary 2-simplex ρ, we have $\overline{F}[\overline{\rho}] \subseteq A_\rho$ by (38). Also by (38), we know that each of the sets A_ρ is a subset of A_τ. Therefore the two-dimensional extension of $\overline{G}$ maps the boundary of τ into A_τ. Since A_τ is convex, and hence an AE, there is a continuous extension of $\overline{G}$ from the closed 3-simplex $\overline{\tau}$ into A_τ.

If we consider two closed 3-simplices, $\overline{\tau}$ and $\overline{\tau'}$, then their overlap consists of closed 2-simplices; hence the extensions to $\overline{\tau}$ and $\overline{\tau'}$ agree on this overlap, Thus the union of all such extensions is a well-defined continuous function as desired. The desired condition $\overline{G}[\overline{\tau}] \subseteq A_\tau$ was automatically fulfilled as we went along. Continuing in this manner to 4-simplices, we obtain a continuous operation $\overline{G}$ that agrees with $\overline{F}$ on all vertices of the triangulation.

It remains to see that this operation can be made to satisfy idempotence and symmetry. As for idempotence, since $\overline{F}$ satisfies $F(x, x) \approx x$, we can easily define $\overline{G}(x, x)$ to be x. This may be taken as the definition of $\overline{G} \restriction \sigma$, for each simplex σ of the diagonal subcomplex. We already have that $\overline{F}[\overline{\sigma}] \subseteq A_\sigma$ for all σ, including those on the diagonal. Since $\overline{G} = \overline{F}$ on the diagonal, we also have the required condition that $\overline{G}[\overline{\sigma}] \subseteq A_\sigma$. Incorporating this special case into our definition of $\overline{G}$, we now have a continuous idempotent operation.

As for symmetry, we merely need, for each simplex σ, to define $\overline{G}$ on σ and $\iota[\sigma]$ at the same time. (If σ is a simplex of the diagonal subcomplex, then $\sigma = \iota[\sigma]$, and so this condition has already been met.) Inductively, we may assume that $\overline{G} = \overline{G} \circ \iota$ on all the boundary simplices of σ. Thus we simply define $\overline{G}$ on σ as we did above, and on $\iota[\sigma]$ we define $\overline{G}$ by the formula $\overline{G} = \overline{G} \circ \iota$. Clearly all the conditions are met, and we now have a continuous, symmetric, idempotent binary operation on S^2. This contradiction completes the proof of the theorem. ▮

2.6.1 Comment on the proof of Theorem 7.

In some ways the proof of Theorem 7 may be more comprehensible than those that we have supplied for Theorems 4, 5 and 6, in §2.3, §2.4 and §2.5. In those proofs we supplied a grid, which is tantamount to a triangulation, but we needed to work with details of that grid (often speaking, for instance, of i and $i+1$, etc.). In our proof of Theorem 7, we use the general and inclusive notion of triangulation, which can be discussed without reference to the detailed configuration of a given triangulation.

It now seems right to conjecture that the method will go a lot further than we have seen it here so far.

2.7 An auxiliary theory.

In 1986—see [32, §3.18, page 35]—we introduced the following equational theory, known here as Σ_1:

$$F(\phi^k(x), x, y) \approx x \tag{39}$$

$$F(x, x, y) \approx y, \tag{40}$$

for $k \in \omega$, $k \geq 1$. We proved [*loc. cit.*] that it is incompatible with every compact Hausdorff space A of more than one element. In [36, §3.3.9] we proved that Σ_1 has a λ-value (§0.2) at least as large as diameter$(A)/4$. Here we prove

Theorem 9 *If A is compact, then $\mu(A, \Sigma_1) \geq$ diameter$(A)/2$.*

Proof. To prove the theorem by contradiction, we may suppose that $\mu(A, \Sigma_1) <$ diameter$(A)/2$. In a manner by now familiar, there exist (discontinuous) operations $\overline{F}$ and $\overline{\phi}$ modeling (39–40) on A, and positive real numbers $\delta_0 \leq \delta_1 <$ diameter$(A)/2$, such that $\overline{F}$ and $\overline{\phi}$ are each constrained by (δ_0, δ_1).

Since A is compact, there exist $a, b \in A$ with $d(a, b) =$ diameter(A). Choose arbitrary $q \in A$. By compactness, the sequence $\overline{\phi}^{\,n}(q)$ has a convergent subsequence:

$$\lim_{i \longrightarrow \infty} \overline{\phi}^{\,n(i)}(q) = c \in A.$$

By the triangle inequality, either $d(b, c) \geq$ diameter$(A)/2$ or $d(a, c) \geq$ diameter $(A)/2$. Without loss of generality, we will assume that $d(b, c) \geq$ diameter$(A)/2$. By (40),

$$\overline{F}\big(\overline{\phi}^{\,n(i+1)}(q),\, \overline{\phi}^{\,n(i)}(q),\, b\big) = \overline{\phi}^{\,n(i)}(q) \tag{41}$$

for all i, and hence this sequence has c as limit. On the other hand, according to Lemma 15, the sequence in (41) is eventually within δ_1 of $\overline{F}(c, c, b) = b$

(by (39)). Therefore, $d(b,c) \leq \delta_1 < \text{diameter}(A)/2$, contrary to our assumption. This contradiction completes the proof of the theorem. ∎

We note that in the proof the (δ_0, δ_1)-constraint on $\overline{\phi}$ was never used.

2.8 A second auxiliary theory.

In [36, §3.3.9] we introduced the following theory, known here as Σ_2:

$$G(\psi_{m+k}(x,y),\, \psi_m(x,y),\, x,\, y) \;\approx\; x \tag{42}$$

$$K(x,y) \;\approx\; G(u,u,x,y) \;\approx\; K(y,x), \tag{43}$$

for $m, k \in \omega$, with $k \geq 1$. We proved [loc. cit.] that Σ_2 is incompatible with any compact A with more than one element. More precisely, we proved that $\lambda_A(\Sigma_2) \leq \text{diameter}(A)/4$. Here we prove something similar for μ.

Theorem 10 *If A is compact, then $\mu(A, \Sigma_2) \;\geq\;$ diameter$(A)/2$.*

Proof. To prove the theorem by contradiction, we may suppose that $\mu(A, \Sigma_2) < \text{diameter}(A)/2$. In a manner by now familiar, there exist (discontinuous) operations $\overline{G}$, $\overline{K}$ and $\overline{\psi}_m$ modeling (42–43) on A, and positive real numbers $\delta_0 \leq \delta_1 < \text{diameter}(A)/2$, such that $\overline{G}$, $\overline{K}$ and $\overline{\psi}_m$ are each constrained by (δ_0, δ_1).

Let a and b be points of A with $d(a,b)$ equal to the diameter of A. By the triangle inequality, we have either $d(a, \overline{K}(a,b) \geq \text{diameter}(A)/2$ or $d(b, \overline{K}(a,b) \geq \text{diameter}(A)/2$. Without loss of generality, we shall assume that

$$d(b, \overline{K}(a,b)) \;\geq\; \text{diameter}(A)/2. \tag{44}$$

Consider the sequence $\overline{\psi}_i(b,a)$; by compactness it has a convergent subsequence:

$$\lim_{i \longrightarrow \infty} \overline{\psi}_{n(i)}(b,a) \;=\; c \in A.$$

By (42),

$$\overline{G}(\overline{\psi}_{n(i+1)}(b,a),\, \overline{\psi}_{n(i)}(b,a),\, b,\, a) \;=\; b \tag{45}$$

for all i. On the other hand, according to Lemma 15, the sequence in (45) is eventually within δ_1 of $\overline{G}(c,c,b,a) = \overline{K}(a,b)$ (by (42)). Therefore, $d(b, \overline{K}(a,b)) \leq \delta_1 < \text{diameter}(A)/2$, contrary to (44). This contradiction completes the proof of the theorem. ∎

Notice that the proof of Theorem 10 does not mention the (δ_0, δ_1)-constraint on ψ_m, for any m. Ignoring this constraint, we obtain the following sharper version:

Theorem 11 *If A is a compact metric space of more than one element, then there is no algebra $\mathbf{A} = \langle A, \overline{G}, \overline{K}, \overline{\psi}_m\rangle_{m\in\omega}$ such that $\chi(\overline{G})$ and $\chi(\overline{K})$ are both $<$ diameter$(A)/2$, and $\mathbf{A} \models \Sigma_2$.*

3 Dealing with composite operations.

There may be a problem in carrying some of the results to equations Σ that involve composite operations. Suppose, for example that $\overline{f}$ is unary and $\chi(\overline{f}) = \varepsilon$. If s lies between $\overline{f}(a)$ and $\overline{f}(c)$ on a segment, then there exists b such that $\overline{f}(b)$ lies with ε of s. Our equation of interest may, however, involve $\overline{g}(\overline{f}(b))$, and we might like to know that this value is near to $\overline{g}(s)$. With what we have so far, we cannot conclude anything about the distance between these two $\overline{g}$-values.

3.1 n-iterated jumps

Let (A, d) be a metric space, and $\mathbf{A} = (A, \overline{F}_t)_{t\in T}$ an algebra based on A. Recalling χ from §0.3, we define

$$\chi_n(\mathbf{A}, d) \;=\; \sup_{\tau} \; \chi(\overline{\tau}).$$

Where τ ranges over all terms in operation symbols F_t that have depth $\leq n$, and where, for each τ, $\overline{\tau}$ denotes the term operation corresponding to τ in the algebra $\mathbf{A} = (A, \overline{F}_t)_{t\in T}$. We may also write $\chi_\infty(\mathbf{A}, d)$ for the same supremum, taken over all terms τ.

When the metric d is clear from the context, we may write $\chi_n(\mathbf{A})$ for $\chi_n(\mathbf{A}, d)$.

Finally, for (A,d) a metric space, and Σ a set of equations of similarity type $\langle n_t \,:\, t \in T\rangle$, we define

$$\mu_n(A,d,\Sigma) \;=\; \inf\{\, \chi_n(\mathbf{A},d) \,:\, \mathbf{A} = (A,\overline{F}_t)_{t\in T} \models \Sigma \,\}; \tag{46}$$

in other words, it is the infimum taken over all algebras built on A that satisfy Σ. When the metric d is clear from the context, we may write $\mu_n(A,\Sigma)$ for $\mu_n(A,d,\Sigma)$. We may also write $\mu_\infty(A,d,\Sigma)$ for the corresponding infimum of χ_∞-values.

Obviously there is a uniform version

$$\chi_n^u(\mathbf{A},d) \;=\; \sup_{\tau}\; \chi_u(\overline{\tau}),$$

and likewise for μ_n^u. Most of this paper deals with compact metric spaces, on which the two concepts coincide, so we will rarely mention χ_n^u.

It is not hard to see that $\mu_n \le \mu_{n+1} \le \mu_\infty$ for all n, and moreover we generally expect that $\mu_n < \mu_{n+1} < \mu_\infty$. Therefore, concerning estimates from below, viz. $\mu_n > \varepsilon$, one should assert this for n as small as possible, in order to convey the most information. On the other hand, such an estimate for a larger value of n may be all that is available, hence very valuable in itself.

3.2 Iterated (δ,ε)-closeness.

Let us say that a function $f : A \longrightarrow B$ is *constrained by* (δ,ε), or (δ,ε)-*constrained* iff it satisfies

$$\text{if } d(x,x') < \delta, \text{ then } d(f(x),f(x')) < \varepsilon.$$

The notion is of course familiar, in that f is defined to be uniformly continuous iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that f is (δ,ε)-constrained.

In working with a finite direct power A^n of a metric space (A,d), let us agree to give A^n the following adjusted version of the sum metric:

$$d\big((a_1,\ldots,a_n),\,(b_1,\ldots,b_n)\big) \;=\; \frac{1}{n}\,\big(d(a_1,b_1) \,+\, \ldots \,+\, d(a_n,b_n)\big). \tag{47}$$

This definition has the advantage that if diameter$(A) = 1$, then diameter$(A^n) = 1$. It also figures in the detailed proof of Lemma 13 just below.

Now suppose that there are positive reals $\delta_0,\delta_1,\ldots,\delta_n$ such that every operation of $(A,F_t)_{t\in T}$ is constrained by the n pairs (δ_0,δ_1), $(\delta_1,\delta_2),\ldots,$

(δ_{n-1}, δ_n). In this case, we say that $(A, F_t)_{t \in T}$ *is n-constrained by* (δ_0, δ_n). The first lemma says that we may always assume that the δ's form an increasing sequence.

Lemma 12 *If* $\mathbf{A} = (A, F_t)_{t \in T}$ *is n-constrained by* (δ_0, δ_n)*, then there are positive reals* δ'_i *(for* $0 \leq i \leq n$*) such that* $\delta'_n = \delta_n$*, such that* $\delta'_n \geq \delta'_{n-1} \geq \cdots \geq \delta'_1 \geq \delta'_0$*, and such that* $(A, F_t)_{t \in T}$ *is constrained by the n pairs* (δ'_0, δ'_1)*,* $(\delta'_1, \delta'_2), \ldots,$ $(\delta'_{n-1}, \delta'_n)$*.*

Proof. If the given δ_i do not already form a monotone increasing sequence, then for some i we have

$$\delta_i > \delta_{i+1} \leq \delta_{i+2} \leq \cdots \leq \delta_{n-1} \leq \delta_n.$$

Let us define

$$\begin{aligned} \delta'_0 \;=\; \delta'_1 \;=\; \cdots \;=\; \delta'_i &\;=\; \delta_{i+1} \\ \delta'_j &\;=\; \delta_j \quad (\text{for } i < j \leq n). \end{aligned}$$

It is clear that these values of δ'_j have the required properties. ∎

Lemma 13 *If* $\mathbf{A} = (A, F_t)_{t \in T}$ *is n-constrained by* (δ_0, δ_n)*, then* $\chi_n(A, F_t)_{t \in T} \leq \delta_n$*.* ∎

We then define

$$\chi^\star_n(\mathbf{A}) \;=\; \inf\{\delta_n : (\exists \delta_0 > 0)\ \mathbf{A} = (A, F_t)_{t \in T} \text{ is } n\text{-constrained by } (\delta_0, \delta_n)\}; \tag{48}$$

$$\mu^\star_n(A, \Sigma) \;=\; \inf\{\, \chi^\star_n(\mathbf{A}, d) \;:\; \mathbf{A} = (A, \overline{F}_t)_{t \in T} \models \Sigma \,\}. \tag{49}$$

Lemma 13 then implies the first inequality of

Lemma 14 $\chi_n(\mathbf{A}) \leq \chi^\star_n(\mathbf{A})$ *and* $\mu_n(\mathbf{A}, \Sigma) \leq \mu^\star_n(\mathbf{A}, \Sigma)$*. If* $A \models \Sigma$*, then these last two μ-values are both zero.*

In the sections that follow, we will be able to prove that $\mu^\star_n(A, \Sigma) \geq K$ for certain A, Σ and $K > 0$. While this information is obviously less informative than it would be to have $\mu_n(A, \Sigma) \geq K$, it nevertheless has the virtues of being provable and of being a non-trivial quantitative version of $A \not\models \Sigma$. In one case (see §3.5) we have $\mu^\star_2(A, \Sigma) \geq K$ while $\mu(A, \Sigma) = 0$. In this case, $\mu^\star_2$ obviously conveys the greater amount of information.

3.3 Some consequences of (δ, ε)-closeness.

Lemma 15 (Limit theorem, approximate version.) *$f(x_i)$ approaches $f(\lim x_i)$ within ε.*

Lemma 16 (Intermediate Value Theorem, approximate version.) *(Move here from Lemma 29, §3.8.)*

Lemma 17 (Brouwer Fixed-Point Theorem, approximate version.)

Lemma 18 (Borsuk-Ulam Theorem, approximate version.)

Lemma 19 *Suppose that A is a triangulable compact metric space (i.e. the geometric realization of a finite simplicial complex). Let S^1 be the ordinary 1-sphere with arc-length distance, scaled to have diameter 1. Suppose that $\overline{F}: A \longrightarrow S^1$ is (δ, ε)-constrained, where $0 < \delta$ and $0 < \varepsilon < 2/3$. Then there exists a continuous function $\overline{G}: A \longrightarrow S^1$ such that $d(\overline{F}(a), \overline{G}(a)) < \varepsilon$ for all $a \in A$.*

Sketch of proof. The proof is much like that of Theorems 4, 5 and 6, in §2.3, §2.4 and §2.5, and especially like that of Theorem 7 in §2.6 (even though this last result is officially about the 2-sphere). ∎

Corollary 20 *Suppose that Σ is a simple theory, and $\mu(S^1, \Sigma) < 2/3$. Then $\lambda_{S^1}(\Sigma) \leq 2\,\mu(S^1, \Sigma)$.*

Corollary 21 (Conjectured.) *Suppose that each equation of Σ equates two terms of depth no more that k, and that $\mu_k^\star(S^1, \Sigma) < 2/3$. Then $\lambda_{S^1}(\Sigma) \leq 2\,\mu_k^\star(S^1, \Sigma)$.*

Results like Corollaries 20–21 must be relatively abundant. We will extend their range to other spaces as tools become available.

3.4 Revisiting §2.8: lattice-ordered groups.

Following [36, §3.3.10] we define $\Lambda\Gamma$ to be the following (doubly infinite) set of equations:

$$x \;\approx\; x \wedge [(z_{m+k} - z_m) + (x \wedge y)] \tag{50}$$

$$x \wedge y \;\approx\; x \wedge [(u - u) + (x \wedge y)] \;\approx\; y \wedge x, \tag{51}$$

where z_n ($n \in \omega$) are terms defined recursively as follows:

$$z_0 \;=\; 0; \qquad\qquad z_{n+1} \;=\; (z_n \;+\; (x - (x \wedge y))).$$

In [36, §3.3.10] we gave an easy proof that *lattice-ordered groups* satisfy (50–51); in other words Equations (50–51) are among the consequences of the equational axioms of lattice-ordered group theory (which we do not state here in detail). Thus any result of the form $\mu_n(A, \Lambda\Gamma) \geq K$ or $\mu_n^{\star}(A, \Lambda\Gamma) \geq K$—such as Theorem 23 just below—implies the same result for the theory of lattice-ordered groups.

The incompatibility of compact Hausdorff spaces with lattice-ordered groups was proved by M. Ja. Antonovskiĭ and A. V. Mironov [3] in 1967. For compact metric spaces, a positive value for $\lambda_A(\Lambda\Gamma)$ was established by W. Taylor [*loc. cit.*]. Here in §3.4 we prove a positive value for $\mu_3(A, \Lambda\Gamma)$.

Our method for estimating $\mu_3(A, \Lambda\Gamma)$ is to connect $\Lambda\Gamma$ with the Σ_2 appearing in Equations (42–43) of §2.8. Lemma 22 below will establish an *interpretation*[8] (in the sense of [22, 14]) of Σ_2 in $\Lambda\Gamma$. (Thus Σ_2 is *a fortiori* interpretable in lattice-ordered groups.) For every algebra $\mathbf{A} \;=\; \langle A, \overline{\wedge}, \overline{\vee}, \boxplus, \boxminus \rangle$ in the similarity type of $\Lambda\Gamma$, we define a new algebra $\mathbf{A}' = \langle A, \overline{G}, \overline{K}, \overline{\psi}_m \rangle_{m\in\omega}$, in the similarity type of Σ_2, as follows. For $a, b, c, d \in A$, we let

$$\begin{aligned} \overline{G}(a,b,c,d) \;&=\; c \,\overline{\wedge}\, [(a \boxminus b) \boxplus (c \,\overline{\wedge}\, d)], \\ \overline{K}(a,b) \;&=\; a \,\overline{\wedge}\, b \\ \overline{\psi}_m(a,b) \;&=\; \overline{z_n}(a,b), \end{aligned}$$

where z_n is as above, and $\overline{z_n}$ is the term operation associated to z_n.

As noted above, the following lemma and theorem hold *a fortiori* for lattice-ordered groups.

Lemma 22 *If* $\mathbf{A}$ *satisfies* $\Lambda\Gamma$*, then* $\mathbf{A}'$ *satisfies* Σ_2*.*

Proof. We need to see that Equations (42) and (43) hold in $\mathbf{A}'$. We look at

$$G(\psi_{m+k}(x,y),\, \psi_m(x,y),\, x,\, y) \;\approx\; x \tag{42}$$

[8]At some point it may become appropriate to add a section on the persistence of μ-values under interpretation.

in detail. To prove its satisfaction in $\mathbf{A}'$, we need to substitute our definitions of $\overline{G}$ and $\overline{\psi}_m$ into (42) and verify the resulting equation under $\Lambda\Gamma$. The reader may check that the resulting equation is tantamount to (50), which is one of the defining equations of $\Lambda\Gamma$. Thus (42) holds in $\mathbf{A}'$. The proof for (43) is similar. ∎

Theorem 23 *If A is a compact metric space of more than one point, then $\mu_3(A, \Lambda\Gamma) \geq$ diameter$(A)/2$.*

Proof. To prove the theorem by contradiction, we may suppose that $\mu_3(A, \Lambda\Gamma) <$ diameter$(A)/2$. By the definition (46), there exist (discontinuous) operations $\overline{\wedge}$, $\overline{\vee}$, $\boxplus$, $\boxminus$ on A such that $\mathbf{A} = \langle A, \overline{\wedge}, \overline{\vee}, \boxplus, \boxminus\rangle$ satisfies $\Lambda\Gamma$, and such that $\chi_3(\mathbf{A}) <$ diameter$(A)/2$. This means that

$$\chi(\overline{\tau}) \;<\; \text{diameter}(A)/2 \tag{52}$$

for every term-operation $\overline{\tau}$ of $\mathbf{A}$ having depth ≤ 3.

Now the algebra $\mathbf{A}'$ of Lemma 22 is a model of Σ_2, whose operations $\overline{G}$ and $\overline{K}$, each being a term-operation of depth ≤ 3, both have χ-value $<$ diameter$(A)/2$. This contradiction to Theorem 11 completes the proof of our theorem. ∎

3.5 Revisiting §2.1: the injective binary operation.

3.5.1 $A = [0, 1]$.

We return our attention to Equations (8) of §2.1, which we repeat here for convenience:

$$F_0(G(x_0, x_1)) \;\approx\; x_0, \qquad F_1(G(x_0, x_1)) \;\approx\; x_1. \tag{8}$$

Moreover, we again let $A = [0, 1]$ with the ordinary Euclidean metric. In §2.1 we proved that $\mu(A, \Sigma) = 0$. Here we shall prove that $\mu_2^\star(A, \Sigma) = 1$. In fact, we shall prove it in a somewhat broader context.

Theorem 24 *Let $A = [0, 1]$ be given any metric that induces the usual topology. Then $\mu_2^\star(A, \Sigma)$ = diameter(A).*

Proof. We note first that clearly $\mu_2^\star(A,\Sigma) \leq \text{diameter}(A)$ for any A and any Σ. Thus to prove the theorem by contradiction, we may suppose that $\mu_2^\star(Y,\Sigma) < \text{diameter}(A)$. By Definitions (48–49) there exist (discontinuous) operations $\overline{F}_0$, $\overline{F}_1$ and $\overline{G}$ modeling (8) on A, and positive real numbers $\delta_0 \leq \delta_2 < \text{diameter}(A)$ such that $(A, \overline{F}_0, \overline{F}_1, \overline{G})$ is 2-constrained by (δ_0, δ_2). Thus there exists a further positive real δ_1 such that

$$\overline{F}_0,\ \overline{F}_1 \text{ and } \overline{G} \text{ are each constrained by } (\delta_0, \delta_1) \text{ and by } (\delta_1, \delta_2). \tag{53}$$

Since $[0,1]$ is compact, there exist $a_0, a_1 \in A$ with $d(a_0, a_1) = \text{diam}(A)$. For flexibility of notation, we take two such pairs: $d(a_0, a_1) = d(b_0, b_1) = \text{diam}(A)$. Considering the four real numbers

$$\overline{G}(a_0, b_0),\ \ \overline{G}(a_1, b_0),\ \ \overline{G}(a_0, b_1),\ \ \overline{G}(a_1, b_1),$$

we may assume, without loss of generality, that the smallest among them is $\overline{G}(a_0, b_0)$. Again without loss of generality, we may assume that $\overline{G}(a_1, b_0)) \leq \overline{G}(a_0, b_1)$. In other words, we have

$$\overline{G}(a_0, b_0) \ \leq\ \overline{G}(a_1, b_0) \ \leq\ \overline{G}(a_0, b_1).$$

Thus, along the segment $\overline{(a_0, b_0)(a_0, b_1)}$ in the square $[0,1]^2$, the (δ_0, δ_1)-constrained function $\overline{G}$ takes values that are above and below the value $\overline{G}(a_1, b_0)$. By Lemma 29, there exists $e \in [0,1]$ such that

$$d\left(\overline{G}(a_0, e),\, \overline{G}(a_1, b_0)\right) \ <\ \delta_1.$$

For the (δ_1, δ_2)-constrained function $\overline{F}_0$ we now calculate, using Σ:

$$d(a_0, a_1) \ =\ d\big(\overline{F}_0(\overline{G}(a_0, e)),\, \overline{F}_0(\overline{G}(a_1, b_0))\big) \ <\ \delta_2 \ <\ \text{diameter}(A).$$

This contradiction to our choice of a_0, a_1 completes the proof. ∎

We notice that in this proof we needed the (δ_0, δ_1)-constraint only for the binary operation $\overline{G}$, and the (δ_1, δ_2)-constraint only for the unary operations $\overline{F}_0$, $\overline{F}_1$. (In other words, (55) contains more information than necessary.) It would thus be possible to give Theorem 25 a slightly sharper statement by modifying the hypotheses according to this observation. Similar remarks apply elsewhere in the paper. As far as we can see for now, such an endeavor merits neither the effort involved nor the cumbersome statements that would result.

3.5.2 Comments on the proof of Theorem 25

Our estimate is made for $\chi_2^\star$ only. This proof does not yield information on χ_2. The reason is that we must be able to estimate the effect of applying $\overline{F}_0$, $\overline{F}_1$ and $\overline{G}$ to the number e that is supplied by Lemma 29. Such an e is not necessarily[9] in the range of our operations, so that we cannot make the necessary estimate simply by applying some term-operation $\overline{\tau}$.

Comparing this proof with the corresponding proof for λ that appears in [36], we note a lot of similarity. In fact this proof is the same almost verbatim.

3.5.3 $A = [0,1]^2$.

Once again, we work with these equations:

$$F_0(G(x_0, x_1)) \;\approx\; x_0, \qquad F_1(G(x_0, x_1)) \;\approx\; x_1. \tag{8}$$

We shall suppose that the usual topology of $[0,1]$ is given by a metric d_0 with the property that $d_0(0,1) \geq 1$. We then let $A = [0,1]^2$ with the metric defined as a sum (taxi-metric, L_1-norm): $d((a,b),(c,d)) = d_0(a,c) + d_0(b,d)$.

Theorem 25 $\mu_2^\star(A, \Sigma) \;\geq\; 1$.

Proof. To prove the theorem by contradiction, we may suppose that $\mu_2^\star(A, \Sigma) \;<\; 1$. In a manner by now familiar, there exist (discontinuous) operations $\overline{F}_0$, $\overline{F}_1$ and $\overline{G}$ modeling (8) on A, and positive real numbers $\delta_0 \leq \delta_1 \leq \delta_2 < 1$ such that

$$\overline{F}_0,\ \overline{F}_1 \text{ and } \overline{G} \text{ are each constrained by } (\delta_0, \delta_1) \text{ and by } (\delta_1, \delta_2). \tag{54}$$

Now $B^2 \;=\; [0,1]^4$, and so the boundary of this space is a three-sphere S^3. Let us consider the action of $\overline{G}$ on this three-sphere. Since $\overline{G}$ is (δ_0, δ_1)-constrained, it takes on δ_1-close values at two antipodal points, by our version of the Borsuk-Ulam Theorem (Theorem 18). Without loss of generality, two antipodal points have the form $((0, x_1), (y_0, y_1))$ and $((1, u_1), (v_0, v_1))$. We thus have

$$d\big(\,\overline{G}\big((0, x_1), (y_0, y_1)\big),\, \overline{G}\big((1, u_1), (v_0, v_1)\big)\,\big) \;<\; \delta_1\,.$$

[9] Objection: if we look at the proof of Lemma 29, we see that e really is in the range. This needs to be sorted out before publication.

Since $\overline{F}_0$ is (δ_1, δ_2)-constrained, Equations Σ yield

$$\begin{aligned} 1 \ &\leq\ d\big((0,x_1),(1,u_1)\big) \\ &=\ d\big(\,\overline{F}_0\overline{G}\big((0,x_1),(y_0,y_1)\big),\ \overline{F}_0\overline{G}\big((1,u_1),(v_0,v_1)\big)\,\big)\ <\ \delta_2. \end{aligned}$$

∎

3.6 Group theory on spaces with the fixed-point property.

In this section we let Γ stand for any equational theory whose models are groups. (Some variation is possible in choice of primitive operations and axioms, but any such theory will do.) We will assume that binary $+$ and unary $-$ are available, either as primitives or as derived operations.

A will be a metric space that has the *fixed-point property*: if $f : A \longrightarrow A$ is continuous, then there exists $e \in A$ such that $f(e) = e$. Until we know the full scope of Theorem 17, we will state and prove it only for a power $[0,1]^n$, which is to say, for an n-simplex. A corresponding result for λ was proved in §3.3.1 of [36].

Theorem 26 *Let* $A = [0,1]^n$ *be given any metric that induces the usual topology, and let* Γ *denote group theory. Then* $\mu_2^\star(A,\Gamma) = \mathrm{diameter}(A)$.

Proof. We note first that clearly $\mu_2^\star(A,\Gamma) \leq \mathrm{diameter}(A)$ for any A and any Γ. Thus to prove the theorem by contradiction, we may suppose that $\mu_2^\star(A,\Gamma) < \mathrm{diameter}(A)$. By Definitions (48–49) there exist (discontinuous) group operations $\boxplus$ and $\boxminus$ on A, and positive real numbers $\delta_0 \leq \delta_2 < \mathrm{diameter}(A)$, such that $(A, \boxplus, \boxminus)$ is 2-constrained by (δ_0, δ_2). Thus there exists a further positive real δ_1 such that

$$\boxplus \text{ and } - \text{ are each constrained by } (\delta_0, \delta_1) \text{ and by } (\delta_1, \delta_2). \tag{55}$$

Since A is compact, there are points a_0, $a_1 \in A$ with $d(a_0, a_1) = \mathrm{diameter}(A)$. Consider the function $f : A \longrightarrow A$ defined by $f(x) = (a_1 \boxminus a_0) \boxplus x$. Since f is (δ_0, δ_1)-constrained, Theorem (17) yields $e \in A$ such that $d(e, f(e)) < \delta_1$. Now let $g : A \longrightarrow A$ be defined by $g(x) = x \boxplus (\boxminus e \boxplus a_0)$. Since g is (δ_1, δ_2)-constrained, we have

$$d(a_0, a_1)\ =\ d\big(g(e), g(f(e))\big)\ <\ \delta_2\ <\ \mathrm{diameter}(A).$$

This contradiction to the choice of a_0 and a_1 completes the proof of the theorem. ∎

For unary operations of the form $x \longmapsto x + a$ and $x \longmapsto x - b$, the constraints in (55) are redundant, since $x - b$ is the same as $x + a$, where $a = -b$. (For the full binary operations, they may not be redundant.) This redundancy may be seen in the proof, in the fact that we applied the constraints (55) only to operations of the form $x + a$. Thus (55) turns out to contain more information than is necessary for the proof.

3.7 Groups of exponent N on $\mathbb{R}$.

In this section we let Γ_N stand for any equational theory whose models are (additively written) groups satisfying $x + \cdots + x \approx 0$ (where x appears N times on the left of this equation). For $N = 2$, this theory was known [36, §3.3.7] to be incompatible with $\mathbb{R}$.

We will use the fact that any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that cycles a set of N elements must have an approximate fixed point, by a minor variation on Theorem 29.

Theorem 27 *Let $\mathbb{R}$ be given any metric that induces the usual topology, and let Γ_N denote group theory with exponent N. Then* $\mu_2^\star(\mathbb{R}, \Gamma_N) \geq \mathrm{radius}(A)$.

Proof. To prove the theorem by contradiction, we may suppose that $\mu_2^\star(A, \Gamma_N) < \mathrm{radius}(A)$. By Definitions (48–49) there are a (discontinuous) exponent-N group operation $\boxplus$ on A, and positive real numbers $\delta_0 \leq \delta_2 < \mathrm{radius}(A)$, such that $(A, \boxplus)$ is 2-constrained by (δ_0, δ_2). Thus there exists a further positive real δ_1 such that

$$\boxplus \text{ is constrained by } (\delta_0, \delta_1) \text{ and by } (\delta_1, \delta_2).$$

Let $\overline{0}$ be the unit element of the group $(\mathbb{R}, \boxplus)$. Since $\delta_2 < \mathrm{radius}(A)$, there exists $a \in \mathbb{R}$ such that $d(a, \overline{0}) > \delta_2$. We consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x \boxplus a$. Clearly $f(\overline{0}) = a$, $f(a) = 2a$, $f(2a) = 3a$, ...and $f((N-1)a) = \overline{0}$. Since f is (δ_0, δ_1)-constrained, (a variant on) Theorem 29 yields $e \in \mathbb{R}$ such that $d(e, f(e)) < \delta_1$. Let $g(x) = (N-1)e \boxplus x$. Since g is (δ_1, δ_2)-constrained, we have

$$d(\overline{0}, a) \;=\; d(g(e), g(f(e))) \;<\; \delta_2,$$

in contradiction to our choice of a. This contradiction completes the proof of the theorem. ▮

3.8 $A = Y$, the triode; $\Sigma =$ lattice theory.

Let A, B, C, D be four non-collinear points in the Euclidean plane, with D in the interior of $\triangle ABC$. Our space Y is defined to be the union of the three (closed) segments AD, BD and CD, called *legs*, with the topology inherited from the plane. In fact, in order to give Y a definite metric d, we will further require that $\triangle ABC$ be equilateral with D at its center, and that each leg have unit length. We then let d be the metric of the plane, as inherited by Y.

For §3.8 we let Σ consist of axioms for lattice theory (expressed in terms of $\wedge$ and $\vee$). It was proved by A. D. Wallace in the mid-1950's (see [38, *Alphabet Theorem*, page xx] for a statement of the result) that the triode Y is not compatible with Σ. Taking $\mu_3^\star$ as defined in §3.2, we shall prove the sharper result that

Theorem 28 $\mu_3^\star(Y, \Sigma) \geq 0.5$.

Before proving Theorem 28 we state and prove one Lemma. It is our discontinuous approximate replacement for the Intermediate Value Theorem.

Lemma 29 *Suppose that f maps a convex subset of $\mathbb{R}$ into $\mathbb{R}$, and that f is (δ, ε)-constrained for some $\delta, \varepsilon > 0$. If $a < c$ and s is between $f(a)$ and $f(c)$, then there exists b with $a \leq b \leq c$ and with $d(f(b), s) < \varepsilon/2$.*

Proof. Consider a finite sequence of reals that begins with a and ends with c, and such that every step is smaller than δ. The corresponding function-values take steps smaller than ε while traversing the interval between $f(a)$ and $f(c)$. Moreover s must lie in one of these f-intervals smaller than ε; hence the conclusion. ▮

Proof of Theorem 28. For a contradiction, suppose that $\mu_3^\star(Y, \Sigma) < 0.5$. By Definitions (48–49) there exist (discontinuous) lattice operations $\overline{\wedge}$ and $\overline{\vee}$ on Y, and positive real numbers δ_0 and δ_3, such that $(Y, \overline{\wedge}, \overline{\vee})$ is 3-constrained by (δ_0, δ_3), and moreover such that $\delta_3 < 0.5$. Thus there exist further positive reals δ_1, δ_2 such that

$$\overline{\wedge} \text{ and } \overline{\vee} \text{ are each constrained by } (\delta_0, \delta_1), \text{ by } (\delta_1, \delta_2), \text{ and by } (\delta_2, \delta_3). \quad (56)$$

By Lemma 12 we may assume that $\delta_0 \leq \delta_1 \leq \delta_2 \leq \delta_3$.

Part 1. We shall prove that either $A \mathbin{\overline{\wedge}} D$ or $A \mathbin{\overline{\vee}} D$ lies in the leg AD (and similarly for B and D). If $A \mathbin{\overline{\wedge}} D$ does not lie in AD, then we have D between $A = A \mathbin{\overline{\wedge}} A$ and $A \mathbin{\overline{\wedge}} D$. Consider the function of meeting with A, viz. $X \longmapsto X \mathbin{\overline{\wedge}} A$. Since D is between two of its values, we may apply Lemma 29 to obtain $E \in AD$ with $d(A \mathbin{\overline{\wedge}} E, D) < \delta_1$. Now, joining with A, we have $d(A \mathbin{\overline{\vee}} (A \mathbin{\overline{\wedge}} E), A \mathbin{\overline{\vee}} D) < \delta_2$; by Σ this may be simplified to $d(A, A \mathbin{\overline{\vee}} D) < \delta_2$. In other words $A \mathbin{\overline{\vee}} D$ lies in AD as desired.

Part 2. We shall prove that either $A \mathbin{\overline{\wedge}} D$ or $A \mathbin{\overline{\vee}} D$ lies within δ_2 of A (and similarly with A changed to B and to C). By Part 1, the three points A, $A \mathbin{\overline{\wedge}} D$ and $A \mathbin{\overline{\vee}} D$ lie along a segment. Without loss of generality we have $A \mathbin{\overline{\vee}} D$ between $A = A \mathbin{\overline{\wedge}} A$ and $A \mathbin{\overline{\wedge}} D$ on that segment. If we consider the function of meeting with A (as in Part 1), then Lemma 29 again yields E such that $d(A \mathbin{\overline{\wedge}} E,\ A \mathbin{\overline{\vee}} D) < \delta_1$. As in Part 1, joining with A again yields $d(A, A \mathbin{\overline{\vee}} D) < \delta_2$.

Part 3. From Part 2, we may assume, without loss of generality, that

$$d(A, A \mathbin{\overline{\vee}} D) < \delta_2 \quad \text{and} \quad d(B, B \mathbin{\overline{\vee}} D) < \delta_2. \tag{57}$$

(The two vertices might be A and C or B and C, and both operations might be meets rather than joins, but surely two of the three end-vertices must have the same pattern.)

Part 4. $A \mathbin{\overline{\vee}} B$ cannot lie in both of the disjoint sets $[A, D)$ and $[B, D)$ (these are two of the legs, minus the endpoint D). Without loss of generality we will assume that $A \mathbin{\overline{\vee}} B$ is not in $[B, D)$. Therefore D lies between $B = B \mathbin{\overline{\vee}} B$ and $A \mathbin{\overline{\vee}} B$. By a familiar argument (this time involving joining with B) we obtain $d(D, E \mathbin{\overline{\vee}} B) < \delta_1$ for some E. Now meeting with B, we have $d(B \mathbin{\overline{\wedge}} D, B) < \delta_2$.

Part 5. Taking the conclusion of Part 4, and joining with D, yields $d(D, D \mathbin{\overline{\vee}} B) = d(D \mathbin{\overline{\vee}} (B \mathbin{\overline{\wedge}} D), D \mathbin{\overline{\vee}} B) < \delta_3$. Combining this with (57), we have

$$d(B, D) \ \leq\ d(B, D \mathbin{\overline{\vee}} B) \ +\ d(D \mathbin{\overline{\vee}} B, B) \ \leq\ \delta_3 + \delta_2 \ <\ 1.$$

Here we have a contradiction to the fact that $d(B, D) = 1$ (see the start of §3.8), which completes the proof of Theorem 28. ∎

3.8.1 Comments on the proof of Theorem 28

Our estimate is made for $\chi_3^\star$ only. This proof does not yield information on χ_3. The reason is that we must be able to estimate the effect of applying $\overline{\wedge}$ and $\overline{\vee}$ to the points called E in Parts 1, 2 and 4. Such an E is not necessarily in the range of our operations, so that we cannot make the necessary estimate simply by applying some term-operation $\overline{\tau}$.

Comparing this proof with the corresponding proof for λ that appears in [36], we note a lot of similarity. It seems as though we could work out a theory for a composite measure, including the possibility of limited jumps and of approximate satisfaction. This will have to await a later date.

3.9 A very special space.

Section under construction.

For α any real number with $0 < \alpha < 1$, we define

$$\begin{aligned} A_\alpha \;=\; \{\, (x,y,z) \,:\, & x^2 + y^2 = 1 \;\& \\ & (-\alpha x \le z \le \alpha x \;\text{ or }\; z = 0)\,\} \;\subseteq\; \mathbb{R}^3. \end{aligned}$$

This space may easily be sketched as a subset of a cylinder in $\mathbb{R}^3$. We give it the rectangular or taxicab metric in that space: $d(\mathbf{x},\mathbf{y}) = \sum |x_i - y_i|$. (Notice that the spaces A_α are all homeomorphic one to another, but the homeomorphisms are not isometries.)

Notice that for $(x,y,z) \in A_\alpha$ with $x < 0$, the definition yields $z = 0$ as the only possible value for z. Thus A_α contains the circle

$$C \;=\; \{\, (x,y,0) \,:\, x^2 + y^2 = 1 \,\},$$

and for negative x, these are the only points in A_α. For positive x, there are other points (x,y,z). The farthest of these from the circle C are $(1,0,\alpha)$ and $(1,0,-\alpha)$. Thus α is a measure of how far A_α extends away from the circle C.

For future reference, we define a closed curve f in A_α (for $0 \le t \le 2\pi$), as follows:

$$f(t) \;=\; \begin{cases} (\cos t,\ \sin t,\ -\alpha\cos t) & \text{if } \cos t \ge 0 \\ (\cos t,\ \sin t,\ 0) & \text{if } \cos t \le 0. \end{cases}$$

(f maps, so to speak, to the lower periphery of A_α.) Concerning the point $A = (1, 0, \alpha) \in A_\alpha$, We proved in [36, *loc. cit.*] that A has distance at least 2α from every point $B = (\cos t, \cdots)$ in the image of f.

For this section, when we refer to a closed curve, we mean a continuous map with domain S^1. We view our $f(t)$ as such a closed curve, by representing S^1 as $\mathbb{R}/2\pi$, and relying on the periodicity of the trigonometric functions. Finally when we say that closed curves $g_0(t)$ and $g_1(t)$ in A are homotopic, we mean that there exists a map $G{:}\, S^1 \times [0, 1] \longrightarrow A$ such that $G(t, i) = g_i(t)$ for $t \in S^1$ and $i \in \{0, 1\}$.

For $\mu_2^\star(A_\alpha, \Gamma) \geq \alpha$, we restrict Γ to be a set of equations in the operations $+$ and $-$ that contains the three equations

$$(x + y) + (-y) \;\approx\; x \tag{58}$$

$$x + 0 \;\approx\; x; \qquad 0 + x \;\approx\; x. \tag{59}$$

We shall prove that for such a Γ,

$$\mu_2^\star(A_\alpha, \Gamma) \;\geq\; \alpha. \tag{60}$$

The proof is by contradiction. To this end, we assume now that $\mu_2^\star(A_\alpha, \Gamma) < \alpha$.

By definition of $\mu_2^\star$, there exist positive reals $\delta_0 \leq \delta_1 \leq \delta_2$ and (discontinuous) operations $\boxplus, \boxminus$ obeying Γ, with both operations constrained by (δ_0, δ_1) and by (δ_1, δ_2). Let $0 = A_0, A_1, A_2, \ldots, A_k = A$ be a sequence of members of A_α, with $d(A_i, A_{i+1}) < \delta_0$ for each i. By the (δ_0, δ_1)-constraint, we have

$$d(A_i \boxplus f(t),\; A_{i+1} \boxplus f(t)) \;<\; \delta_1$$

for each i and each t. By (a version of) Lemma 19, for each i there is a continuous curve $\overline{g}_i{:}\, S^1 \longrightarrow A_\alpha$

$$d(\overline{g}_i(t), A_i \boxplus f(t)) \;<\; \delta_1 \tag{61}$$

for each i and each t. Combining the last two inequalities, we have

$$d(\overline{g}_i(t), \overline{g}_{i+1}(t)) \;<\; 3\,\delta_1$$

for each i and each t. Assuming now that $3\delta_1$ is less than the diameter of a circle in our model, we know that all the functions g_i are homotopic. In particular g_k is homotopic to $f(t)$ and hence maps onto $\{f(t) : \pi/2 \leq t \leq 3\pi/2\}$. By surjectivity and (61) there exists t_0 such that

$$d(A \boxplus f(t_0),\, (-1,0,0)) \;<\; \delta_1. \tag{62}$$

By reasoning similar to that for (62), except using the first equation of (59), we have successively nearby points $0 = B_0, B_1, \ldots, B_m = f(t_0)$ and maps h_i, each close to the corresponding $f(t) \boxplus B_i$. In this way, we arrive at the existence of s_0 with

$$d(f(s_0) \boxplus f(t_0),\, (-1,0,0)) \;<\; \delta_1. \tag{63}$$

Now from (62) and (63), from the group equation (58), and from the (δ_1, δ_2)-constraint, we have

$$\begin{aligned} &d\big(A, (-1{,}0,0) \boxminus f(t_0)\big) \\ &\qquad = d\big((A \boxplus f(t_0)) \boxminus f(t_0), (-1,0,0) \boxminus f(t_0)\big) \;<\; \delta_2; \\ &d\big(f(s_0), (-1{,}0,0) \boxminus f(t_0)\big) \\ &\qquad = d\big((f(s_0) \boxplus f(t_0)) \boxminus f(t_0), (-1,0,0) \boxminus f(t_0)\big) \;<\; \delta_2. \end{aligned}$$

By the triangle inequality, $d(A, f(s_0)) < 2\delta_2 < 2\alpha$. This contradicts our earlier assertion that every point in the image of f is at least 2α from A, and thus the the result is proved.

Also note that A_α is compatible with H-space theory. (For the moment this is left to the reader.)

ONE FINAL PIECE would be to work out the diameter of A_α. Then check out the range of normalized values of λ. It looks like we would still get a large range of λ-values.

References

[1] J. F. Adams, On the non-existence of elements of Hopf-invariant one, *Mathematische Annalen* (2) 72 (1960), 20–104.

[2] R. M. Anderson, “Almost” implies “near,” *Transactions of the American Mathematical Society* 296 (1986), 229–237.

[3] M. Ja. Antonovskiĭ and A. V. Mironov, On the theory of topological l-groups. (Russian. Uzbek summary) Dokl. Akad. Nauk UzSSR 1967, no. 6, 6–8. MR 46 #5528.

[4] H.-J. Bandelt and J. Hedlíková, Median algebras, *Discrete Mathematics* 45 (1983), 1–30.

[5] K. Borsuk, Drei Sätze über die n-dimensionale euklidische Sphäre, Fund. Math. 20 (1933), 177–190.

[6] ——— *Theory of retracts*, Monografie Matematyczne, Tom 44, Państwowe Wydawnictwo Naukowe, Warsaw, 1967, 251 pages.

[7] R. Bott, On symmetric products and the Steenrod squares, *Annals of Mathematics* 57 (1953), 579–590.

[8] B. F. Caviness and J. R. Johnson, eds., *Quantifier Elimination and Cylindrical Algebraic Decomposition. Texts and Monographs in Symbolic Computation*, New York: Springer-Verlag, 1998.

[9] T. H. Choe, On Compact Topological Lattices of Finite Dimension, *Transactions of the American Mathematical Society* 140 (1969), 223–237.

[10] A. H. Clifford, Connected ordered topological semigroups with idempotent endpoints, I. *Transactions of the American Mathematical Society* 88 (1958), 80–98.

[11] G. E. Collins, Quantifier Elimination for the Elementary Theory of Real Closed Fields by Cylindrical Algebraic Decomposition, *Lecture Notes in Computer Science* 33 (1975), 134-183.

[12] J. Dugundji and A. Granas, *Fixed-point theory I*, Państwowe Wydawnictwo Naukowe, Warsaw, 1982.

[13] W. M. Faucett, Compact semigroups irreducibly connected between two points; Topological semigroups and continua with cutpoints. *Proceedings of the American Mathematical Society* 6 (1955), 741–756.

[14] O. C. García and W. Taylor, *The lattice of interpretability types of varieties*, Memoirs of the American Mathematical Society, Number 305, iii+125 pages. MR 86e:08006a.

[15] S. T. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965.

[16] I. M. James, Multiplication on spheres, I, II, *Proceedings of the American Mathematical Society* 13 (1957), 192–196 and *Transactions of the American Mathematical Society* 84 (1957), 545–558.

[17] I. Kaplansky, Topological Rings, *American Journal of Mathematics* 69 (1947), 153–183.

[18] R. J. Koch and A. D. Wallace, Admissibility of semigroup structures on continua, AMS Transactions 88 (1958), 277–287.

[19] ——— Topological semilattices and their underlying spaces, Semigroup Forum 1 (1970), 209–223.

[20] J. Matoušek, *Using the Borsuk-Ulam Theorem, Lectures on Topological Methods in Combinatorics and Geometry*, Springer-Verlag, Berlin, 2000.

[21] R. McKenzie, On spectra, and the negative solution of the decision problem for identities having a non-trivial finite model, *J. Symbolic Logic* 41 (1975), 186–196.

[22] W. D. Neumann, On Mal'cev conditions, *Journal of the Australian Mathematical Society* 17 (1974), 376–384.

[23] M. B. Pour-El and J. I. Richards, *Computability in analysis and physics.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989. xii+206 pp.

[24] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[25] H. Steinlein, Borsuk's theorem and its generalizations and applications: a survey, pages 166–235 in A. Granas, ed., *Méthodes topologiques en analyse non linéaire (Séminaire scientifique OTAN)*, Les Presses de l'Université de Montréal, 1985.

[26] A. Tarski, Sur les ensembles définissables de nombres réels, *Fundamenta Mathematicae* 17 (1931), 210–239.

[27] ——— A Decision Method for Elementary Algebra and Geometry, RAND Corp. monograph, 1948.

[28] ——— *A Decision Method for Elementary Algebra and Geometry*, 2nd ed. Berkeley, CA: University of California Press, 1951.

[29] W. Taylor, Characterizing Mal'cev conditions, *Algebra Universalis* 3 (1973), 351–397.

[30] ——— The fine spectrum of a variety, *Algebra Universalis* 5 (1975), 263–303.

[31] ——— Varieties obeying homotopy laws, *Canadian Journal of Mathematics* 29 (1977), 498–527.

[32] ——— *The clone of a topological space*, Volume 13 of *Research and Exposition in Mathematics*, 95 pages. Heldermann Verlag, 1986.

[33] ——— *The Geometry of Computer Graphics*, Wadsworth and Brooks-Cole, Pacific Grove, CA, 1992.

[34] ——— Spaces and equations, *Fundamenta Mathematicae* 164 (2000), 193–240.

[35] ——— Equations on real intervals, *Algebra Universalis* 55 (2006), 409–456.

[36] ——— Approximate satisfaction of identities, 98 pp., 2010, to appear.

[37] J. van Mill, A topological group having no homeomorphisms other than translations. *Transactions of the American Mathematical Society* 280 (1983), 491–498.

[38] A. D. Wallace, The structure of topological semigroups, *Bulletin of the American Mathematical Society* 61 (1955), 95–112.

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