

Possible classification of finite-dimensional compact Hausdorff topological algebras

by

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With gratitude to Professor Don Pigozzi

Abstract. The time may be right to ask for an enumeration or classification of all the reasonably small topological algebras. Here the terrain is surveyed, and a program of investigation is proposed.

0 Introduction.

This paper is part of a continuing investigation—see the author’s papers [48] (1986), [49] (2000), [50] (2006), [52] (2010), and [53] (2011)—into the compatibility relation, which is described in (2) below. A. D. Wallace defined the inquiry succinctly in 1955, when he asked [58, p. 96], “Which spaces admit what structures?” By “structure,” he meant the existence of continuous operations identically satisfying certain equations: e.g., the structure of a topological group or a topological lattice, and so on. Here we survey the current state of knowledge in this area, especially for finite simplicial complexes, and ask some refined versions of Wallace’s question.

0.1 Role of this investigation in mathematics.

We see such topological structure as fundamental to mathematics. Generally, there seem to be two ways to put an infinite number system on a firm logical and practical foundation. The first is through recursion, the method that underlies calculations with integers and rational numbers (for example). The second is through continuity of operations, as in the real number system \mathbb{R} , where we can meaningfully calculate values like $\sin(\pi/3)$ or $\sqrt[3]{2}$ by approximation. Here we are looking into the possibilities of calculation through continuity.

For these two modalities to be available in any practical way, we must, at the very least, be talking about a topological space that has a countable dense subset—the first axiom of countability. Thus, for example, discrete

topological spaces are compatible with any consistent set of equations, but a discrete topological algebra is of no use in pursuing calculations through continuity. Discrete spaces play no role in the rest of this paper.

0.2 Limited focus of this investigation.

The investigation emphasizes topological algebras that satisfy some non-trivial (in the sense of §6.1) equations. We do not wish to diminish the importance of other algebras—for instance some topological semigroups are of paramount importance, and yet the associative law remains trivial in the sense of §6.1. But the main unknown, under the present focus, is the identical satisfaction of equations.

In keeping with the ideas of §0.1, we limit our attention to first-countable spaces. In fact we mostly limit ourselves to very simple spaces: finite simplicial complexes. First of all, much of the variation and mystery of the subject already lies in this seemingly elementary domain. Secondly, as soon as we admit infinite polyhedra, any consistent set of equations can be modeled (see §5 below).

Most of our attention will be to connected simplicial complexes. Every finite model of an equation-set Σ may be viewed as a topological model of Σ based on a zero-dimensional complex, which is of course disconnected. These models, and any other disconnected topological models, are generally not under consideration. This exclusion is most important, and will be reiterated, in the open questions of §9.

0.3 Layout of the paper.

The beginning sections lay out concepts relevant to our results and problems. The reader who is proficient in these notions may proceed to §7, and examine the examples shown there, which are the heart of the paper. After that, one might read the comments and questions that arise in §§8–9. These sections comprise the main new material of this paper.

0.4 Acknowledgments

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of equational logic, the central focus of this article. This is a field where he clarified many issues; see for instance his articles with W. J. Blok [8] and with K. Palasińska [39].

1 Satisfaction of equations by operations.

Readers with some familiarity with logic or general algebra can easily skip §1, at least on a first reading.

1.1 Terms and equations

A *similarity type* consists of a set T and a function $t \mapsto n_t$ from T to natural numbers. A *term of type* $\langle n_t : t \in T \rangle$ is recursively either a variable or a formal expression of the form $F_t(\tau_1, \dots, \tau_{n_t})$ for some $t \in T$ and some shorter terms τ_i of this type. An **equation** of this type is a formal expression $\tau \approx \sigma$ for terms τ and σ of this type. A formal equation makes no assertion, but merely presents two terms for consideration. The actual mathematical assertion of equality is made (in a given context) by the satisfaction relation \models (see (1) below). We mostly work with a set Σ of equations, finite or infinite, and tacitly assume that there is a similarity type $\langle n_t : t \in T \rangle$ such that each equation in Σ is of this type.

Examples: In almost any concrete example of interest, the foregoing formality is not really necessary for comprehension. It suffices to give, for example, the familiar assertion that “ Σ has two binary operations \wedge and \vee ,” instead of insisting on e.g. “ $\wedge = F_1$ and $\vee = F_2$, where $T = \{1, 2\}$ and $n_1 = n_2 = 2$.” In such a simplified context, formal equations may be written like ordinary equations in standard lattice theory. (One should be careful, however, about writing informal terms such as “ $x \wedge y \wedge z$,” which is not meaningful in the absence of associativity.)

1.2 Satisfaction of equations

Given a set A and for each $t \in T$ a function $\overline{F}_t : A^{n(t)} \rightarrow A$ (called an *operation*), we say that the operations \overline{F}_t *satisfy* Σ and write

$$(A, \overline{F}_t)_{t \in T} \models \Sigma, \tag{1}$$

iff for each equation $\sigma \approx \tau$ in Σ , both σ and τ evaluate to the same function when the operations \overline{F}_t are substituted for the symbols F_t appearing in σ and τ .

A structure of the form $(A, \overline{F}_t)_{t \in T}$ (as in (1)) is called an *algebra*; the set A is called the *universe* of $(A, \overline{F}_t)_{t \in T}$. Often, if the context permits, we denote $(A, \overline{F}_t)_{t \in T}$ by the bold letter corresponding to the letter denoting the universe, and so on. Then we can express (1) by saying that the algebra \mathbf{A} *satisfies* (or *models*) Σ .

In discussing satisfaction of equations, it is standard (and helpful) to distinguish as we have done between an operation symbol F_t and an operation \overline{F}_t interpreting the symbol.¹ Nevertheless in keeping with the last part of §1.1 above, we may sometimes omit the bar from familiar operations like $+$, \wedge and so on.

2 Compatibility of a space with a set of equations.

Given a *topological space* A and a set of equations Σ , we write²

$$A \models^{\text{ctn}} \Sigma, \tag{2}$$

and say that A and Σ are *compatible*, iff there exist *continuous* operations \overline{F}_t on A satisfying Σ , in other words iff (1) holds with continuous operations \overline{F}_t . (Here we mean that each function $\overline{F}_t: A^{n_t} \rightarrow A$ should be continuous relative to the usual product topology formed on the direct power A^{n_t} .)

One may also read (2) as “ A topologically models Σ ,” or “ A continuously models Σ .”

Given operations \overline{F}_t on a topological space A , we may of course form the algebra $\mathbf{A} = (A; \overline{F}_t)$; if in addition each \overline{F}_t is continuous, we may say that this \mathbf{A} is a *topological algebra based on the space* A . With this vocabulary, the compatibility relation (2) may be rephrased as follows: *there exists a topological algebra satisfying Σ that is based on the space* A .

¹Obviously the simple notation \overline{F}_t will be inadequate if more than one operation interprets F_t in a given discussion.

²In a context that contains little possibility of confusing (1) and (2) one may omit the designation “ctn” for continuous modeling, and simply write $A \models \Sigma$ for (2). We have done this in [48], [49], [50], [52] and [53].

Thus, for instance, A is compatible with group theory if and only if A is the underlying space of some topological group. If desired, one may skip to §7 on a first reading, for a much longer list of examples.

3 General results on compatibility.

While the definitions are simple, the relation (2) remains mysterious. Two results, one fifty years old, the other recent, point toward this mystery. First, the algebraic topologists have long known that the n -dimensional sphere S^n is compatible with H-space theory ($x \cdot e \approx x \approx e \cdot x$) if and only if $n = 1, 3$ or 7 . (There is a large literature on this topic; one landmark paper was Adams [1].) Second, for $A = \mathbb{R}$, the relation (2) is algorithmically undecidable for Σ — see [50]; i.e. there is no algorithm that accepts as input an arbitrary finite Σ , and outputs the truth value of (2) for $A = \mathbb{R}$. In any case, (2) appears to hold only sporadically, and with no readily discernible pattern.

The mathematical literature contains numerous but scattered further examples of the truth or falsity of specific instances of (2). The author’s earlier papers [48], [49], [50], [52] collectively refer to most of what is known, and in fact many of the earlier examples illustrating incompatibility are recapitulated throughout the long article [52]. The present article will cover most of the known compatibilities for finite simplicial complexes.

4 Compatibility and the interpretability lattice.

Here we review a notion introduced by W. D. Neumann in 1974 (see [38]), and further studied by O. C. García and W. Taylor in 1981 (see [20]). (In 1968 J. Isbell [25] constructed quasi-orderings of arbitrary categories; the ordering we use—i.e. Neumann’s—can be seen as arising from Isbell’s ordering.)

4.1 Interpretability as an order.

We introduce an order on the class of all sets $\Sigma, \Gamma \dots$ of equations, as follows. Let us suppose that the operation symbols of Σ are F_s ($s \in S$), and the operation symbols of Γ are G_t ($t \in T$). We say that Σ is *interpretable in* Γ ,

and write $\Sigma \leq \Gamma$, iff there are terms α_s ($s \in S$) in the operation symbols G_t such that, if $(A, \overline{G}_t)_{t \in T}$ is any model of Γ , then $(A, \overline{\alpha}_s)_{s \in S}$ is a model of Σ .

A typical example has Γ defining Boolean algebras and Σ defining Abelian groups with operations $+$, $-$, 0 . Here the terms α_+ and α_- are both equal to the so-called *symmetric difference* $\alpha_+(x, y) = \alpha_-(x, y) = (x \wedge (\neg y)) \vee (y \wedge (\neg x))$. (It is worthwhile noticing that this interpretation is neither one-one on the class of all BA's nor onto the class of all Abelian groups.) For further concrete examples, see §7.2.5, §7.5.4 and §8.3 below.

Strictly speaking, we need to observe that, so far, our relation \leq is not anti-symmetric. It is easy to find distinct sets Σ_1 and Σ_2 that are mutually related by \leq . It is however a quasi-order, and when we speak of an order, or a least upper bound, and so on, we are referring to the order formed in the usual way modulo the equivalence relation that includes the pair (Σ_1, Σ_2) whenever the two Σ_i are as above, i.e. $\Sigma_1 \leq \Sigma_2 \leq \Sigma_1$. We generally will leave this fine point unexpressed.

4.2 Interpretability defines a lattice

Given sets Σ and Γ of equations, there is a set $\Sigma \wedge \Gamma$ that is a greatest lower bound of Σ and Γ in the \leq -ordering of §4. For a precise definition, including an axiomatization of $\Sigma \wedge \Gamma$, the reader may consult R. McKenzie [37] or O. García and W. Taylor [20].

We describe here the (algebraic) models of $\Sigma \wedge \Gamma$. We make the inessential assumption that the operation symbols of Σ (resp. Γ) are F_s ($s \in S$) (resp. F_t ($t \in T$)), with S disjoint from T . The operation symbols of $\Sigma \wedge \Gamma$ are F_j ($j \in S \cup T$), together with a new binary operation symbol p . The models of $\Sigma \wedge \Gamma$ are precisely all algebras isomorphic to a product $\mathbf{A} \times \mathbf{B}$, where

- (i) $\mathbf{A} \models \Sigma$.
- (ii) $\mathbf{B} \models \Gamma$.
- (iii) For each $t \in T$, $\mathbf{A} \models F_t(x_1, \dots, x_n) \approx x_1$ and $\mathbf{A} \models p(x_1, x_2) \approx x_1$.
- (iii) For each $s \in S$, $\mathbf{B} \models F_s(x_1, \dots, x_n) \approx x_1$ and $\mathbf{A} \models p(x_1, x_2) \approx x_2$.

For instance, to see that $\Sigma \wedge \Gamma \leq \Sigma$, we define an interpretation as follows. For $s \in S$, the term α_s is $F_s(x_1, x_2, \dots)$; for $t \in T$, the term α_t is x_1 , and $\alpha_p(x_1, x_2)$ is x_1 . For any $(A, \overline{F}_s)_{s \in S}$, the interpreted algebra $(A, \overline{F}_j)_{j \in S \cup T}$

clearly has the form $\mathbf{A} \times \mathbf{B}$ described above, with \mathbf{B} a singleton. *Mutatis mutandis*, we have $\Sigma \wedge \Gamma \leq \Gamma$. For the fact that $\Sigma \wedge \Gamma$ is a *greatest* lower bound, let us suppose that Φ is a set of equations with operation symbols G_i ($i \in I$), and that there are terms α_i (resp. β_i) interpreting Φ in Σ (resp. Γ). It is not hard to see that the terms $p(\alpha_i, \beta_i)$ will interpret Φ in $\Sigma \wedge \Gamma$.

Continuing our inessential assumption that $S \cap T = \emptyset$, it is not hard to see that $\Sigma \cup \Gamma$ is a *least upper bound*³ of Σ and Γ , which we may also denote $\Sigma \vee \Gamma$.

4.2.1 The spaces compatible with $\Delta \wedge \Gamma$.

For an arbitrary topological space C , and for arbitrary sets Σ and Γ of equations, $C \models^{\text{ctn}} \Sigma \wedge \Gamma$ if and only if C is homeomorphic to a product space $A \times B$, where $A \models^{\text{ctn}} \Sigma$ and $B \models^{\text{ctn}} \Gamma$.

Proof. We continue the notation and assumptions of §4.2. Given C equal (or homeomorphic) to the product space $A \times B$, and topological algebras \mathbf{A} and \mathbf{B} modeling Σ and Γ , respectively, we extend \mathbf{A} and \mathbf{B} to $(S \cup T)$ -algebras using clauses (iii) and (iv) of §4.2, and then take their product. This product has underlying space C and models $\Sigma \wedge \Gamma$. Thus C is compatible with $\Sigma \wedge \Gamma$.

Conversely, given a space C compatible with $\Sigma \wedge \Gamma$, the existence of the corresponding spaces A and B is proved in [20, Proposition 5, p. 22].

4.3 For each space, compatibility defines an ideal of the lattice.

Let A be an arbitrary topological space. We will see that *the class of all Σ that are compatible with A forms an ideal in the interpretability lattice*. In this report we shall denote this ideal by $I(A)$.

First, let us suppose that $A \models^{\text{ctn}} \Gamma$ and that $\Sigma \leq \Gamma$. By definition of \models^{ctn} , there is a topological algebra $(A, \overline{G}_t)_{t \in T}$, that models Γ . By the definition of $\Sigma \leq \Gamma$, we have that $(A, \overline{\alpha}_s)_{s \in S}$ models Σ , with S and T disjoint. The operations $\overline{\alpha}_s$ are built using composition from the continuous operations \overline{G}_t , hence are continuous themselves. In other words, $(A, \overline{\alpha}_s)_{s \in S}$ is a topological algebra that models Σ . Therefore $A \models^{\text{ctn}} \Sigma$, as desired.

³For any set A of sets of equations (with all their types disjoint), the union $\bigcup A$ is a least upper bound of the family A . However the lattice is a proper class, and there may exist a subclass that has no join.

Next, given Σ and Γ , each compatible with the space A , we must show that $\Sigma \vee \Gamma = \Sigma \cup \Gamma$ (described at the end of §4.2) is compatible with A . This result is immediate from the definitions involved.

Thus each space A yields an ideal in the interpretability lattice, which is denoted $I(A)$.

4.3.1 $I(A)$ is principal: the theory Σ_A .

Given a space A , we define a theory Σ_A as follows. For each continuous function $\mu : A^n \rightarrow A$, there is an n -ary operation symbol F_μ . For $1 \leq i \leq n < \omega$ we let $\pi_i^n : A^n \rightarrow A$ be the continuous function defined by $\pi_i^n(a_1, \dots, a_n) = a_i$. For a continuous function $\lambda : A^n \rightarrow A^m$, and for $1 \leq i \leq m$, we let λ_i denote the continuous function $\pi_i^m \circ \lambda$. Now we define Σ_A to consist of the equations

$$F_{\pi_i^n}(x_1, \dots, x_n) \approx x_i \quad (3)$$

$$F_\mu(F_{\lambda_1}(x_1, \dots, x_m), \dots, F_{\lambda_n}(x_1, \dots, x_m)) \approx F_{\mu \circ \lambda}(x_1, \dots, x_m). \quad (4)$$

for all $1 \leq i \leq n$ and all pairs of continuous functions $A^m \xrightarrow{\lambda} A^n \xrightarrow{\mu} A$. We shall see that Σ_A generates the ideal $I(A)$. (This was asserted without proof in [20, Proposition 11].)

It is not hard to see that $A \models^{\text{ctn}} \Sigma_A$: for the requisite topological algebra on A , one simply takes $\overline{F_\mu} = \mu$, for all $A^n \xrightarrow{\mu} A$. Thus $\Sigma_A \in I(A)$, and so the principal ideal generated by Σ_A is a subset of $I(A)$. For the reverse inclusion, let us consider an arbitrary $\Sigma \in I(A)$. This means that $A \models^{\text{ctn}} \Sigma$; i.e., there exists a topological algebra $\mathbf{A} = (A, \overline{G_s})_{s \in S}$ satisfying Σ . We construct an interpretation of Σ in Σ_A as follows. For each $n = 1, 2, \dots$ and each n -ary $s \in S$ we define the term α_s to be $F_\lambda(x_1, \dots, x_n)$, where λ is the operation $\overline{G_s} : A^n \rightarrow A$. It is not hard to see that the terms α_s form an interpretation of Σ in Σ_A . (The proof uses the given fact that $A \models^{\text{ctn}} \Sigma$, together with an inductive argument on all the subterms of terms appearing in Σ .) Thus $\Sigma \leq \Sigma_A$; i.e., Σ lies in the desired principal ideal. Thus the two sets are equal: $I(A)$ is the principal ideal generated by Σ_A .

Nevertheless, the equation-set Σ_A is large and unwieldy. In a few cases, we know a simple finite generator of $I(A)$. For example if A is any of the spaces mentioned in §6.1 below, then $I(A)$ is the principal ideal generated by $F(x) \approx F(y)$, as one may easily see from the results cited in §6.1. For

such a space A and a finite exponent k , the ideal $I(A^k)$ is also principal, as is proved in [49, Theorem 2 and §11.4].

If A is the one-sphere S^1 , then $I(A)$ is the principal ideal generated by Abelian group theory [49, Theorems 42–43]. If A is the dyadic solenoid, then $I(A)$ is the principal ideal generated by the theory of $\mathbb{Z}[1/2]$ -modules⁴ [49, Theorems 46–47]. For both $I(S^1)$ and $I(S)$ (S the solenoid), the ideal generator can be taken as a finite set of equations.

For any given A , we generally do not know whether $I(A)$ is generated by a single finite set Σ of equations. Further speculations on the generators (e.g. whether there exists such a Σ that is a recursive set of equations) remain equally opaque.

4.3.2 Unions of chains

If Λ_2 is a set of equations, and if Λ_1 is an arbitrary subset of Λ_2 , then $\Lambda_1 \leq \Lambda_2$ in our lattice. The converse is far from true: even if $\Lambda_1 \leq \Lambda_2$, it may be true that the two Λ_i have disjoint similarity types. Thus the consideration of the union of a chain (under inclusion) is not central to our main topic. Nevertheless, we include one small observation.

The ideal $I(A)$ may not be closed under unions of chains. One may have $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \subseteq \dots$, with $A \models^{\text{ctn}} \Lambda_k$ for each k , but $A \not\models^{\text{ctn}} \Lambda = \bigcup \Lambda_k$. Such Λ_k —with A taken as a closed interval of the real line—may be seen in §7.2.2 below. (The example comes from [47, p. 525].) In other words, every finite subset of Λ lies in $I(A)$, but Λ does not.

Incidentally, this example shows that while the union of a chain (under inclusion) is an upper bound of that chain, it need not be a *least* upper bound.

4.3.3 Sometimes $I(A)$ is a prime ideal.

If C is a product-indecomposable space, then Σ_C is meet-prime, which further implies that $I(C)$ is a prime ideal in the lattice.

Proof. Suppose that $\Sigma \wedge \Gamma \leq \Sigma_C$. By §4.2.1, C is homeomorphic to a product space $A \times B$ with $A \models^{\text{ctn}} \Sigma$ and $B \models^{\text{ctn}} \Gamma$. Since C is product-indecomposable, either A or B is a singleton. Thus C is homeomorphic to A or to B . In the former case $\Sigma \leq \Sigma_C$, and in the latter $\Gamma \leq \Sigma_C$.

⁴ $\mathbb{Z}[1/2]$ is the ring of all rationals with denominator a power of 2.

4.3.4 The complementary filter.

If A is product-indecomposable, then by §4.3.3 the complement of $I(A)$ is a *filter* which we will denote $F(A)$. This consists of all Σ that are not compatible with A . By §4.3.2, when A is a closed interval of \mathbb{R} , there is a set $\Sigma \in F(A)$ such that no finite subset of Σ is in $F(A)$. Hence this filter is generally not Mal'tsev-definable (see [45]). It is unknown whether it might be subject in some cases to a syntactic definition (such as a weak Mal'tsev condition). (Exception: in [45] we gave a Mal'tsev condition describing $F(A)$ for A a two-element discrete space.)

4.4 The ideal of a product of two spaces.

Let A and B be topological spaces. We saw in §4.2.1 that *if A and B are compatible with Σ and Γ , respectively, then the product space $A \times B$ is compatible with the meet $\Sigma \wedge \Gamma$.*

Now if we have $\Delta \in I(A) \cap I(B)$, then Δ is compatible with both A and B . By the previous paragraph, $A \times B$ is compatible with $\Delta \wedge \Delta$, which is co-interpretable with Δ , hence equal to Δ in the lattice. In other words, we now have $I(A) \cap I(B) \subseteq I(A \times B)$. They are not generally equal. For instance, if A is not homeomorphic to a perfect square, then, though $I(A \times A)$ will contain the perfect-square equations (§7.5 below), $I(A) \cap I(A)$ will not.

5 Note on free topological algebras.

Let A be a metrizable space, and Σ a finite or countable set of equations that is consistent (does not entail $x \approx y$). Considering A purely as a set, one of course has the free algebra $\mathbf{F}_\Sigma(A)$; it has A embedded as a subset, and satisfies the equations Σ . In 1964, S. Świerczkowski showed [44] how to topologize (even metrize) $\mathbf{F}_\Sigma(A)$ in such a way that A is embedded as a subspace, and each operation is continuous. Thus in particular, Σ is compatible with the topological space that underlies $\mathbf{F}_\Sigma(A)$.

We mention this example of compatibility to illustrate the fact that, beyond consistency, there is no apparent constraint on the Σ that can appear in the compatibility relation $A \models^{\text{ctn}} \Sigma$, even when we require A to satisfy the first countability axiom, as described in §0.1.

The topological spaces defined by Świerczkowski are large and non-compact. If A is a CW-complex, then so is $\mathbf{F}_\Sigma(A)$ (see Bateson [6]), but the

construction of the algebra $\mathbf{F}_\Sigma(A)$ is inherently infinitary, and so the complex structure is, to our knowledge, almost always infinite. It is only for very special and somewhat trivial equation-sets Σ that $\mathbf{F}_\Sigma(A)$ turns out to be finitely triangulable.⁵ By way of contrast, our main proposal in this report (see §6 below) will be to consider $I(A)$ when A is a finite simplicial complex. Here the compatible Σ appear to be more limited.

6 Restrictions on compatibility for a finite complex.

We turn our attention toward compact Hausdorff spaces, mostly limiting it to those connected spaces that have the form of a finite simplicial complex. The latter form the most down-to-earth geometric corner of topology, and hopefully our understanding could be rooted there. For simplicity, we will refer to a space as *finite* if it has a finite triangulation, and as *compact* if it is compact and Hausdorff.

Our starting point is the impression that the various Σ that have been observed on finite connected complexes often fall into several broad categories: lattice-related equations, group-related equations, $[k]$ -th power equations, simple equations, and a few special equation-sets. In this section we review a few incompatibility results that make such a division slightly more plausible.

6.1 Undemanding sets of equations.

A set Σ is called *undemanding* if it can be satisfied on some set of more than one element—equivalently, on any set—by taking each operation to be either a projection function or a constant function. Such operations are continuous, and hence if Σ is undemanding, then Σ is compatible with every space A . Taylor proved [49] a sort of converse result: that *many finite spaces A have the property that $A \models^{\text{ctn}} \Sigma$ only for undemanding sets Σ .* (The proofs apply algebraic topology of the sort used in analyzing H-spaces, as mentioned in §3.)

In other words, such an A is compatible with *no* interesting Σ ! The list of such A contains, for instance, all spheres other than S^1 , S^3 and S^7 , the Klein bottle, the projective plane, a one-point join of two 1-spheres, and several

⁵For example, for Σ defining G -sets over a finite group G .

others. It appears that the proofs could be extended to many other finite A , but no one has carried out this job. From these considerations it appears that for many finite A , perhaps for most, the situation is totally arid.

For such a space A , the ideal $I(A)$ is the smallest ideal containing every undemanding set of equations. In fact this ideal is generated by the single undemanding equation $f(x) \approx f(y)$ (which postulates the existence of a constant function).

Among those Σ that have at least one constant function, any undemanding Σ is least in the interpretability ordering.

(For a k -dimensional counterpart of “undemanding,” see §7.5.3 below.)

6.1.1 “Undemanding” is an algorithmic property.

There is an easy algorithm that accepts any finite set Σ of equations as input, and halts with output 1 or 0, depending whether Σ is undemanding. We will describe this algorithm informally.

Given Σ , it has a finite similarity type $n:T \rightarrow \mathbb{Z}$. We now consider an arbitrary finite set K of equations of the form

$$F_t(x_1, \dots, x_{n(t)}) \approx x_j \tag{5}$$

or of the form

$$F_t(x_1, \dots, x_{n(t)}) \approx C, \tag{6}$$

where our formal language has been augmented to include a single new constant symbol C . For each $t \in T$, our K must include a single equation involving F_t ; that one equation must be either Equation (6) or one instance of Equation (5)—thereby choosing a value of j in that equation.

If σ is any term in the language of Σ , the equations in K will immediately imply either $\sigma \approx x_j$ for some unique j , or $\sigma \approx C$. For each $\sigma \approx \tau$ occurring in Σ , we may check whether σ and τ both reduce to the same x_j or else both to C . If this happens for all equations in Σ , we say that K is consistent with Σ .

We now undertake to do this for all of the (finitely many) possibilities for K . If one K turns out to be consistent with Σ , we may say Σ is undemanding. Otherwise, all such K turn out to be inconsistent with Σ , in which case we conclude that Σ is demanding.

If all the operations are for instance binary, then the number of sets K is obviously $3^{|T|}$; we see therefore that the algorithm is exponential in $|T|$.

Nevertheless, in many cases of interest $|T|$ is small, and the algorithm is easily carried out. The reader is invited to try his/her hand at equations (25–26) in §7.6.2.

6.2 Not both groups and semilattices

The incompatibility of (non-trivial) compact Hausdorff spaces with lattice-ordered groups was proved by M. Ja. Antonovskii and A. V. Mironov [3] in 1967. Therefore, of course, if Σ is an axiom-set for LO-groups, we will not have $A \models^{\text{ctn}} \Sigma$ for any non-trivial finite space A . If A is also connected, we in fact have the stronger conclusion—a consequence of J. D. Lawson and B. Madison [33]—that A is not compatible both with group theory and with lattice theory.

Of course, from the perspective of the present investigation, it would be very desirable to have a stronger version of this result, where group theory and lattice theory are replaced by lower elements of the interpretability lattice. In any case we will use §6.2 as a rough guide in organizing §7 which follows, separating group-like topological algebras from lattice-like ones. (In §7.3, however, we find some examples that lie on the overlap.)

6.2.1 Proof of the assertion in §6.2.

In §6.2.1, all citations refer to items in Lawson and Madison [33] (1970).

Our proof (that no nontrivial connected compact Hausdorff space is compatible with both group theory and lattice theory) is by contradiction. We will assume that A is a non-trivial compact connected Hausdorff space, and that there are continuous operations \wedge , \vee , \cdot and $^{-1}$ such that (A, \wedge, \vee) is a topological lattice and $(A, \cdot, ^{-1})$ is a topological group.

By compactness, (A, \wedge, \vee) has 0 and 1. Therefore (A, \wedge) is certainly a non-trivial compact connected nontrivial idempotent semigroup with 0. By Corollary 2.12 on page 135, each maximal idempotent of (A, \vee) is marginal. Therefore A has at least one marginal element.

Marginal elements are defined in Definition 1.1 on page 129. Peripheral elements are defined in Definition 1.2. On the bottom of page 129 we are told that every marginal element is peripheral, with a reference to another paper of Lawson and Madison. Also in Definition 1.2 we have that a point is *inner* if and only if it is not peripheral. Therefore our space A contains one point that is not inner.

On the other hand, Theorem 1.6 on page 130 tells us that if A is a finite dimensional locally compact Hausdorff space, then the set of inner points of A is dense in A . And hence non-empty. Thus our space A has one point that is inner and one point that is not inner.

On the other hand $(A, \cdot, {}^{-1})$ is a topological group, and hence A is homogeneous. Therefore, either all points of A are inner or no points of A are inner. This contradiction completes the proof.

7 $A \models^{\text{ctn}} \Sigma$ for Σ non-trivial and A given by a finite complex.

We present essentially all the examples that we know for sure. Our rough division into types of Σ is partly based on the results mentioned in §6.2.

7.1 Σ related to group theory.

7.1.1 Grouplike algebra on spheres.

We look at one strengthening of group theory (i.e. higher in the lattice), and two weakenings.

The one-dimensional sphere S^1 is compatible with Abelian group theory. (The Abelian group may be modeled as the set of unit-modulus complex numbers under multiplication, or as the set of orthogonal 2×2 real matrices of determinant 1.) On the other hand, S^3 is the space underlying the group of unit quaternions, which is not Abelian. (R. Bott proved in 1953 that S^3 is not compatible with Abelian group theory—see [10].) S^7 has the multiplication of unit octonians. With this multiplication, S^7 forms an H-space (see §3), which in fact satisfies the alternative laws (associativity on all two-generated subalgebras). S^7 does not, however, have a multiplication forming an associative H-space (monoid), as was proved by I. M. James in 1957 (see [27]). Thus, for k any positive integer with $k \neq 1, 3, 7$, we have the set inequalities

$$\begin{aligned} I(S^k) &= I(S^k) \cap I(S^7) \cap I(S^3) \cap I(S^1) \subset \\ &I(S^7) \cap I(S^3) \cap I(S^1) \subset I(S^3) \cap I(S^1) \subset I(S^1). \end{aligned}$$

The four ideals are separated by H-space theory, associative H-space theory (monoids) (or by group theory), and Abelian group theory, using the results

cited here and in §3. (Recall that $I(S^1)$ is described near the end of §4.3.1, and $I(S^k)$ is described in §6.1.)

7.1.2 Other groups.

There are various compact Lie groups (orthogonal, special orthogonal, and so on). Matrix multiplication (which is inherently continuous) is often the basic operation. Their various underlying spaces appear to be very sparse among the class of all compact manifolds. The underlying spaces of compact Lie groups may be finitely triangulated (see [34] and references given there).

7.2 Σ derived from lattice theory.

7.2.1 Distributive lattices (with 0 and 1).

A real interval $[a, b]$ has a well-known distributive lattice structure. Therefore each simplex $[a, b]^n$ has compatible distributive lattice operations, as does any of its sublattices. In the compact realm, every compatible lattice has a zero (bottom) and a one (top). The compact subuniverses of $([0, 1], \bar{\wedge}, \bar{\vee}, 0, 1)^2$ appear to be limited in their possible shapes, although a full description of the limitations has not yet been discovered.

For the first such limitation, we note that in 1959 Dyer and Shields proved [15] that every compact connected metric topological lattice is contractible and locally contractible. In particular a finite graph (one-dimensional complex) with a lattice structure must be acyclic.

For a further limitation on finite graphs, we note that if A is an acyclic non-linear finite graph, i.e., a one-dimensional compact connected simplicial complex that does not define a line segment—e.g. if A is a Y-shaped space—then A is not compatible with lattice theory, and hence cannot be such a subuniverse of $[0, 1]^2$. See §7.2.4 below for further incompatibilities.

The aforementioned result, of non-compatibility between lattice theory and an acyclic non-linear finite graph A , comes in essence from Wallace [58], although only a weaker theorem is stated, and without proof. (See the “Alphabet Theorem” on page 107 of [58].) We shall thus include a short proof here, by contradiction. So suppose that $\mathbf{A} = (A, \bar{\wedge}, \bar{\vee})$ is a topological lattice. Since our space A is compact, \mathbf{A} has a 0 and a 1. Since A is not homeomorphic to a segment, there is a point E of A such that $A \setminus E$ has at least three components $S_0, S_1, S_2, \dots, S_N$. Clearly there is one S_k that

contains neither 0 nor 1.

We now take $P \in S_k$; clearly $P \neq E$. We consider the map $X \mapsto X\bar{\vee}P$. It maps 0 to P and 1 to 1. By connectedness, and the fact that every path from P to 1 must pass through E , we see that our map must have E in its range. In other words, for some X , $P\bar{\vee}X = E$. Thus $P \leq E$.

A dual argument shows that $P \geq E$; by anti-symmetry $P = E$. This is the contradiction that establishes our result.

It is worth mentioning, for future reference (§8.2) that the lattice operations on a real interval are *piecewise linear*:

$$x\bar{\wedge}y = x \text{ if } x \leq y; \quad y \text{ if } x \geq y, \quad (7)$$

and similarly for join.

7.2.2 One can go higher in $I([0, 1])$.

For this section, we let Λ_0 be a finite equational axiom system for distributive lattice theory with zero and one. For each integer $n \geq 1$ we let Λ_n be Λ_0 augmented with a unary operation symbol f and constant symbols a_1, \dots, a_n , and extended with the following axioms:

$$\begin{aligned} a_1 \wedge a_2 &\approx a_1, & a_2 \wedge a_3 &\approx a_2, & \dots, & a_{n-1} \wedge a_n &\approx a_{n-1} \\ f(0) &\approx 0, & f(a_1) &\approx 1, & f(a_2) &\approx 0, & f(a_3) &\approx 1, & \dots \\ f(1) &\approx 1 \text{ if } n \text{ is even, } & & & & & & & 0 \text{ otherwise.} \end{aligned}$$

One easily checks that, in the interpretability lattice

$$\Lambda_0 < \Lambda_1 < \dots < \Lambda_n < \Lambda_{n+1} < \dots$$

(For non-interpretability of Λ_{n+1} in Λ_n , we note that, modulo equational deductions, Λ_n has only $n + 2$ constant terms, whereas any interpretation of Λ_{n+1} will require $n + 3$ logically distinct constant terms.)

Compatibility of Λ_n with a closed interval is easiest if we use the interval $[-1, 1]$. Then the desired function \bar{f} can be taken as the Chebyshev polynomial T_{n+1} of degree $n + 1$. (Or one can simply take \bar{f} to be piecewise linear as specified by our equations.)

We therefore have an ω -chain of sets in the ideal $I([0, 1])$, going upward from the theory of distributive lattices with zero and one (§7.2.1).

7.2.3 Lattices (with 0 and 1).

Lattice theory lies strictly below modular lattice theory in the interpretability lattice. Nevertheless, we do not know any space B that is compatible with lattice theory (with or without zero and one), and yet is not compatible with modular lattice theory. (It is possible that, for $\Sigma =$ lattice theory, and for suitably chosen A , the space of the free algebra $\mathbf{F}_\Sigma(A)$ (see §5) might be such a B . Furthermore, in [7], G. Bergman has suggested several finite spaces that are compatible with lattice theory, but may fail to be compatible with modular lattice theory.)

We do know a space B that separates modular lattice theory from distributive lattice theory in this manner, namely such that B is compatible with modular lattice theory but not with distributive lattice theory. If B is the union of three closed 2-simplices along one common edge, then B has the announced properties. The proof, by G. Bergman and W. Taylor, appears in [7, §3.1]. See also §9.4.7 for a more general question about the separation of one variety from another by the compatibility relation

In the nineteen-fifties A. D. Wallace conjectured that every compact, connected topological lattice $(L, \bar{\wedge}, \bar{\vee})$ is distributive. This was disproved in 1956 by D. E. Edmondson [16], who gave a non-modular example⁶ with L homeomorphic to $[0, 1]^3$. (Of course this space is compatible with distributive lattice theory.) Wallace's conjecture holds for $L = [0, 1]^2$ (see [2]) and for modular lattices with $L = [0, 1]^3$ (see [21]).

7.2.4 Semilattices (with 0 and 1).

By contrast with §7.2.1, every finite tree (see §7.2.1) is compatible with semilattice theory—as may be seen in §3.7 of W. Taylor [48]—even semilattice theory with **0** and **1**. (And it is not hard to see from the proof that the semilattice operation may be taken to be piecewise linear, i.e. simplicial.)

Taylor proved in 1977 (see [47]) that if A is a topological semilattice, then the homotopy group $\pi_n(A, a)$ is trivial for every $n \geq 1$ and every $a \in A$. (In 1965 (see [11]) D. R. Brown had obtained the same conclusion for a different equation-set: $x \wedge x \approx x, x \wedge 0 \approx 0 \wedge x \approx 0$. In the compact case Brown's result already applies to a semilattice A , since A will then have a zero.)

⁶G. Bergman has recently found a simpler construction of an example with these properties—see §4.2 of [7].

(In 1959 Dyer and Shields had proved [15] that every compact connected metric topological lattice is contractible and locally contractible.)

7.2.5 Majority operations and median algebras.

It is well known that if (A, \wedge, \vee) is a lattice, then the derived operation defined by the term

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \quad (8)$$

satisfies the *majority equations*

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x. \quad (9)$$

Thus the majority equations lie below lattice theory in the interpretability lattice, and so are compatible with the space of any topological lattice (§7.2.3).

Moreover the majority equations are also compatible with the finite trees mentioned in §7.2.1 and §7.2.4. The idea (due to M. Sholander in 1954—see [43]) is very simple. Given such a tree T , for any two points $a, b \in T$, there is a smallest connected subset containing the two, which will be denoted $[a, b]$. Moreover, Sholander proved, for any three points a, b and $c \in T$, the intersection $[a, b] \cap [b, c] \cap [c, a]$ is a singleton. Taking its lone member as the value of $\bar{m}(a, b, c)$, we obtain a symmetric, continuous operation $\bar{m}: T^3 \rightarrow T$ that satisfies Equation (9). Finally, we remark here that the \bar{m} so defined on a tree T satisfies a stronger set of equations, the axioms of *median algebra*—see e.g. the 1983 treatise by Bandelt and Hedlíková [5], or the 1980 treatise by Isbell [26].

In fact, it was proved in 1979–82 by J. van Mill and M. van de Vel [56, 57] that, among finite-dimensional spaces, the ones compatible with the majority equations are precisely the absolute retracts. (They refer to a continuous majority operation as a “mixer.”)

7.2.6 Multiplication with one-sided unit and zero.

One very weak consequence of semilattice theory with zero and one—or of ring theory—is the following set of two equations:

$$x \wedge 0 \approx 0, \quad x \wedge 1 \approx x.$$

These equations lie quite low in the interpretability lattice; hence it is not hard to find contractible spaces that model them. (For example see e.g. §7.2.1 and §7.2.4.) On the other hand, as was mentioned in §3.6 of [48], it is easy to see that if A is a path-connected finite space compatible with these equations, then A is contractible.

7.3 Below both groups and lattices: H-spaces

H-spaces (multiplication with two-sided unit element), and *associative H-spaces* (otherwise known as *monoids*) were mentioned in §7.1.1; their theories lie well below group theory. It is interesting to note that both of these theories also lie below \wedge -semilattices with $\mathbf{1}$ (§7.2.4).

For example, we may let $\mathbf{S}^1 = (S^1, \cdot, e)$ denote the circle group, with unit element e . We may let $\mathbf{I} = (I, \cdot, 1)$ denote the unit interval, where \cdot is the usual semilattice operation, and 1 is the top element, and also the unit element for this algebra. Then $\mathbf{S}^1 \times \mathbf{I}$ is also an associative H-space, with two-side unit element $(e, 1)$. One may easily check that

$$P = \{(u, v) \in S^1 \times I : u = e \text{ or } v = 0\}$$

is a subuniverse of $\mathbf{S}^1 \times \mathbf{I}$. It is homeomorphic to the pointed union of the pointed spaces (S^1, e) and $(I, 0)$. (In other words, the space P is homeomorphic to the letter P.) Thus the space P is, for example, compatible with monoids. (This example appeared in [48], and is derived from work of Wallace [58].)

If B is any compact metrizable space that is an absolute retract among metric spaces, then B is compatible with H-spaces (see §3.2.3 of W. Taylor [51]).

If A is compatible with the Mal'tsev equations, then A is compatible with Σ_H —see §7.4.3 below.

7.3.1 A mysterious theorem.

Algebraic topology has a lot to say about—and methods concerning—H-spaces. As one sample result, we mention this:

J. R. Harper proved in 1972 (*inter alia*, see [22]) that if \mathbf{A} is a finite connected H-space, then the homotopy group $\pi_4(A)$ obeys the law $x^2 = 1$. ($A = S^3$ is an example of such an H-space with $\pi_4(A) \neq 0$.)

7.3.2 Digression on homotopy

One may examine *satisfaction up to homotopy*. In the case of H-space theory, one asks for a continuous map $F : A^2 \rightarrow A$, and an element $e \in A$, such that the maps $x \mapsto F(x, e)$ and $x \mapsto F(e, x)$ are each homotopic to the identity map $x \mapsto x$.

Many of the spaces of interest in this investigation are contractible, and therefore model $x \approx y$ up to homotopy — which means that they model any Σ up to homotopy. We therefore will not pursue this notion here, except to report that if A is a CW-complex, and if A is compatible with H-space theory up to homotopy, then⁷ in fact A is an H-space.

7.4 Σ consisting of simple equations.

If A is an absolute retract in the class of metric spaces, and if Σ is a consistent set of simple equations, then A is compatible with Σ (see Taylor [51]). A term σ is *simple* iff there is at most one operation symbol F_t in σ , appearing at most once. An equation $\sigma \approx \tau$ is simple iff both terms σ and τ are simple. For example, the majority equations (9) are simple,

For absolute retracts, consult works by Borsuk [9] and Hu [24]. For example, the finite trees defined in §7.2.1 are absolute retracts (among, e.g., metric spaces). Thus the result of this section extends the compatibility results of §7.2.5.

Moreover, if Σ is a consistent set of simple equations in a finite similarity type, and if A is an absolute extensor (see [24]) in the class of completely regular spaces, then there is a topological algebra $\mathbf{A} = (A, \dots F_t \dots)$ whose simple identities are precisely the simple consequences of Σ (see [51, Theorem 7(b)]). This is the rare case where we have some control over equations *not* holding in an algebra \mathbf{A} constructed in this report.

If Σ is a finite (or recursive) set of simple equations, and if A is a finite (or recursive) tree, and if we know some computable (hence continuous) operations modeling Σ on a closed interval, then there are computable (hence continuous) operations modeling Σ on A . The method is described in §4.2 of [51]; it probably can be extended to an arbitrary absolute retract which is a finite complex. A special case of the method is given in detail in §7.4.2 below. (For computability of real functions, see [41].)

⁷See Whitehead [59, Theorem III.4.7, page 117]. In fact Whitehead proves this under assumptions weaker than what we have stated here.

7.4.1 Minority equations on a closed interval.

As an example of simple equations, we consider the *ternary minority equations*

$$q(x, x, y) \approx q(x, y, x) \approx q(y, x, x) \approx y. \quad (10)$$

A closed real interval $[a, b]$ is well known to be an absolute retract, so by §7.4 there exists a ternary operation \bar{q} on $[a, b]$ satisfying (10). We can, however, define such an operation directly, without reference to §7.4. A minority operation \bar{q} may be defined by the following two conditions:

- (i) If $u \leq v \leq w$, then $\bar{q}(u, v, w) = u - v + w$.
- (ii) \bar{q} is completely symmetric in its three variables.

It is worth noting that there is a single formula defining this \bar{q} , namely

$$\bar{q}(u, v, w) = u \wedge v \wedge w - \bar{m}(u, v, w) + u \vee v \vee w, \quad (11)$$

where \bar{m} is the ternary majority operation defined in Equation (8).

If A is a space homeomorphic to an interval, then of course our definition of \bar{q} may be transferred to A by laying down coordinates. Any non-linear change of coordinates will effect the values of the resulting $\bar{q}^A : A^3 \rightarrow A$, but Equation (10) will not be affected. Linear changes of coordinates will not affect any values of \bar{q}^A .

(A very different—and more complicated— \bar{q} was described in Equation (71) of §9.3 of [50].)

7.4.2 Minority equations on a tree.

Here we will illustrate one way to satisfy the minority equations (10) on a simple tree—as mentioned in §7.2.1 and §7.2.4 and §7.2.5. Specifically let Y stand for the Y-shaped space that is formed by joining three closed intervals with the amalgamation of one endpoint each. Y is an absolute retract; hence compatible with the minority equations (10) by §7.4. We can, however, define such an operation directly, without reference to §7.4.

Let Y_1, Y_2, Y_3 be the three subsets of Y that can be formed by joining two out of three of the constituent intervals. The significant facts about the Y_i are these:

- (i) Each element of Y belongs to at least two of the Y_i .
- (ii) Each Y_i is homeomorphic to an interval, and hence has a minority operation \bar{q}_i by §7.4.1.
- (iii) For each i there is a continuous function \bar{p}_i retracting Y onto Y_i .

Let \bar{m} be a majority operation on Y —whose existence is assured by §7.2.5. We now define $\bar{Q}: Y^3 \rightarrow Y$ as follows:

$$\begin{aligned} \bar{Q}(a, b, c) = \bar{m}(\bar{q}_1(\bar{p}_1(a), \bar{p}_1(b), \bar{p}_1(c)), \\ \bar{q}_2(\bar{p}_2(a), \bar{p}_2(b), \bar{p}_2(c)), \bar{q}_3(\bar{p}_3(a), \bar{p}_3(b), \bar{p}_3(c))). \end{aligned}$$

From points (i)–(iii) it follows easily that \bar{Q} is a minority operation on Y .

As mentioned at the end of §7.4, the methods of §4.2 of [51]—a recursive invocation of the methods here—will allow one to construct a ternary majority operation on any finite tree.

7.4.3 Mal'tsev operations.

The *Mal'tsev equations* are

$$p(x, x, y) \approx p(y, x, x) \approx y. \quad (12)$$

One may say that their study initiated the investigation of relative strengths of equation-sets, ultimately leading to the lattice of §4.2. Equations (12) obviously lie below the minority equations (10) in the lattice. Thus Mal'tsev operations are found on a closed interval and on any finite tree (by §7.4.1 and §7.4.2).

Moreover, in any group $(A, \cdot, {}^{-1})$, the formula

$$\bar{p}(a, b, c) = a \cdot b^{-1} \cdot c \quad (13)$$

defines a Mal'tsev operation on A . Therefore S^1, S^3 have Mal'tsev operations.

As a sort of hybrid example, we look at the cylinder $[a, b] \times S^1$. It has a Mal'tsev operation as does any (necessarily closed) subset onto which the entire space $[a, b] \times S^1$ retracts. (E.g. a belt around the cylinder that is pinched so as to be one-dimensional in spots and two-dimensional in other spots.)

Notice that any space A that has a Mal'tsev operation is an H-space (§7.3): if $\bar{p}: A^3 \rightarrow A$ satisfies (12), and if $e \in A$, we may then define a multiplication $x \cdot y = \bar{p}(x, e, y)$. This multiplication has e as a two-sided unit.

7.4.4 Two-thirds minority operations.

The *two-thirds minority equations* are

$$t(x, x, y) \approx t(y, x, x) \approx y; \quad t(x, y, x) \approx x. \quad (14)$$

Clearly they lie higher in the lattice than the Mal'tsev equations (12). (Strictly higher because they are not interpretable in Abelian group theory—cf. §4.3.1.) Equations (14) also lie above the ternary majority equations (9): $\bar{p}(x, y, z) = \bar{t}(x, \bar{t}(x, y, z), z)$ defines a majority operation, as one may easily check. Equations (14) play a significant role in the study of *arithmetic varieties* (varieties that are congruence-permutable and congruence-distributive)—see e.g. A. F. Pixley [40].

Of course an interval $[a, b]$ or a tree has a continuous two-thirds minority operation by the general results of §7.4. One choice for \bar{t} on $[a, b]$ is this:

$$\bar{t}(u, v, w) = u - \bar{m}(u, v, w) + w,$$

whose form has much in common with Equations (11) and (13). For the tree Y one may use the method of §7.4.2.

7.5 Σ defining $[k]$ -th powers.

For each set Σ of equations, and for each $k = 2, 3, \dots$, there exists a set $\Sigma^{[k]}$ with the following property: an arbitrary topological space A is compatible with $\Sigma^{[k]}$ if and only if there exists a space B such that $B \models^{\text{ctn}} \Sigma$ and A is homeomorphic to the direct power B^k . If Σ is finite (resp. recursive, resp. r.e., etc.), then $\Sigma^{[k]}$ may be taken as finite (resp. recursive, resp. r.e., etc.).

From the definition (which we have skipped) it is immediate that $\Sigma^{[k]} \geq \Sigma$ in our lattice (§4). The theory $\Sigma^{[k]}$ was developed in 1975 by R. McKenzie [37]; see also [46, pp. 268–269] or §10.1 of [50]. The connection of $\Sigma^{[k]}$ with topological spaces was perhaps first noted in [20].

Obviously, if $\Gamma^{[k]} \in I(A)$, then $\Gamma \in I(A)$ and A is a k -th power. The converse is false,⁸ even when $k = 2$: take A to be a four-element discrete space, and Γ to be the $\Sigma^{[2]}$ of §7.5.2 below. Then $\Gamma \in I(A)$ and A is a square, but $\Gamma^{[2]} \notin I(A)$ (for then, by §7.5.2, A would be the square of a square, which it is not).

⁸This observation thanks to G. M. Bergman.

In this context, of course, every example adduced so far in §7 yields further examples for each $k = 2, 3, \dots$. If B is known to be compatible with Σ , then $A = B^k$ is known to be compatible with $\Sigma^{[k]}$. In the opposite direction, we of course need to know all possible factorizations of A as homeomorphic to some B^k . If each such B is incompatible with Σ , then we know that A is not compatible with $\Sigma^{[k]}$. (This of course includes the case where no such factorization exists.)

7.5.1 The operations of $\Sigma^{[k]}$.

Given operations $\overline{F}_1, \dots, \overline{F}_k$ on a set B , each of arity nk , we may define an n -ary operation \overline{F} on the set B^k as follows:

$$\begin{aligned} \overline{F}((b_1^1, \dots, b_1^k), \dots, (b_n^1, \dots, b_n^k)) \\ = (\overline{F}_1(b_1^1, \dots, b_n^k), \dots, \overline{F}_k(b_1^1, \dots, b_n^k)). \end{aligned} \quad (15)$$

Clearly, if B has a topology, and if each \overline{F}_j is continuous, then \overline{F} is continuous. One may think of $\Sigma^{[k]}$ as having one such n -ary operation symbol for each k -tuple of nk -ary term operations of Σ . More usually, we take only these special cases as fundamental operations of $\Sigma^{[k]}$:

$$\overline{H}((b_1^1, \dots, b_1^k), \dots, (b_k^1, \dots, b_k^k)) = (b_1^1, \dots, b_k^k); \quad (16)$$

$$\overline{d}((b_1^1, \dots, b_1^k)) = (b_1^2, \dots, b_1^k, b_1^1); \quad (17)$$

$$\overline{G}_t((b_1^1, \dots, b_1^k), \dots, (b_n^1, \dots, b_n^k)) = (\overline{F}_t(b_1^1, \dots, b_n^1), \dots, \overline{F}_t(b_1^k, \dots, b_n^k)), \quad (18)$$

where \overline{F}_t ($t \in T$) are the fundamental operations of Σ . (The other operations (15) can be formed from these.)

7.5.2 Squares— Σ empty and $k = 2$.

For Σ empty, $\Sigma^{[2]}$ may be axiomatized as:

$$\begin{aligned} H(x, x) &\approx x \\ H(x, H(y, z)) &\approx H(x, z) \approx H(H(x, y), z) \\ d(d(x)) &\approx x \\ d(H(x, y)) &\approx H(d(y), d(x)). \end{aligned}$$

If A is the square of another space B , i.e. $A = B^2$ with the product topology, then A is compatible with $\Sigma^{[2]}$ in the following manner. We define operations \overline{H} and \overline{d} on B^2 via

$$\overline{H}((b_1, b_2), (b_3, b_4)) = (b_1, b_4) \quad (19)$$

$$\overline{d}((b_1, b_2)) = (b_2, b_1), \quad (20)$$

for all $b_1, \dots, b_4 \in B$. These operations are obviously continuous, and it is easy to check by direct calculations that they obey $\Sigma^{[2]}$. Thus $B^2 \models^{\text{ctn}} \Sigma^{[2]}$. (Equations (19–20) are special cases of Equations (16–17) above.)

Conversely, it is not hard to prove that if A is any space with continuous operations H' and d' modeling this $\Sigma^{[2]}$, then there exist a space B and a bijection $\phi : A \rightarrow B^2$ that is both a homeomorphism of spaces and an isomorphism of (A, H', d') with $(B^2, \overline{H}, \overline{d})$, with \overline{H} and \overline{d} defined as above. (One begins by defining B to be the subspace $\{a \in A : d'(a) = a\}$.)

Thus this $\Sigma^{[2]}$ is compatible with A if and only if A is homeomorphic to a square, as claimed.

7.5.3 Squares of spaces.

If B is any space and \overline{F}_i is a $2n$ -ary operation on B ($i = 1, 2$), then—as a special case of (15)—one has an n -ary operation \overline{F} defined on $A = B^2$ as follows:

$$\overline{F}((b_1^1, b_1^2), \dots, (b_n^1, b_n^2)) = (\overline{F}_1(b_1^1, \dots, b_n^1), \overline{F}_2(b_1^2, \dots, b_n^2)). \quad (21)$$

If each \overline{F}_i is continuous, then \overline{F} is continuous.

For most spaces B , there are many continuous operations on B^2 besides those described in Equation (21), and there is little or no real restriction on the equation-sets that may be compatible with B^2 . But for certain spaces, notably those described at the start of §6.1, the compatible equation-sets are very limited.

In Theorem 2 of [49] it was proved that if B is one of these spaces, such as a figure-eight or a sphere S^n ($n \neq 1, 3, 7$), then a set Σ is compatible with B^2 only if Σ is interpretable in operations of type (21), where each \overline{F}_i is either a coordinate projection function or a constant. Such a set Σ is called 2-undemanding. There is an algorithm to determine if a finite set is 2-undemanding.

The reader may easily imagine the corresponding definition for k -undemanding sets. Then a k -th power such as $(S^n)^k$ ($n \neq 1, 3, 7$) is compatible with Σ only if Σ is k -undemanding.

7.5.4 Below squares in the interpretability lattice.

Let Γ consist of the single equation

$$(x \star y) \star (y \star z) \approx y. \quad (22)$$

In the context of §7.5.2, if we define

$$x \star y = d(H(y, x)), \quad (23)$$

then it is not hard to check that Equation (22) follows from the equations $\Sigma^{[2]}$ of §7.5.2. In other words, Γ is interpretable in $\Sigma^{[2]}$ (where Σ is empty). Therefore, by §7.5.2 and by §4.3, if A is the square of another space B , then $A = B^2 \models^{\text{ctn}} \Gamma$.

In fact, if we apply the definition (23) to our operations \bar{d} and \bar{H} of §7.5.2, we obtain the following concrete definition of a continuous $\bar{\star}$ modeling Γ on any square B^2 :

$$(b_1, b_2) \bar{\star} (b_3, b_4) = (b_2, b_3). \quad (24)$$

(And the fact that $(B^2, \bar{\star}) \models \Gamma$ can be reconfirmed by an easy calculation.)

Thus (22) is an example of an equation that is 2-undemanding (§7.5.3) but is not undemanding (§6.1).

(This discussion of Γ and $\bar{\star}$ is due in part to T. Evans [17]. Equation (22) was also discussed on pages 202–203 of [49].)

7.5.5 A special case: $A = \mathbb{R}^k$.

We mentioned at the start of §7.5 that one might need to know all topological factorizations of A as a power B^k in order to assess the truth of $B \models^{\text{ctn}} \Sigma^{[k]}$. There is one case where all such factorizations are known, namely $A = \mathbb{R}^k$. If \mathbb{R}^k is homeomorphic to B^k for some B , then B is homeomorphic to \mathbb{R} [50, Lemma 29] (based on F. B. Jones and G. S. Young [28]). (In other words, topologically the space \mathbb{R}^k has unique k -th roots.)

The remarks in §7.5 now yield that, *for any finite k and any set Σ of equations, $\Sigma^{[k]}$ is compatible with \mathbb{R}^k if and only if Σ is compatible with \mathbb{R} .* (This result first appeared as [50, Corollary 30].)

Few other k -th power spaces are known to have unique k -th roots, and so the result stated here cannot be generalized very far. It does, however, hold for powers $[0, 1]^k$.

7.5.6 The $[k]$ -th root of a theory.

It is possible to turn the tables and define a theory $\sqrt[k]{\Sigma}$ such that an arbitrary space A is compatible with $\sqrt[k]{\Sigma}$ if and only if the space A^k is compatible with Σ . The theory $\sqrt[k]{\Sigma}$ was defined by R. McKenzie in 1975 (see [37]); it is also briefly discussed on page 68 of [20].

We will exhibit $\sqrt[k]{\Sigma}$ for $k = 2$ and Σ the theory of H-spaces (binary multiplication with two-sided unit element, §7.2.6). Here is $\sqrt[2]{\Sigma}$; it has two constants and two 4-ary operations:

$$\begin{aligned} f_1(x_1, x_2, c_1, c_2) &\approx x_1 \\ f_2(x_1, x_2, c_1, c_2) &\approx x_2 \\ f_1(c_1, c_2, x_1, x_2) &\approx x_1 \\ f_2(c_1, c_2, x_1, x_2) &\approx x_2. \end{aligned}$$

It should be clear that if operations \bar{f}_i, \bar{c}_i ($i = 1, 2$) satisfy these equations on A , then one may define an H-space operation on A^2 via

$$\bar{F}((a_1, a_2), (a_3, a_4)) = (\bar{f}_1(a_1, \dots, a_4), \bar{f}_2(a_1, \dots, a_4))$$

for all $a_1, \dots, a_4 \in A$. The general method of defining $\sqrt[k]{\Sigma}$ should be clear from here.

Obviously in general $I(A) \subseteq I(A^k)$, and the reverse inclusion may fail; for example, if $\Sigma = \Delta^{[k]}$ for some Δ and if A is not homeomorphic to a k -th power, then $\Delta^{[k]} \in I(A^k)$ but $\Delta^{[k]} \notin I(A)$ (for Δ taken as, say, the empty theory). In terms of this section, we may equivalently say that if $A \models^{\text{ctn}} \Sigma$, then $A \models^{\text{ctn}} \sqrt[k]{\Sigma}$, but not always conversely.

J. van Mill exhibited [54] a space V such that V is not compatible with group theory, but V^2 is compatible. In other words group theory lies in $I(V^2)$ but not $I(V)$. Nevertheless, the space V seems far from being a finite space, and we do not expect examples of this type to play a big role in the analysis of compatibility for finite spaces.

If Σ is a set of *simple equations* (see §7.4), then $\sqrt[k]{\Sigma}$ is equivalent to Σ in the interpretability lattice, which entails that $I(A^k) = I(A)$ and that

$A^k \models^{\text{ctn}} \Sigma$ implies $A \models^{\text{ctn}} \Sigma$. This theorem was proved in 1983 by B. Davey and H. Werner [14], and about the same time by R. McKenzie [unpublished]. A later proof appears in [20, Prop. 39, p. 69].

7.6 Miscellaneous Σ .

7.6.1 Exclusion of fixed points.

We consider the equation-set

$$F(x, x, y) \approx y; \quad F(\phi(x), x, y) \approx x.$$

If A is a space of more than one element that has the *fixed-point property* (each continuous self-map has a fixed point), then, applying that property to a given $\bar{\phi}$, we clearly see that these equations are not compatible with A . Such spaces include the closed simplex of each finite dimension (Brouwer fixed-point Theorem).

The equations also fail to be compatible with S^1 —which obviously does not have the fixed-point property. Indeed, in §4.3.1, $S^1 \models^{\text{ctn}} \Sigma$ if and only if, in our lattice, Σ lies below the theory of Abelian groups. Thus ϕ will be interpreted as a unary Abelian group operation. All such operations have 0 as a fixed point, and so the fixed-point argument may be applied again.

It is easy to find operations that show \mathbb{R} to be compatible with the equations, but in fact I do not know of any finite complex that is compatible.

In the reverse direction, one may note that in 1959 E. Dyer and A. Shields proved [15] that if A is a finite-dimensional compact connected space compatible with lattice theory, then A has the fixed-point property.

7.6.2 One-one but not onto.

We consider the equations

$$F(x, y, 0) \approx x, \quad F(x, y, 1) \approx y, \tag{25}$$

$$\psi(\theta(x)) \approx x, \quad \phi(\theta(x)) \approx 0, \quad \phi(1) \approx 1, \tag{26}$$

which first appeared in [48, §3.17]. Clearly this set is demanding (see §6.1.1). In a non-singleton model $\mathbf{A} = (A, \bar{F}, \bar{\psi}, \bar{\theta}, \bar{\phi}, \bar{0}, \bar{1})$, Equations (25) imply that $\bar{0} \neq \bar{1}$. The next equation tells us that θ is one-to-one, and the last two tell us (using $\bar{0} \neq \bar{1}$) that the range of θ is not all of A . Every one-one continuous

self-map of the sphere S^n ($n = 1, 2, \dots$) maps onto S^n (for example, by the Invariance of Domain Theorem). Therefore these equations are incompatible with spheres S^n . (For most spheres, we already knew this, by §6.1. For S^1 , S^3 and S^7 , the result is new in this section; for all spheres, the proof here is much easier than the proof referenced in §6.1.)

On the other hand, it is not hard to satisfy the equations with continuous operations on the closed interval $[0, 1]$:

$$\bar{0} = 0, \quad \bar{1} = 1, \quad \bar{F}(a, b, c) = (1 - c)a + cb \quad (27)$$

$$\bar{\theta}(a) = a/2, \quad \bar{\psi}(a) = 2a \wedge 1, \quad \bar{\phi}(a) = (2a - 1) \vee 0. \quad (28)$$

We would also like to see that Equations (25–26) can be satisfied on $[0, 1]$ with (continuous) piecewise linear operations. The operations in Line (28) are already piecewise linear; we need only add a piecewise linear definition for (a new) \bar{F} that satisfies (25). The reader may check that the following definition suffices:

$$\bar{F}(a, b, c) = \begin{cases} a \vee 2c & \text{if } c \leq 1/2 \\ b \vee (2 - 2c) & \text{if } c \geq 1/2. \end{cases}$$

A slight variant of Equations (25–26) replaces Equations (25) with the equations of §7.2.6. These equations serve, again, to separate 0 from 1 in any algebra of more than one element. They are satisfied on $[0, 1]$ by using (28) together with the ordinary meet operation on $[0, 1]$.

7.6.3 Entropic operations on $[0, 1]$.

In 1974 Fajtlowicz and Mycielski (see [18]) considered continuous affine combinations on $[0, 1]$, i.e. functions that have this form:

$$\bar{F}_\alpha(a, b) = \alpha a + (1 - \alpha)b, \quad (29)$$

one such operation for each $\alpha \in [0, 1]$. Such an operation is easily seen to satisfy the equations

$$F_\alpha(x, x) \approx x, \quad F_\alpha(F_\alpha(x, y), F_\alpha(u, v)) \approx F_\alpha(F_\alpha(x, u), F_\alpha(y, v))$$

The first of these is the idempotent law; the second is the *entropic law*. They also proved that if α is transcendental, then $([0, 1], \bar{F}_\alpha)$ satisfies no equations

other than the logical consequences of idempotence and entropicity. These equations are obviously undemanding (see the algorithm in §6.1.1), and hence not interesting for the present investigation.

On the other hand, they proved that if α is algebraic, then $([0, 1], \overline{F}_\alpha)$ satisfies some equations beyond the logical consequences of idempotence and entropicity. Regrettably, I don't know which values of α yield an equation set that is demanding. (E.g. when $\alpha = 1/2$, we have the equation $F_\alpha(x, y) \approx F_\alpha(y, x)$, which renders the equations demanding. I don't know other examples.)

One may further consider two or more \overline{F}_α in the same term. For instance, for any α and β we clearly have the *mixed entropic law*

$$F_\alpha(F_\beta(x, y), F_\beta(u, v)) \approx F_\beta(F_\alpha(x, u), F_\alpha(y, v)).$$

Moreover, one can consider affine combinations with more than two variables. We do not emphasize such combinations, since each of them can be formed by concatenating binary affine combinations. For example, given positive reals μ, ν, λ that sum to 1, if we let $\alpha = \mu + \nu$ and $\beta = \mu/(\mu + \nu)$, then we have

$$\overline{F}_\alpha(\overline{F}_\beta(x, y), z) = \mu x + \nu y + \lambda z.$$

7.6.4 Some twisted ternary operations on $[0, 1]$.

Let $\overline{R}_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation of 3-space through angle θ , whose axis is the line joining $(0, 0, 0)$ to $(1, 1, 1)$. (For example, when $\theta = 2\pi/3$ this rotation cyclically permutes the three positive coordinate axes.) Further, let \overline{m} be the ternary majority operation on \mathbb{R} that is defined by Equation (8) of §7.2.5. Here we consider the composite ternary operation on \mathbb{R} , defined by

$$\overline{F}_\theta = \overline{m} \circ \overline{R}_\theta.$$

As established in [50, §9.4], the interval $[0, 1]$ is a subuniverse of $(\mathbb{R}, \overline{F}_\theta)$, and moreover the operation \overline{F}_θ satisfies the equations

$$F_\theta(x, x, x) \approx x, \quad F_\theta(x, y, z) \approx F_\theta(z, x, y).$$

Moreover, the derived operation

$$\overline{p}_\theta(a, b) = \overline{F}_\theta(a, a, b)$$

turns out to be an affine combination on $[0, 1]$ (as defined in Equation (29)). Therefore p_θ obeys the idempotent and entropic equations of §7.6.3, plus further equations if the coefficients of p_θ are algebraic. These easily translate to further laws for \overline{F}_θ .

We do not have a clear idea of how high in the lattice these examples might lie.

8 The operations needed for the examples in §7.

Somewhat surprisingly, the concrete examples of compatibility provided throughout §7 require operations only of a very unsophisticated design. (A few examples above, such as the P in §7.3, are originally formed as products. In such a case, the following analysis should be seen as applying to the two factors separately.)

8.1 Multilinear Operations

Let A be a topological subspace of some power \mathbb{R}^k . For $\alpha \in \mathbb{R}$ and K a subset of $\{1, \dots, n\} \times \{1, \dots, k\}$ we consider continuous functions $\overline{F}: A^n \rightarrow \mathbb{R}$ of the form

$$\overline{F}((x_{(1,1)}, \dots, x_{(1,k)}), \dots, (x_{(n,1)}, \dots, x_{(n,k)})) = \alpha \prod_{(i,j) \in K} x_{(i,j)}$$

uniformly over A^n . We usually restrict our attention to sets K with the property that for each $i \in \{1, \dots, n\}$ there is at most one j with $(i, j) \in K$. For such a set K , we will say that this \overline{F} , or any linear combination of such \overline{F} 's for varying α and K , is *multilinear*.

If $k = 1$ (in which case, each $j = 1$) we may say that \overline{F} is *linear*, resp. *bilinear*, if each K is a singleton, resp. has two elements.

Then for an operation $\overline{F}: A^n \rightarrow A \subseteq \mathbb{R}^k$, we say that \overline{F} is multilinear (resp. linear) iff each of its components is multilinear (resp. linear).

Given a continuous operation $\overline{F}: A^n \rightarrow A$, if the space A^n may be triangulated so that on each simplex \overline{F} is multilinear, then we may say that \overline{F} is *piecewise multilinear*. Similarly *piecewise linear*, *piecewise bilinear*, and so on.

8.2 Piecewise linear operations seem to suffice on $[0, 1]$.

Let us first look at $I([0, 1])$, the sets Σ known to be compatible with the interval $[0, 1]$. In fact, *piecewise linear operations suffice* for all the concrete examples included in §7. The piecewise linearity is made explicit in Equation (7) of §7.2.1, in points (i) and (ii) of §7.4.1 and in §7.6.2; elsewhere it may be easily inferred from the context.

In detail, the operations in §7.2.1 are piecewise linear, by Equation (7). The equations in §7.2.2 can be modeled either with piecewise linear functions or with Chebyshev polynomials (among infinitely many possibilities). The equation-sets below lattice theory—semilattices in §7.2.4, majority operations in §7.2.5 and 0,1-multiplication in §7.2.6—are *a fortiori* satisfied with piecewise linear operations on $[0, 1]$. And then the minority operation \bar{q} defined in Equation (11) of (§7.4.1), the Mal'tsev operation of §7.4.3, and the two-thirds minority operation \bar{t} of §7.4.4 are linear combinations of operations defined earlier, and hence still piecewise linear.

Finally, it is not hard to check that the entropic operations in §7.6.3, and the twisted operations in §7.6.4 are all piecewise linear. As for the composite ring-lattice operations in §7.6.2, we gave two ways to define \bar{F} , one piecewise linear, and one not.

In the first sentence of §7.4 we cited only an existence proof for operations on $[0, 1]$ to make A compatible with Σ . To constructively provide such operations would require solving the word problem for free Σ -algebras, and then analyzing the topological structure of $\mathbf{F}_\Sigma([0, 1])$.

$\Sigma_{[0,1]}$ obviously defines a huge and complicated mathematical structure; complete knowledge of it may be impossible (unless, for example, we are so lucky as to find a simple finite generator for $I([0, 1])$). We do, however, know something about it. In several places—notably §7.2, §7.4 and §7.6—we have reported on positive findings of $[0, 1] \models^{\text{ctn}} \Sigma$ for various sets Σ . Each of these reports amounts to a description of a finite piece of $\Sigma_{[0,1]}$.

8.3 Some further piecewise bilinear operations on a closed interval.

In this speculative section we note the possibility that for A a closed interval of the real line, there may exist $\Sigma \in I(A)$ that is higher than any other such Σ that we have considered so far in this account.

In this context it works best to consider the interval $[-1, 1]$. The operations we will consider, beyond the ordinary join \vee and meet \wedge and constants 0 and 1 that we have already considered, are these:

- (i) Ordinary multiplication, $x \cdot y$
- (ii) Truncated addition: $x \boxplus y$, to mean $[(x + y) \wedge 1] \vee (-1)$
- (iii) Shrinking: $\overline{F}(x)$ to mean⁹ $x/3$.

Besides the distributive-lattice equations for \wedge, \vee , and commutativity and associativity for $x \cdot y$, the equations satisfied by our operations include¹⁰ these:

$$x \boxplus y \approx y \boxplus x; \quad (F(x) \boxplus F(y)) \boxplus F(z) \approx F(x) \boxplus (F(y) \boxplus F(z)) \quad (30)$$

$$(F(x) \boxplus F(x)) \boxplus F(x) \approx x \quad (31)$$

$$x \cdot (F(y) \boxplus F(z)) \approx (x \cdot F(y)) \boxplus (x \cdot F(z)) \quad (32)$$

$$(x \vee 0) \cdot (y \wedge z) \approx (x \vee 0) \cdot y \wedge (x \vee 0) \cdot z \quad (33)$$

$$((x \vee 0) \boxplus y) \boxplus (z \vee 0) \approx (x \vee 0) \boxplus (y \boxplus (z \vee 0)) \quad (34)$$

$$((x \vee 0) \boxplus y) \boxplus (z \wedge 0) \preceq (x \vee 0) \boxplus (y \boxplus (z \wedge 0)) \quad (35)$$

$$F(x \wedge y) \approx F(x) \wedge F(y) \quad (36)$$

$$(x \cdot x) \vee 0 \approx x \cdot x \quad (37)$$

$$(x \wedge 0) \cdot (y \wedge z) \approx ((x \wedge 0) \cdot y) \vee ((x \wedge 0) \cdot z) \quad (38)$$

and the duals of (33–36). (The dual of (37) is **not** included.) For the notation in (35): $\alpha \preceq \beta$ means $\alpha \vee \beta \approx \beta$. Probably the careful reader can find further interesting examples.¹¹

For our context, the question is whether the operations defined here on $[-1, 1]$ satisfy an equation-set that lies higher in the interpretability lattice than (or incomparable with), say, the equations already seen in §7.2.2. Equations (30–36) do *not* have this property: they are (jointly) interpretable in distributive lattice theory by defining $F(x)$ to be x , and defining both $x \boxplus y$ and $x \cdot y$ to be $x \wedge y$. This interpretation does not work for the set of Equations (30–38); we do not know the location in the lattice of this set.

⁹The 3 is somewhat arbitrary here.

¹⁰We thank George M. Bergman for Equations (35) and (38).

¹¹For example the list could be extended by adding 1 (or any other constants), and adding laws satisfied by such constant(s).

8.4 Multilinear maps define many group operations.

The groups on S^1 , S^3 and S^7 (see §7.1.1) all proceed from coordinate systems (pairs, quadruples or octuples of real numbers). The product in S^3 , say, of (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) has four components, each of which is a bilinear function of the x_i and the y_j —a linear combination of the sixteen products $x_i y_j$. Products in S^1 and S^7 are calculated similarly. In all three groups, inverses are calculated by a form of conjugation, which is linear.

The matrix groups (§7.1.2) involve the ordinary product of two $N \times N$ matrices; in the product, each entry is a bilinear function of the entries in the two given matrices. In dealing with unitary matrices, the inverse is simply conjugation, which is linear. For more general non-singular matrices, one will also require the operation of calculating inverses, which can be seen as the calculation of many determinants, followed by non-zero division. Each determinant may be calculated as a multilinear function of the columns.

8.5 Point operations.

Operations such as those defined in Equations (19–20) of §7.5.2 were termed *point operations* by Trevor Evans in [17]. Another point operation may be seen in Equation (24). The definition is that our space is a direct power, and each coordinate of an \bar{F} -value is determined as one of the input coordinates. Obviously point operations are multilinear in the sense of §8.1.

More generally, if each coordinate of an \bar{F} -value is determined as one of the input coordinates *or a constant*, then we have operations that can model the k -undemanding equations in the third paragraph of §7.5.3.

The operation of $\Sigma^{[k]}$ defined in Equation (15) of §7.5.1 may be seen as a hybrid of Evans' pure point operations, and the basic operations of the root variety Σ . If the basic operations \bar{F}_i of §7.5.1 can be taken as multilinear, then the constructed operation \bar{F} defined in Equation (15) will be multilinear as well.

8.6 Operations of arity 4 and higher.

None of our concrete examples mentions an operation of arity 4 or higher. (Of course simple equations (§7.4) can involve operation symbols of any arity.) We therefore do not know of any significant role played by N -ary operations for $N \geq 4$. For example, we do not know whether, for each $N \geq 4$, there

exists a (finite) space A such that any generator of the ideal $I(A)$ (§4.3) must include an operation symbol of arity $\geq N$. (In fact we do not even know whether this holds with $N = 3$; some of our examples involve ternary operations, but in some or all cases they might be dispensable.)

9 Outlook and questions.

From known examples of the compatibility relation $A \models^{\text{ctn}} \Sigma$, and from the many instances in which the relation is known to fail, it may be possible to catalog or classify the possibilities, at least for some finite spaces A (i.e. spaces homeomorphic to the realization of a finite simplicial complex) and for some finite Σ . Or at least to formulate a conjecture as to what is possible.

9.1 Topological models of a given theory Σ .

It may be difficult to characterize or enumerate the finite models of a given Σ . The overall difficulty should be apparent from the surprising results surrounding H-spaces (§7.1.1).

Moreover, there seems to be little structure to the collection (among finite complexes) of all topological groups, say, or all topological semilattices, etc. Algebraically, the collection is a category and a variety, and products are of some use—e.g. the product of two finite complexes is a finite complex. But subalgebras and homomorphic images of finite complexes are not usually finite complexes.

There are, of course, a few exceptional cases where the topological spaces compatible with Σ can be expressly described or classified. Such are for example the squaring equations of §7.5.3 (and analogous k -th power equations), and also the majority operations of §7.2.5.

9.2 The theories compatible with a given space A .

In a few places—such as §7.5.3 and §4.3.1—we have seen a space A for which the compatible theories Σ can be described or enumerated, such as $A = S^1$. For general A , however, the task eludes us.

More precisely, we are asking for some description of the ideal $I(A)$ of §4.3 and §4.3.1. We thus have the lattice-theoretic structure to help formulate a description. In particular, we know (§4.3.1) that $I(A)$ is principal. The task

here will be to find a generator, or generating family, that is (in some sense) small and easily understood.

For a relatively simple space like $[0, 1]$ or its finite powers, it may be possible to refine our understanding of $I(A)$. It seems interesting that all the known theories compatible with $[0, 1]$ are very simple (or lie below some simpler compatible theory). This points either to an inherent simplicity of $I([0, 1])$, or to a large misapprehension on the part of those who have studied it. Hopefully, the former.

9.3 The theories compatible with *any* finite space.

Let I be the union of the ideals $I(A)$ for all finite complexes A . By §6.2 it is not an ideal, but it is down-closed. Remarkably, it again seems that everything we know to be in I is relatively simple, or at least lies below a fairly simple set of equations. The upper boundary of I may be easier to define than the boundaries of an individual $I(A)$. (We have no conjecture as to a possible form.)

9.4 Specific questions.

9.4.1 Thoroughness of §7.

*Does §7 include, at least implicitly, all the **known** examples of equation-sets Σ that hold on a finite topological space A ?*

(In saying “implicitly,” we allow for example that Σ might lie below some theory mentioned in our text, or that A might be a direct product or a finite power.) (And of course, this could change with time; again please let the author know of any new discoveries.)

9.4.2 Completeness of §7.

Does §7 include, at least implicitly, all equation-sets Σ that hold on a finite topological space A ?

In other words, we are asking about the down-set I described in §9.3. The answer here may surely be “no,” even after §9.4.1 may have been corrected. It may, however, be true that we are close to a full knowledge of I .

9.4.3 What is $I = \bigcup \{I(A) : A \text{ any finite complex}\}$?

For example, *Does there exist a recursive sequence $\Sigma_0, \Sigma_1 \dots$ (with each Σ_n a finite set of equations) such that $\Sigma \in I$ if and only if for some n , $\Sigma \leq \Sigma_n$ in the interpretability lattice?*

9.4.4 What operations are needed for I ?

For each $\Sigma \in I$ (as defined in §9.4.3), do there exist a finite complex A and continuous piecewise multilinear operations \bar{F}_t on A such that $(A, \bar{F}_t)_{t \in T} \models \Sigma$?

We suspect the answer here is “no,” but we have no counterexample.

If not, does there exist some reasonable enlargement of the category “piecewise multilinear” for which the answer is yes?

9.4.5 Algorithmic questions: fixed space.

Given a fixed finite space A , does there exist an algorithm that inputs a finite set Σ of equations, and outputs whether $A \models^{\text{ctn}} \Sigma$?

Given a fixed finite space A , is the set of all finite Σ with $A \models^{\text{ctn}} \Sigma$ recursively enumerable?

(We assume that one can work out a language to code a set of equations.)

In special cases, an algorithm for $A \models^{\text{ctn}} \Sigma$ exists and is implicit in what we have already written. For A one of the spaces of §6.1, the algorithm would check whether Σ is undemanding. For a k -th power of one of those spaces, the algorithm would check whether Σ is k -undemanding (§7.5.3). For the sphere S^1 , one would check whether Σ can be modeled by linear operations with integer coefficients (see §4.3.1). For the majority of finite spaces, however, the answer is unknown. In fact, we know of no finite A for which we can say that the answer to either question is negative. By contrast, for $A = \mathbb{R}$, we do know that there is no algorithm (see [50]).

(The proof in [50] of the algorithmic undecidability of $\mathbb{R} \models^{\text{ctn}} \Sigma$ seems to require a non-compact space, where some periodic functions can be found to live.)

9.4.6 Algorithmic questions: fixed theory.

For a fixed set Σ of equations, does there exist an algorithm to decide, for a finite complex A , whether $A \models^{\text{ctn}} \Sigma$?

Is the set of such A recursively enumerable?

We advise the reader that some very simple questions on finite complexes—such as the question of the simple connectedness of a 2-complex—can fail to have an algorithmic solution. (See [36] or [42] for examples.)

9.4.7 How well can $A \models^{\text{ctn}} \Sigma$ separate two theories?

We restate a question from §7.2.3:

Does there exist a space A that is compatible with lattice theory but not with modular lattice theory? Is there a finite space A with this property?

Obviously the corresponding question may be asked for any two Σ that are distinct in the interpretability lattice. The specific question posed here remains open, and may be taken as an indicator of how little we know in this area. For a similar question, that of separating modular lattice theory from distributive lattice theory, an example is known. If B is the topological space formed by joining three triangles (closed 2-simplices) along a single common edge, then B is compatible with modular lattice theory, but not with distributive lattice theory. The proof—to which the present author contributed—may be found in §3.1 of G. Bergman [7]; see especially Theorem 7 there.

9.4.8 Description of $I([0, 1])$?

(i) Does §7 give a thorough description of all known Σ compatible with $[0, 1]$?

(ii) Is there a finite Σ that generates the ideal $I([0, 1])$ of all theories compatible with the interval $[0, 1]$? If so, attempt to exhibit a specific finite generator Σ .

(iii) If so, will the Σ that is implicit in §7 suffice for this purpose? Would it help to include the operations shown in §8.3?

(iv) Can one recursively enumerate a set of finite Σ 's that collectively generate the ideal $I([0, 1])$?

(v) In question (ii) or (iv), can one find such a Σ (or Σ 's) that can be modeled with piecewise linear operations on $[0, 1]$?

(vi) In question (ii) or (iv), can one find such a Σ (or Σ 's) whose operation symbols all have arity ≤ 3 ? What about ≤ 2 ?

9.4.9 Description of $F([0, 1])$?

Describe the filter $F([0, 1])$ of all theories not compatible with the space $[0, 1]$. If possible, frame this description as a weak Mal'tsev condition [45].

As mentioned in §4.3.4, $F([0, 1])$ is not a Mal'tsev filter.

9.4.10 Other spaces A .

The questions in §9.4.8 may be asked for any space A , and we consider them to be on the table, especially for A a finite complex. (“Linearity” may require a specified coordinate system.) With a few exceptions (such as $A = S^1$), we do not expect them to be any easier than the questions for $A = [0, 1]$.

If A is product-indecomposable, then the questions of §9.4.9 may also be asked for A .

9.4.11 How dense are the non-trivial finite complexes?

Among those complexes that have at most m simplices, of dimension at most n , what fraction are compatible with some demanding theory (§6.1)?

We expect a meaningful answer only in the limit as m , or as m and n together, approach infinity. The precise method of counting complexes (simply by raw data, or by isomorphism types of complex, or by homeomorphism types of space, for example), is certainly part of the problem. We would not be surprised if the limiting fraction turns out to be zero.

References

- [1] J. F. Adams, On the non-existence of elements of Hopf-invariant one, *Mathematische Annalen* (2) 72 (1960), 20–104.
- [2] L. W. Anderson, On the distributivity and simple connectivity of plane topological lattices, *Transactions of the American Mathematical Society* 91 (1959), 102–112.
- [3] M. Ja. Antonovskii and A. V. Mironov, On the theory of topological l -groups. (Russian. Uzbek summary) *Dokl. Akad. Nauk UzSSR* 1967, no. 6, 6–8. MR 46 #5528.

- [4] Kirby A. Baker and Albert R. Stralka, Compact, distributive lattices of finite breadth, *Pacific J. Math.* 34 (1970) 311–320. <http://projecteuclid.org/euclid.pjm/1102976425> . MR0282895
- [5] H.-J. Bandelt and J. Hedlíková, Median algebras, *Discrete Mathematics* 45 (1983), 1–30.
- [6] A. Bateson, Fundamental groups of topological R -modules, *Transactions of the American Mathematical Society* 270 (1982), 525–536.
- [7] G. Bergman, Simplicial complexes with lattice structures. Preprint 2015; to appear. https://math.berkeley.edu/~gbergman/papers/geom_lat.pdf
- [8] Willem J. Blok and D. Pigozzi, Alfred Tarski’s Work on General Metamathematics, *Journal of Symbolic Logic* 53 (03) 1988, 36–50.
- [9] K. Borsuk, *Theory of retracts*, Monografie Matematyczne, Tom 44, Państwowe Wydawnictwo Naukowe, Warsaw, 1967, 251 pages.
- [10] R. Bott, On symmetric products and the Steenrod squares, *Annals of Mathematics* 57 (1953), 579–590.
- [11] D. R. Brown, Topological semilattices on the two-cell, *Pacific Journal of Mathematics* 15, 1965, 35–46.
- [12] Tae Ho Choe, The breadth and dimension of a topological lattice, *Proc. Amer. Math. Soc.* 23 (1969) 82–84. MR0248760
- [13] A. H. Clifford, Connected ordered topological semigroups with idempotent endpoints, I. *Transactions of the American Mathematical Society* 88 (1958), 80–98.
- [14] B. Davey and H. Werner, Dualities and equivalences for varieties of algebras, *Colloq. Math. Soc. János Bolyai* 33 (1983), 181–275.
- [15] E. Dyer and A. Shields, Connectivity of topological lattices, *Pacific Journal of Mathematics* 9 (1959), 443–448.
- [16] D. E. Edmondson, A nonmodular compact connected topological lattice, *Proceedings of the American Mathematical Society* 7 (1956), 1157–1158.

- [17] T. Evans, Products of points — some simple algebras and their identities. *Amer. Math. Monthly* 74 (1967), 362-372.
- [18] S. Fajtlowicz and J. Mycielski, On convex linear forms. *Algebra Universalis* 4 (1974), 244–249.
- [19] W. M. Faucett, Compact semigroups irreducibly connected between two points; Topological semigroups and continua with cutpoints. *Proceedings of the American Mathematical Society* 6 (1955), 741–756.
- [20] O. C. García and W. Taylor, *The lattice of interpretability types of varieties*, Memoirs of the American Mathematical Society, Number 305, 1984, iii+125 pages. MR 86e:08006a.
- [21] G. Gierz and A. R. Stralka, Modular lattices on the 3-cell are distributive, *Algebra Universalis* 26 (1989), 1–6.
- [22] J. R. Harper, Homotopy groups of finite H-spaces, *Bulletin of the American Mathematical Society* 78 (1972), 532–534.
- [23] R. Hedegaard and E. W. Weisstein, H-space, <http://mathworld.wolfram.com/H-Space.html>
- [24] S. T. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965.
- [25] J. R. Isbell, Remarks on decompositions of categories, *Proceedings of the American Mathematical Society* 19 (1968), 899-904.
- [26] ——— Median algebra, *Trans. Amer. Math. Soc.* (2) 260 (1980), 319–362.
- [27] I. M. James, Multiplication on spheres, I, II, *Proceedings of the American Mathematical Society* 13 (1957), 192–196 and *Transactions of the American Mathematical Society* 84 (1957), 545–558.
- [28] F. B. Jones and G. S. Young, Product spaces in n -manifolds, *Proceedings of the American Mathematical Society* 10 (1959), 307-308.
- [29] I. Kaplansky, Topological Rings, *American Journal of Mathematics* 69 (1947), 153–183.

- [30] R. J. Koch and A. D. Wallace, Admissibility of semigroup structures on continua, *AMS Transactions* 88 (1958), 277–287.
- [31] ——— Topological semilattices and their underlying spaces, *Semigroup Forum* 1 (1970), 209–223.
- [32] A. N. Kolmogorov, On the representation of continuous functions of many variables by superposition of continuous functions of one variable and addition. (Russian), *Dokl. Akad. Nauk SSSR* 114 (1957), 953–956.
- [33] J. D. Lawson and B. Madison, Peripherality in semigroups, *Semigroup Forum* 1 (1970), 128–142.
- [34] Lie Groups, Wikipedia article, [http://en.wikipedia.org/wiki/Triangulation_\(topology\)](http://en.wikipedia.org/wiki/Triangulation_(topology))
- [35] A. A. Mal'tsev and B. S. Nurutdinov, Iterative algebras of continuous functions (Russian), *Dokl. Akad. Nauk SSSR* 224 (1975), 532–535.
- [36] A. Markov, The insolubility of the problem of homeomorphy. (Russian) *Dokl. Akad. Nauk SSSR* 121 (1958) 218220.
- [37] R. McKenzie, On spectra, and the negative solution of the decision problem for identities having a non-trivial finite model, *J. Symbolic Logic* 41 (1975), 186–196.
- [38] W. D. Neumann, On Mal'cev conditions, *Journal of the Australian Mathematical Society* 17 (1974), 376–384.
- [39] K. Palasińska and D. Pigozzi, Gentzen-style axiomatizations in equational logic, *Algebra Universalis* 34 (1995), 128–143.
- [40] A. F. Pixley, Characterizations of arithmetical varieties. *Algebra Universalis* 9 (1979), 87–98.
- [41] M. B. Pour-El and J. I. Richards, *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989. xii+206 pp.
- [42] Recursion theory, *Handbook of mathematical logic*, Part C, pp. 525–815. *Studies in Logic and the Foundations of Math.*, Vol. 90, North-Holland, Amsterdam, 1977.

- [43] M. Sholander, Medians and betweenness, *Proceedings of the American Mathematical Society* 5 (1954), 801–807; Medians, lattices and trees, *ibid.*, 808–812.
- [44] S. Świerczkowski, Topologies in free algebras, *London Math. Soc. Proceedings* (3) 14 (1964), 566–576.
- [45] W. Taylor, Characterizing Mal’cev conditions, *Algebra Universalis* 3 (1973), 351–397.
- [46] ——— The fine spectrum of a variety, *Algebra Universalis* 5 (1975), 263–303.
- [47] ——— Varieties obeying homotopy laws, *Canadian Journal of Mathematics* 29 (1977), 498–527.
- [48] ——— *The clone of a topological space*, Volume 13 of *Research and Exposition in Mathematics*, 95 pages. Heldermann Verlag, 1986.
- [49] ——— Spaces and equations, *Fundamenta Mathematicae* 164 (2000), 193–240.
- [50] ——— Equations on real intervals, *Algebra Universalis* 55 (2006), 409–456.
- [51] ——— Simple equations on real intervals, *Algebra Universalis* 61 (2009), 213–226.
- [52] ——— Approximate satisfaction of identities, 98 pp., 2010, to appear. See <http://euclid.colorado.edu/~wtaylor/approx.pdf>. See also <http://arXiv.org/abs/1504.01165>
- [53] ——— Discontinuities in the identical satisfaction of equations, 75 pp., 2011, to appear. See <http://euclid.colorado.edu/~wtaylor/jumps.pdf>. See also <http://arXiv.org/abs/1504.01445>
- [54] J. van Mill, A rigid space X for which $X \times X$ is homogeneous, *Proceedings of the American Mathematical Society* 83 (1981), 597–600.
- [55] ——— A topological group having no homeomorphisms other than translations. *Transactions of the American Mathematical Society* 280 (1983), 491–498.

- [56] J. van Mill and M. van de Vel, On an internal property of absolute retracts, *Topology Proceedings* 4 (1979), 193–200.
- [57] ——— On an internal property of absolute retracts II, *Topology and its applications* 13 (1982), 59–68.
- [58] A. D. Wallace, The structure of topological semigroups, *Bulletin of the American Mathematical Society* 61 (1955), 95–112.
- [59] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, Volume 61, Springer-Verlag, 1978.

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