Approximate satisfaction of identities

by

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Abstract. For a metric space (A, d), and a set Σ of equations, a quantity is introduced that measures how far continuous operations must deviate from satisfying Σ on (A, d).

Contents

0 Introduction.

This paper is part of a continuing investigation—see the author's papers [29] (1986), [30] (2000), and [31] (2006)—into the *compatibility* relation (see (2) below) between a topological space A and a set Σ of equations, which we will briefly review in §0.1.

0.1 Compatibility—context and background.

In this context, Σ typically denotes a set (finite or infinite) of equations[1], which are understood as universally quantified. We usually expect that Σ has a specified similarity type. This means that we are given a set T and whole numbers $n_t \geq 0$ $(t \in T)$, that for each $t \in T$ there is an operation symbol[2] F_t of arity $n(t)$, and that the operation symbols of Σ are included among these F_t.

Given a set A and for each $t \in T$ a function $\overline{F_t} : A^{n(t)} \longrightarrow A$ (called an operation), we say that the operations $\overline{F_t}$ *satisfy* Σ and write

$$(A, \overline{F}_t)_{t \in T} \models \Sigma, \tag{1}$$

if for each equation $\sigma \approx \tau$ in Σ, both σ and τ evaluate to the same function when the operations $\overline{F_t}$ are substituted for the symbols F_t appearing in σ and τ. Given a *topological space* A and a set of equations Σ, we write

$$A \models \Sigma, \tag{2}$$

and say that A and Σ are *compatible*, iff there exist *continuous* operations $\overline{F_t}$ on A satisfying Σ.

While the definitions are simple, the relation (2) remains mysterious. The algebraic topologists long knew that the n-dimensional sphere S^n is compatible with H-space theory (see §3.2.2) if and only if $n = 1, 3$ or 7. For

[1] A (formal) equation is an ordered pair of terms (σ, τ), more frequently written $\sigma \approx \tau$. As such it makes no assertion, but merely presents two terms for consideration. The actual mathematical assertion is made by the satisfaction relation $\models$.

[2] In our examples we give the operation symbols familiar names like $+$ or $\wedge$. These may be thought of as colloquial expressions for the more formal F_t.

$A = \mathbb{R}$, the relation (2) is algorithmically undecidable for Σ [31]; i.e. there is no algorithm that inputs an arbitrary finite Σ and outputs the truth value of (2) for $A = \mathbb{R}$. In any case, (2) appears to hold only sporadically, and with no readily discernable pattern.

The mathematical literature contains many scattered examples of the truth or falsity of specific instances of (2). The author's earlier papers [29], [30], [31], collectively refer to most of what is known. We will therefore not attempt to write a list of examples for this introduction. In any case, many of the known results are recapitulated later in the paper, as we endeavor to find specific metrical versions of (2) or its negation. The reader is invited to peruse the two Figures in §1.2 as a starting point for this information.

0.2 Metric approximation to compatibility.

In this paper we examine and refine the compatibility relation (2) when the topology of A is given by a metric d. If (2) holds, that is if $A \models \Sigma$, then there is little more to say in the context of this paper. On the other hand, if (2) fails, that is, if Σ cannot be modeled on A with continuous operations, then we ask whether it is possible to model Σ *approximately* with respect to the metric d. More precisely, for real $\varepsilon > 0$, we write

$$A \models_{\varepsilon} \Sigma \tag{3}$$

iff there exist continuous operations $\overline{F}_t$ on A such that, for each equation $\sigma \approx \tau$ in Σ, the terms σ and τ evaluate to functions that are within ε of each other. We seek information about the number $\lambda_A(\Sigma)$ that is defined as follows: $\lambda_A(\Sigma)$ *is the smallest non-negative*[3] *real such that* $A \models_{\varepsilon} \Sigma$ *for every* $\varepsilon > \lambda_A(\Sigma)$.

(The reader may check that λ_A is characterized by the validity for all real ε of the following two statements: (a) if $0 < \varepsilon < \lambda_A(\Sigma)$, then $A \not\models_{\varepsilon} \Sigma$; (b) if $\lambda_A(\Sigma) < \varepsilon$, then $A \models_{\varepsilon} \Sigma$. It therefore makes little difference whether $\models_{\varepsilon}$ refers to distances $< \varepsilon$ or $\leq \varepsilon$. Seldom in the paper do we refer to $\models_{\varepsilon}$ as such, although see §0.4, Problem 6 in §0.5, the proof of Lemma 2 in §2.4, and §3.4.3.)

The paper contains some general facts about $\lambda_A(\Sigma)$. For example, we have studied how $\lambda_A(\Sigma)$ depends on deductions, on interpretation of operations

[3] If there is any real number satisfying this condition, then there is a smallest one, by completeness. If there is no such real number, then we let $\lambda_A(\Sigma) = \infty$.

by terms, on choice of metric for A's topology, and so on. We have included a small catalog of estimates for $\lambda_A(\Sigma)$, along with some indication of various methods that are sometimes applicable. On the theoretical side, near the end of the paper we show that if the space A has a finite triangulation, then simplicial maps can be used for the approximate satisfaction of Σ. With this tool we see that, for α a computable real, the collection of finite sets Σ with $\lambda_A(\Sigma) < \alpha$ is recursively enumerable (Corollary 37).

0.3 The structure of the paper.

In §1 we state again the definition of $\lambda_A(\Sigma)$ and some further auxiliary definitions. At this point, statements of the form $\lambda_A(\Sigma) < M$ or $\lambda_A(\Sigma) > M$ will be intelligible to the reader; two charts in §1.2 contain a number of such results, selected from later in the paper (mostly from §3). §2 contains theoretical results about $\lambda_A(\Sigma)$ and its calculation.

The long §3 contains our proofs of estimates on $\lambda_A(\Sigma)$ (both from above and from below). §3 is organized mostly by a rough typology of the various Σ's that can appear: inconsistent, group-theoretic, lattice-theoretic, and so on. Right at the start of §3 we outline an alternate organization of the material, in terms of the different methods of proof that are available.

In §4 we investigate the possibility of eliminating any dependence on the choice of metric, for example by taking the infimum of $\lambda_A(\Sigma)$-values over all metrics that yield A's topology and have diameter 1. Like §3, §4 is mostly a collection of small results and methods. In §5 we briefly examine product varieties: how does $\lambda_A(\Gamma \times \Delta)$ relate to $\lambda_A(\Gamma)$ and $\lambda_A(\Delta)$? It was known that if A is product-indecomposable and $A \models \Gamma \times \Delta$, then $A \models \Gamma$ or $A \models \Delta$. For certain special A, we have an analogous result: if $\lambda_A(\Gamma \times \Delta)$ is small, then $\lambda_A(\Gamma)$ is small or $\lambda_A(\Delta)$ is small. §6 deals with approximate satisfaction by simplicial (piecewise linear) maps: if A is a finite simplicial complex, and if Σ is finite, then in the definition of $\lambda_A(\Sigma)$ we may restrict attention to simplicial maps (Corollary 33). §7 contains the enumerability result mentioned above in §0.2.

§8 concerns a certain kind of filter that can be defined using λ_A on the lattice of interpretability types of varieties. §9 is a very short excursion into the question of approximate satisfaction by differentiable operations.

After the brief introduction in §1, the remaining sections can be read in almost any order. Even §2 may be skipped; one can return there when necessary. §4 may be read independently as a collection of estimates, although most of its proofs depend on §3. §8 requires §5 for a proof and also for an example, but otherwise can be understood independently. §6 should probably precede §7. The other sections are independent of one another. As for the long §3, its subsections (§3.1, §3.2, and so on) can each be read independently.

0.4 Vision and outlook

We propose the incompatibility measure λ_A, along with δ_A^- and δ_A^+ of §4, as a new tool for explaining, systematizing, and perhaps extending known results on the truth or falsity of the compatibility relation (2). In many cases that we have investigated, it turns out that the approximate satisfiability relation (3) is as tractable[4] as the apparently simpler relation (2).

It is our hope that some pattern may emerge from this finer information that will begin to elucidate the overall idea of compatibility. As one example we mention that the algorithmic or recursion-theoretic character of the relation $A \models \Sigma$ (say, for finite Σ and a finite complex A) is not understood.[5] On the other hand we now know (Corollary 38 of §7.2) that, for fixed recursive ε, the relation $A \models_\varepsilon \Sigma$, with A and Σ ranging over finite complexes and finite theories, is recursively enumerable. We also know (Corollary 40 of §7.3) that for fixed finite K, the set of Σ with $\lambda_{|K|}(\Sigma) = 0$ is (at worst) a Π_2-set.

Finally, we remark that, for a fixed set Σ of equations, one may for example define a topological invariant $\mu_\Sigma(A) = \delta_A^-(\Sigma)$ (from §4). It may be that for certain Σ the invariant μ_Σ could play some role in topology, for instance in the classification of spaces A. Certainly some cases may be seen in the paper where a given μ_Σ distinguishes two spaces; each such non-homeomorphism was, however, already known.

0.5 Problems

We have written few explicit problems into the body of this paper, although obviously many things are not yet known. Here we collect a few ideas for further study. In a sense the most important problem is the vague one of elucidating the notion of compatibility between a topological space and a set of equations. However a list of some more focused problems may be of some use.

[4] For three notable exception to this assertion, see §3.3.3, §3.3.5 and §3.3.7.

[5] Although from [31] we know that the relation $\mathbb{R} \models \Sigma$ is not algorithmic.

Problem 1. Of course, every approximation given for some $\lambda_A(\Sigma)$ calls for a better approximation, if not for the exact value. Specific instances of this problem are implicit throughout §3.

Problem 2. We mention an incompatibility result for Boolean algebras in §3.3.5. There we ask whether this result can be sharpened to a positive lower estimate on $\lambda_A(B)$ for B some equations true in Boolean algebras, and A some appropriate space or spaces.

Problem 3. Are there theorems saying that, under certain conditions, $\lambda_A(\Sigma) = 0$ implies $A \models \Sigma$?

Problem 4. Every approximation that one discovers for some $\delta^-(\Sigma)$ or some $\delta^+(\Sigma)$ calls for a better approximation, if not for the exact value. Specific instances of this problem are implicit throughout §4.3.

Problem 5. Does δ^- take any values besides 0, 0.5 and 1? (See the various calculations of δ^- in §4.3. Cf. Theorem 14 of §4.2 for the situation with δ^+.)

Problem 6. The recursive enumerations of §7.2 (mentioned in §0.4) essentially rely on blind luck: try *all* possible piecewise linear operations (in a certain appropriate finite collection), and check each one (using Tarski's algorithm) to see if it yields $\models_\varepsilon$. Is there an algorithm that proceeds more systematically toward the desired results?

Problem 7. Corollary 40 of §7.3 says that for a fixed finite simplicial complex K, the set of Σ with $\lambda_{|K|}(\Sigma) = 0$ is a Π_2-set. Is it actually of a simpler arithmetic character?

Problem 8. In §8 we define $\mathcal{L}_Z$, for a metric space Z, to be the class of deductively closed theories $\Sigma^\star$ such that $\lambda_Z(\Sigma^\star) > 0$. Does $\mathcal{L}_Z$ satisfy the finiteness condition of Mal'tsev conditions [27]? Namely if Σ is in $\mathcal{L}_Z$, does Σ have a finite subset that is also in $\mathcal{L}_Z$? If this condition holds, and if $\mathcal{L}_Z$ is also closed under the formation of product varieties, then $\mathcal{L}_Z$ is definable by a Mal'tsev condition. (The latter condition holds for $Z = [0, 1]$, as proved in §8.)

Problem 9. If the previous problem has a positive solution for $Z = [0, 1]$, then $\mathcal{L}_Z$ is definable by a Mal'tsev condition. The problem here is to give an explicit Mal'tsev condition for $\mathcal{L}_{[0,1]}$.

Problem 10. Consider all finite equation-sets Σ in a fixed recursive similarity type that has infinitely many operation symbols of each finite arity. Is the set K_1 of those Σ that are analytic-compatible with $\mathbb{R}$ recursively inseparable from the set K_0 of those Σ that satisfy $\lambda_{\mathbb{R}}(\Sigma) > 0$? (In [31] we proved the inseparability of K_0 from those Σ that are not compatible with $\mathbb{R}$.)

Problem 11. Is there any set Σ of equations for which the invariant μ_Σ (defined at the end of §0.4) is an applicable topological invariant? Can any connection be found between μ_Σ and classical invariants such as cohomology and dimension?

0.6 Acknowledgments

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1 The quantity $\lambda_A(\Sigma)$ for equations Σ and a metric space (A, d).

In §1.1 — indeed, in a large part of the paper — we shall begin with a topological space A that has been provided with a specific metric (or at least a pseudometric) d. At the same time, we consider a set Σ of equations (in an arbitrary similarity type).

1.1 A pseudometric $\lambda_{\mathbf{A}}$ for term operations on a topological algebra $\mathbf{A}$.

For our given (A, d), we are generally interested in the possible existence of topological algebras $\mathbf{A} = (A, \overline{F}_t)_{t\in T}$ that model Σ approximately. In §1.1 we consider a single such $\mathbf{A}$, and define a measure of how closely $\mathbf{A}$ approximates Σ. The only constraint on $\mathbf{A}$ is that its similarity type should include each operation symbol occurring in the equations Σ that we wish to approximate.

For any term $\sigma = \sigma(x_0, x_1, \ldots)$ (of the appropriate similarity type), let $\sigma^{\mathbf{A}} : A^\omega \longrightarrow A$ (sometimes also denoted by $\overline{\sigma}$) be the corresponding term operation that is recursively defined in the usual way. For two such terms

σ, τ, we define

$$\lambda_{\mathbf{A}}(\sigma,\tau) \;=\; \sup\bigl\{\, d(\sigma^{\mathbf{A}}(\mathbf{a}),\tau^{\mathbf{A}}(\mathbf{a}))\,:\,\mathbf{a}\in A^{\omega}\bigr\} \;\in\; \mathbb{R}^{\geq 0}\cup\{\infty\}. \qquad (4)$$

(If there is some need to specify d, we may write $\lambda_{(\mathbf{A},d)}(\sigma,\tau)$.) Clearly $\lambda_{\mathbf{A}}$ is an L_∞-type distance, and as such it is a pseudometric on the set of all terms σ (of the appropriate type).

Clearly, if d is a metric, then $\lambda_{\mathbf{A}}$ is a metric (in the extended sense) as far as term operations are concerned: $\lambda_{\mathbf{A}}(\sigma,\tau)=0$ iff $\sigma^{\mathbf{A}}=\tau^{\mathbf{A}}$. From this, it is immediate that

$$\mathbf{A} \;\models\; \sigma\approx\tau \qquad \text{if and only if} \qquad \lambda_{\mathbf{A}}(\sigma,\tau) \;=\; 0.$$

Thus $\lambda_{\mathbf{A}}(\sigma,\tau)$ may be considered a measure of the failure of $\mathbf{A}$ to model $\sigma\approx\tau$.

For a (finite or infinite) set Σ of equations (of the appropriate type), we may extend the previous definition to

$$\lambda_{\mathbf{A}}(\Sigma) \;=\; \sup\bigl\{\lambda_{\mathbf{A}}(\sigma,\tau)\,:\,\sigma\approx\tau\,\in\Sigma\bigr\} \;\in\; \mathbb{R}^{\geq 0}\cup\{\infty\}.$$

As before,

$$\mathbf{A} \;\models\; \Sigma \qquad \text{if and only if} \qquad \lambda_{\mathbf{A}}(\Sigma) \;=\; 0.$$

1.2 Definition of λ_A for a space A.

Considering now (A,d) simply as a metric space—in other words not supposing any operations given on A in advance—we may define the minimax quantity

$$\lambda_A(\Sigma) \;=\; \inf\bigl\{\lambda_{\mathbf{A}}(\Sigma)\,:\,\mathbf{A}=(A;\overline{F}_t)_{t\in T},\ \overline{F}_t \text{ any continuous operations}\bigr\}.$$

(Again, we may write $\lambda_{(A,d)}(\Sigma)$ if there is a need to specify d.)

This λ_A—defined for a metric space A either here or by properties (a) and (b) of §0.2—is the central object of study in this paper. A significant part of our work lies simply in estimating λ for enough cases to illustrate the sort of variation that λ can exhibit. Figures 1 and 2 summarize some of our numerical findings. (For the exact statements of the relevant results, see the sections referenced in the two Figures.)

We clearly have: if $A\models\Sigma$, i.e. if A is compatible with Σ, then $\lambda_A(\Sigma)=0$. As may be seen in §3.1, §3.4.4 and §3.4.8, the converse is false: *one may find incompatible A and Σ that nevertheless obey the following property: for every real $\varepsilon>0$ there are continuous operations $\overline{F}_t^{\,\varepsilon}$ $(t\in T)$ on the space A satisfying the equations Σ within ε.*

	$(0,1)$	$\mathbb{R}$	$\mathbb{R}^2$	Y	Y_ε	S^n
$\Sigma\Lambda$	$\models$	$\models$	$\models$	$\models$	$\models$	$\lambda = 1$ §3.2.1
Λ	$\models$	$\models$	$\models$	$\lambda \geq 0.144$ §3.4.6	$0 < \lambda < \varepsilon$ §3.4.7	$\lambda = 1$ §3.2.1
$\Sigma\Lambda_0$	$\lambda = 0$ §3.4.4	$\lambda = \infty$ §3.4.4				$\lambda = 1$ §3.2.1
Γ	$\models$	$\models$	$\models$			$\lambda = 1$ §3.2.1
Γ_2		$\lambda = \infty$ §3.3.6	**?** §3.3.7			$\lambda = 1$ §3.2.1
H	$\models$	$\models$	$\models$	$\models$	$\models$	$\lambda = 1$ §3.2.1

For the given spaces and theories, A is compatible with Σ if and only if $\models$ appears in the corresponding entry of the chart. A blank spot means that we know that $A \models \Sigma$ fails but we have not attempted to estimate the quantity $\lambda_A(\Sigma)$. The single question-mark indicates a λ-value that we have been unable to estimate.

Theories Σ: $\Sigma\Lambda$, semilattices; $\Sigma\Lambda_0$: with zero; Λ lattices; Γ groups; Γ_2, groups of exponent 2; H, H-spaces.

Spaces A: $(0,1)$, $\mathbb{R}$, $\mathbb{R}^2$, with the usual metrics; Y a 120-degree triode with usual planar metric; Y_ε, triode with a specific exotic metric; S^n, the n-sphere (with $n \neq 0, 1, 3, 7$). Except for $\mathbb{R}$ and $\mathbb{R}^2$, all metrics scaled to a diameter of 1.

Figure 1: Some estimates of $\lambda_A(\Sigma)$.

	$\mathbb{R}$	$[0,1]$	$[0,1]_?$	$[0,1]^k$	$[0,1]^k_\varepsilon$
I	$\lambda = 0$ §3.1.2	$\lambda = 0$ §3.1.2	$\lambda = 0$ §3.1.2	$\lambda = 0$ §3.1.2	$\lambda = 0$ §3.1.2
Σ_C		$\lambda = 0$ §3.4.8			
Γ	$\models$	$\lambda = 0.5$ §3.3.1	$\lambda \geq 0.5$ §3.3.1	$\lambda = 0.5$ §3.3.1	$\lambda \geq 0.5$ §3.3.1
Γ_2	$\lambda = \infty$ §3.3.6	$\lambda = 0.5$ §3.3.1	$\lambda \geq 0.5$ §3.3.1	$\lambda = 0.5$ §3.3.1	$\lambda \geq 0.5$ §3.3.1
INJ		$\lambda = 0.5$ §3.5.1	$\lambda \geq 0.5$ §3.5.1	$\lambda = 0.354$ ($k = 2$) §3.5.2	$0 < \lambda < \varepsilon$ §3.5.3
$INJ_{m,k}$		$\lambda = 0.5$ §3.5.4	$\lambda \geq 0.5$ §3.5.4	$\lambda = 0.5$ §3.5.4	$0 < \lambda < \varepsilon$ §3.5.4
$Set^{[n]}$			$\lambda = 0.5$ ($n = 2$) §3.6.3	$\models$ OR	$0 < \lambda < \varepsilon$ ($k \geq n$) §3.6.3

For the given spaces and theories, A is compatible with Σ if and only if $\models$ appears in the corresponding entry of the chart. A blank spot means that we know that $A \models \Sigma$ fails, but we have not attempted to estimate the quantity $\lambda_A(\Sigma)$.

Theories Σ: I is an inconsistent theory from §3.1.2; Σ_C is from §3.4.8; Γ (Γ_2) is groups (of exponent 2); $INJ_{m,k}$ comprises equations defining an embedding of A^m into A^k (with $m > k$); INJ is the special case of $m = 2$, $k = 1$; $Set^{[n]}$ is the theory whose models are n^{th} powers of sets.

Spaces A: $[0,1]_?$ is $[0,1]$ with an *arbitrary* diam-1 metric for the usual topology; $[0,1]^k_\varepsilon$ is $[0,1]^k$ with a specific exotic metric for the usual topology — see §3.5.4.

Figure 2: Further estimates of $\lambda_A(\Sigma)$

2 General and introductory remarks about $\lambda_A(\Sigma)$.

2.1 $\lambda_A(\Sigma)$ and the radius and diameter of A.

For (A, d) any metric space, we let

$$\begin{aligned} \text{diam}((A,d)) &= \sup\{\, d(a_0, a_1) \,:\, a_0, a_1 \in A \,\} \\ \text{radius}((A,d)) &= \inf\{r \,:\, A \subseteq \text{some ball of radius r}\}; \end{aligned}$$

When the context permits, we write diam(A) and radius(A). Clearly

$$\text{radius}(A) \;\leq\; \text{diam}(A) \;\leq\; 2 \cdot \text{radius}(A).$$

For some special spaces, such as an n-dimensional cube, the inequality on the right is an equation; for some other spaces, such as a sphere of dimension n, the inequality on the left is an equation.

Suppose that Σ is a set of equations (in variables x_i $(i \in \omega)$) that does not contain $x_i \approx x_j$ for any $i \neq j$, and that (A, d) is any metric space of finite radius R. We shall see that $\lambda_A(\Sigma) \leq R$.

For each $\varepsilon > 0$, there exists $a \in A$ such that $d(a, x) \leq R + \varepsilon$ for every $x \in A$. Consider the topological algebra $\mathbf{A} \;=\; (A, \overline{F}_t)_{t \in T}$ of type appropriate to Σ that is defined as follows: every $\overline{F}_t$ is a constant operation with value a. It is immediate that every equation of Σ holds within $R + \varepsilon$; hence $\lambda_A(\Sigma) \;\leq\; R + \varepsilon$. Since this holds for every positive ε, we have $\lambda_A(\Sigma) \;\leq\; R$; in other words $\lambda_A(\Sigma) \;\leq\;$ radius(A).

If Σ is any consistent[6] theory, then $\Sigma^\star$ will not contain any equation $x_i \approx x_j$ with $i \neq j$; hence if Σ is consistent, then $\lambda_A(\Sigma^\star) \leq$ radius(A).

The reader may easily check that

$$\lambda_A(x_0 \approx x_1) \;=\; \text{diam}(A).$$

In particular, λ_A cannot take any value strictly between radius(A) and diam(A).

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[6]In the context of equational logic, we define a theory Σ to be *consistent* iff Σ has a model of more than one element. In other words, $x_0 \approx x_1$ is not a consequence of Σ.

2.2 Dependence upon deductions.

It is obvious from the definition that *if* $\Sigma \subseteq \Sigma'$, *then* $\lambda_A(\Sigma) \leq \lambda_A(\Sigma')$. In fact it is possible to have

$$\lambda_A(\Sigma) < \lambda_A(\Sigma'), \tag{5}$$

even though every equation of Σ' is a logical consequence of Σ. For a simple example, let A be a space with radius(A) < diam(A), take Σ to be an inconsistent theory containing no equation of the form $x_i \approx x_j$, and let $\Sigma' = \Sigma \cup \{x_0 \approx x_1\}$. According to §2.1, we have

$$\lambda_A(\Sigma) \leq \text{radius}(A) < \text{diam}(A) = \lambda_A(\Sigma').$$

Thus in fact $\lambda_A(\Sigma)$ is not a logical invariant of Σ; in other words, not an invariant of the equational class defined by Σ. (For another failure of invariance under logical deduction, see §3.1.2.)

One may obtain by fiat a logical invariant of Σ, as follows. Let $\Sigma^\star$ stand for the set of logical consequences of Σ (in a similarity type that is defined by the operation symbols appearing in Σ). If we consider only quantities of the form $\lambda_A(\Sigma^\star)$, then we are indeed considering a logical invariant. The results of §3.1.1, most of §3.2, and §3.3.4 turn out to be essentially of this form.

There are, however, good reasons to consider the logical non-invariant $\lambda_A(\Sigma)$ in its own right. In some cases we can make an upper estimate $\lambda_A(\Sigma) < K$ for a small finite Σ, but are unable to extend that estimate to all the logical consequences of Σ. (An example may be seen in the proof of Part (i) of Theorem 9 in §3.5.4.) This at least points to a further problem.

As for lower estimates — $\lambda_A(\Sigma) > K$ — for some familiar finite axiom systems Σ it happens that

(i) we do not know a positive lower bound on $\lambda_A(\Sigma)$;

(ii) we can prove $\lambda_A(\Sigma \cup \Gamma) > K$ for a certain $K > 0$ and a certain finite set Γ of consequences of Σ;

(iii) therefore $\lambda_A(\Sigma^\star) > K$ by (5).

For instance, consider the lower bounds for $\lambda_Y(\Lambda)$ and $\lambda_{\mathbb{R}}(\Lambda_0)$ that appear in §3.4.6 and §3.4.4, respectively. (Here Λ (resp. Λ_0) is lattice theory (resp. with 0), and Y is a triode (one-dimensional Y-shaped compact subset of a plane).) These results stem from Lemma 7 of §3.4.5; a close examination of that lemma reveals that its assumptions involve equations that are redundant for lattice theory. So, strictly speaking, §3.4.6 does not yield a lower bound for the usual equational-axiomatic formulation of lattice theory, although it does yield a lower bound for lattice theory thought of as the equations true in all lattices. Similarly the lower bound in §3.3.1, requires certain consequences of the axioms of group theory. (Meanwhile, the group axioms appear to be neither necessary nor sufficient, in themselves, to yield this estimate.) Similar remarks could be made about the estimates appearing in §3.3.6 and §3.5.1.

Another example of a lower bound that apparently requires a redundant extension of the original axiom system is Theorem 13 of §3.6.3; its proof uses the (classically) redundant equations (85–86).

Among our non-trivial estimates on $\lambda_A(\Sigma^\star)$ are lower bounds that arise as in (iii) above, and the very crude upper bound radius(A) that occurs in §2.1 (for consistent Σ). Some more interesting upper bounds on $\lambda_A(\Sigma^\star)$ may be found in §3.3.4 and §3.4.7.

For instance, in §3.4.1 and §3.4.4 below, we present examples where we are able to compute that $\lambda_A(\Sigma) \geq \mathrm{diam}(A)$. In these cases, it clearly follows that $\lambda_A(\Sigma) = \lambda_A(\Sigma^\star) = \mathrm{diam}(A)$.

2.3 Monotonicity of λ_A under interpretability.

In 1974, W. D. Neumann [20] introduced[7] a quasi-ordering of equational theories, known as interpretability. The interpretability of Γ in Δ, denoted here $\Gamma \leq \Delta$, is defined as follows.

Let us suppose that the operation symbols of Γ are F_t $(t \in T)$. (These are not necessarily the operation symbols of Σ.) Given terms[8] α_t $(t \in T)$ in the language of Σ, we define, for each term σ in the language of Γ, a term $\sigma^\star$ in the language of Σ. The definition is by recursion in the length of σ:

$$x^\star = x \qquad (x \text{ any variable}), \tag{6}$$

$$F_t(\tau_1, \ldots, \tau_{n(t)})^\star = \alpha_t(\tau_1^\star, \ldots, \tau_{n(t)}^\star) \tag{7}$$

[7]A number of interpretability notions were already current, especially due to A. Tarski and G. Birkhoff, including some that were much like Neumann's. Nevertheless his emphasis on the resulting ordering of theories (varieties) was new in the equational context.

[8]If Σ and Γ have disjoint sets of operation symbols, we may express this relationship informally as "$F_t = \alpha_t$." For an example of this way of expressing an interpretation, see Equations (57–59) of §3.4.1.

(where (7) is formally defined by the logical notion of *simultaneous substitution*). If necessary, we may refer to α_t as the **interpreting term for** F_t and to $\sigma^\star$ as the **interpreting term for** σ.

Following [20] and [12, page 1] we now define $\Gamma \le \Sigma$ (Γ is **interpretable** in Σ) to mean that there exist terms α_t $(t \in T)$ such that for all $\sigma \approx \tau \in \Gamma$ we have $\sigma^\star \approx \tau^\star \in \Sigma$.

(It is important to point out that this definition differs subtly from earlier versions. In most of the earlier contexts, such as [12], one presented the definition of interpretability in terms of the model classes (varieties) Mod Σ and Mod Γ, and thus one worked implicitly with the deductively closed theories $\Sigma^\star$ and $\Gamma^\star$. In such a context it is immaterial whether one says "$\sigma^\star \approx \tau^\star \in \Sigma$" or "$\sigma^\star \approx \tau^\star$ is provable from Σ." (And thus, it may emphasized, all the older results may be interpreted as valid under the definition proposed here.) Here, however, $\lambda_A(\Sigma)$ is not invariant under deductive consequence (§2.2), and so we expressly require $\sigma^\star \approx \tau^\star$ to be a member of Σ. It is only under this strict definition of interpretability that Theorem 1 holds.)

Theorem 1 *Let A be a metric space, and let Σ and Γ be sets of equations (finite or infinite) in any similarity types. If Γ is interpretable in Σ, then $\lambda_A(\Gamma) \le \lambda_A(\Sigma)$.*

Proof. If $\lambda_A(\Sigma) = \infty$, there is nothing more to prove, so we shall assume that $\lambda_A(\Sigma) < \infty$. We select a real $\varepsilon > 0$, and keep it fixed until the last two sentences of the proof. By definition of λ_A, there exists a topological algebra $\mathbf{A}$, of type appropriate to Σ and based on A, such that

$$\lambda_{\mathbf{A}}(\gamma \approx \delta) \; < \; \lambda_A(\Sigma) + \varepsilon \tag{8}$$

for any equation $\gamma \approx \delta$ in Σ.

Let the operation symbols of Γ be F_t $(t \in T)$. Since $\Gamma \le \Sigma$, there exist Σ-terms α_t $(t \in T)$ such that $\sigma^\star \approx \tau^\star$ is in Σ for each equation $\sigma \approx \tau$ of Γ. We now define a topological algebra $\mathbf{A}' = \langle A, \overline{F}_t \rangle_{t \in T}$, *via*

$$F_t^{\mathbf{A}'} \; = \; \alpha_t^{\mathbf{A}} \tag{9}$$

for each $t \in T$. The rest of the proof is based on the claim that for any term σ in the language of Γ, we have

$$\sigma^{\mathbf{A}'} \; = \; (\sigma^\star)^{\mathbf{A}} \tag{10}$$

(i.e. the A'-interpretation of σ is the same operation as the A-interpretation of $\sigma^\star$). The proof is by induction on the length of σ. If σ is a variable, this assertion is immediate from (6). If σ is a composite term, say $\sigma = F_t(\tau_1, \ldots, \tau_{n(t)})$, we first rewrite (7) as

$$\sigma^\star \ = \ \alpha_t(\tau_1^\star, \ldots, \tau_{n(t)}^\star), \tag{11}$$

and then calculate

$$\begin{aligned}(\sigma^*)^{\mathbf{A}}(a_1, a_2, \ldots) \ &= \ \alpha_t^{\mathbf{A}}((\tau_1^\star)^{\mathbf{A}}(a_1, a_2, \ldots), \ldots, (\tau_{n(t)}^\star)^{\mathbf{A}}(a_1, a_2, \ldots)) \\ &= \ F_t^{\mathbf{A}'}(\tau_1^{\mathbf{A}'}(a_1, a_2, \ldots), \ldots, \tau_{n(t)}^{\mathbf{A}'}(a_1, a_2, \ldots)) \\ &= \ \sigma^{\mathbf{A}'}(a_1, a_2, \ldots).\end{aligned}$$

(The first equation uses (11) together with the usual recursive definition of the term-function $(\sigma^\star)^{\mathbf{A}}$. The second equation uses (9); it also uses (10) inductively for $\tau_j^{\mathbf{A}'} = (\tau^\star)^{\mathbf{A}}$ $(1 \leq j \leq n(t))$. The third line is based on the recursive definition of $\sigma^{\mathbf{A}'}$.) This completes our proof of (10).

From (10) it is immediate that if $\sigma \approx \tau$ is any equation in the operation symbols F_t, then

$$\lambda_{\mathbf{A}'}(\sigma \approx \tau) \ = \ \lambda_{\mathbf{A}}(\sigma^\star \approx \tau^\star). \tag{12}$$

If, moreover, $\sigma \approx \tau \in \Gamma$, then (by the first part of this proof) $\sigma^\star \approx \tau^\star \in \Sigma$. Thus (8) and (12) immediately yield

$$\lambda_{\mathbf{A}'}(\sigma \approx \tau) \ < \ \lambda_A(\Sigma) \, + \, \varepsilon. \tag{13}$$

In other words, we have now shown that, for every positive real ε, there is a topological algebra $\mathbf{A}'$ based on A such that (13) holds for every $\sigma \approx \tau \in \Gamma$. Hence by §1.1, for every positive real ε there is a topological algebra $\mathbf{A}'$ based on A such that

$$\lambda_{\mathbf{A}'}(\Gamma) \ \leq \ \lambda_A(\Sigma) \, + \, \varepsilon.$$

Since λ_A (for a space A) is defined as the inf of values of $\lambda_{\mathbf{A}}$ (for topological algebras based on A—see §1.2), we finally have $\lambda_A(\Gamma) \leq \lambda_A(\Sigma)$. ∎

Many applications of Theorem 1 are fairly obvious, and will not require any emphasis. For example, in §3.3.6 we prove that $\lambda_{\mathbb{R}}(\Gamma_2) \geq \text{radius}(\mathbb{R})/2$, for Γ_2 a version of the theory of groups of exponent 2. It is not hard to see (either directly or through Theorem 1) that the same estimate holds for a (strong enough; perhaps redundant) version of Boolean algebra. Many such observations are possible as we go along; we generally will not mention them. Nevertheless Theorem 1 has provided some guidance for our exposition: generally speaking we have striven to attach estimates from below to theories that are low in our quasi-ordering, and to attach estimates from above to high theories.

We shall apply Theorem 1 in §3.4.1, for (a certain set $\Lambda\Gamma$ of equations of) the theory of lattice-ordered groups and A an arbitrary compact metric space. We could of course estimate $\lambda_A(\Lambda\Gamma)$ directly, but it seems more informative to take note of a certain theory $\Sigma_2 \leq \Lambda\Gamma$, for which an estimate of λ_A seems more natural. Σ_2 in effect represents an interesting Mal'tsev condition satisfied by LO-groups, and the estimate seems to relate naturally to this condition.

2.4 Dependence on the choice of metric.

It is apparent from the definitions in §1.1 and §1.2 that λ_A apparently depends on our choice of metric to represent the topology of A, and hence is not an invariant of the topological space A. Indeed numerous examples will confirm that that this dependence is very real, even among metrics that are normalized to be of diameter 1. (See e.g. §3.4.4, §3.4.6 and §§3.5.2–3.5.3 below. In fact §3.3.4 contains an example where the choice of (diameter-1) metric can yield any value strictly between 0 and 1.)

Nevertheless, the metric-based quantity $\lambda_{(A,d)}$ can convey interesting and useful information. We will return in §4 to some definitions (using inf and sup) that extract a topological invariant from the function $d \longmapsto \lambda_{(A,d)}$. Here we shall only prove that, for compact A, the condition $\lambda_A(\Sigma) > 0$ is a topological invariant. (Compactness is essential, as one may see from the two λ-values that are calculated in §3.4.4 below.)

Lemma 2 *If ρ and d are metrics defining one and the same topology on A, if that topology is compact, and if $\lambda_{(A,\rho)}(\Sigma) > 0$, then $\lambda_{(A,d)}(\Sigma) > 0$.*

Proof. The compact space A has finite diameter, and clearly both λ-values are limited by the diameter (§2.1), and hence both are finite. Suppose that $\lambda_{(A,\rho)}(\Sigma) = \varepsilon_2 > 0$, and let d be a metric for the same topology on A. Take

$\varepsilon_1 > 0$ to be a Lebesgue number of the metric space (A, d) for its covering by all $\frac{\varepsilon_2}{2}$-balls of the metric space (A, ρ). (In other words, for each $a \in A$ there exists $a' \in A$ such that $B_d(a, \varepsilon_1) \subseteq B_\rho(a', \varepsilon_2/2)$.) We immediately have[9]

$$\text{if} \quad d(a, b) \leq \varepsilon_1 \quad \text{then} \quad \rho(a, b) \leq \varepsilon_2. \tag{14}$$

To complete the proof, it will be enough to establish that

$$\lambda_{(A,d)}(\Sigma) < \varepsilon_1$$

is impossible. If it held, then we would have $((A, d), \overline{F}_t)_{t\in T} \models_{\varepsilon_1} \Sigma$ for some continuous operations $\overline{F}_t$. By (14), these same operations $\overline{F}_t$ would establish that $\lambda_{(A,\rho)}(\Sigma) < \varepsilon_2$, which contradicts our original assumption about ε_2. This contradiction completes the proof of the Lemma. ▮

2.5 The usual $\mathbb{R}$-metric yields only extreme values for $\lambda_{\mathbb{R}}$.

Theorem 3 *Consider a metric space (A, d) for which there exist functions ϕ and γ with the following properties: $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a monotone increasing function such that, for all x, the sequence $\phi^n(x)$ approaches 0 in $\mathbb{R}$. And suppose that $\gamma : A \longrightarrow A$ is a homeomorphism satisfying*

$$d(\gamma(a), \gamma(b)) \;\leq\; \phi(d(a, b)) \tag{15}$$

for all $a, b \in A$. Then for any Σ, $\lambda_{(A,d)}(\Sigma) \;=\; 0$ or ∞.

Proof. Suppose $\lambda_{(A,d)}(\Sigma) \;<\; \infty$. Then there exist $K < \infty$ and continuous operations $\overline{F}_t$ on A such that each equation of Σ holds for these operations within K. If $K \;=\; 0$, then clearly we are done. Otherwise, consider the operations $\overline{F}_t{}'$, where

$$\overline{F}_t{}'(a_1, a_2, \cdots) \;=\; \gamma\,(\overline{F}_t(\gamma^{-1}(a_1), \gamma^{-1}(a_2), \cdots)).$$

We claim that these operations satisfy Σ within $\phi(K)$.

[9]It may already be in the literature that for all $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ satisfying (14); we have not seen it.

For any term σ, let $\overline{\sigma}$ denote the usual term function associated to σ in the algebra $(A, \overline{F}_t)_{t\in T}$, and let $\overline{\sigma}\,'$ denote the corresponding term function in $(A, \overline{F}_t{}')_{t\in T}$. A straightforward inductive argument establishes that

$$\overline{\sigma}\,'(a_1, a_2, \cdots) \;=\; \gamma\,(\overline{\sigma}(\gamma^{-1}(a_1), \gamma^{-1}(a_2), \cdots)).$$

Now, for any equation $\sigma \approx \tau$ of Σ, we may calculate

$$\begin{aligned} d(\overline{\sigma}\,'(a_1, \cdots), \overline{\tau}\,'(a_1, \cdots)) &= d(\gamma\,(\overline{\sigma}(\gamma^{-1}(a_1), \cdots)), \gamma\,(\overline{\tau}(\gamma^{-1}(a_1), \cdots))) \\ &\leq \phi\,(d(\overline{\sigma}(\gamma^{-1}(a_1), \cdots), \overline{\tau}(\gamma^{-1}(a_1), \cdots))) \\ &\leq \phi(K), \end{aligned}$$

where the second line follows from (15). Thus our claim is verified.

Now this construction may clearly be iterated: there are continuous operations satisfying Σ within $\phi^2(K)$, within $\phi^3(K)$, and so on. Since $\phi^n(K) \to 0$, it is clear that $\lambda_{(A,d)}(\Sigma) \,=\, 0$. ∎

Our main example of a metric space satisfying the hypotheses of Theorem 3 is $\mathbb{R}^n$ with the usual metric. For this space, γ and ϕ can be taken both to be scalar multiplication by 1/2. Thus

Corollary 4 *If d is the usual metric on $\mathbb{R}^n$, and Σ is any set of equations, then $\lambda_{(\mathbb{R}^n,d)}(\Sigma) \,=\, 0$ or ∞.* ∎

On the other hand, if d is the usual metric on $(0, 1)$ (an obvious homeomorph of $\mathbb{R}$), then such ϕ and γ do not exist. (As may be seen from a simple measure-theoretic argument.)

Problem — is Corollary 4 true on $(0, 1)$?

2.6 An inequality for retractions.

Theorem 5, which follows, is typically applied in a context where $\lambda_{(B,d)}$ is known (either exactly or approximately), and an appropriate K-value is known or can be found. Then the inequality (16) is used to supply an upper estimate on $\lambda_{(A,d)}$. Such use of the theorem may be found in §3.2.2, §3.3.4, §3.4.2, §3.4.7, §3.6.3 and §4.3.1.

Theorem 5 *Let (A, d) be a metric space, and B a subset of A. Suppose that there is a continuous map $\psi : A \longrightarrow B$ such that $\psi \restriction B$ is the identity on B (i.e., ψ retracts A onto B), and such that $d(a, \psi(a)) \leq K$ for all $a \in A$. Then*

$$\lambda_{(A,d)}(\Sigma) \;\leq\; \lambda_{(B,d)}(\Sigma) \,+\, K \tag{16}$$

for every consistent Σ.

Proof. Let us assume that the operations appearing in Σ are F_t $(t \in T)$. For each $\varepsilon > 0$, there are continuous operations $F_t^{\mathbf{B}}$ on B, forming an algebra $\mathbf{B}$ such that $\lambda_{\mathbf{B}}(\Sigma) < \lambda_{(B,d)}(\Sigma) + \varepsilon$. We define operations $F_t^{\mathbf{A}}$ on A as follows:

$$F_t^{\mathbf{A}}(a_1, \cdots, a_{n(t)}) \;=\; F_t^{\mathbf{B}}(\psi(a_1), \cdots, \psi(a_{n(t)})). \tag{17}$$

Since B is closed under the operations $F_t^{\mathbf{B}}$, and since ψ retracts A onto B, we readily see that

$$\sigma^{\mathbf{A}}(a_1, \cdots) \;=\; \sigma^{\mathbf{B}}(\psi(a_1), \cdots))$$

for any term σ in our language (other than σ a single variable).

To estimate $\lambda_{\mathbf{A}}(\Sigma)$, we consider a single equation $\sigma \approx \tau$ of Σ. If σ and τ are both variables, they must be the same variable, since Σ is consistent. In this case λ_A and λ_B both take the value 0, and so (16) holds, for this one equation.

If neither σ nor τ is a variable, then for any $a_1, a_2, \cdots \in A$, we have

$$\begin{aligned} d(\sigma^{\mathbf{A}}(a_1, \cdots), \tau^{\mathbf{A}}(a_1, \cdots)) \;&=\; d(\sigma^{\mathbf{B}}(\psi(a_1), \cdots)), \tau^{\mathbf{B}}(\psi(a_1), \cdots)) \\ &<\; \lambda_{(B,d)}(\Sigma) \,+\, \varepsilon. \end{aligned}$$

If, say, τ is the variable x_1, then we have

$$\begin{aligned} d(\sigma^{\mathbf{A}}(a_1, \cdots), \tau^{\mathbf{A}}(a_1, \cdots)) \;&=\; d(\sigma^{\mathbf{B}}(\psi(a_1), \cdots)), a_1) \\ &\leq\; d(\sigma^{\mathbf{B}}(\psi(a_1), \cdots)), \psi(a_1)) \,+\, d(\psi(a_1), a_1) \\ &<\; \lambda_{(B,d)}(\Sigma) \,+\, \varepsilon \,+\, K. \end{aligned}$$

It is now clear that for all $\sigma \approx \tau \in \Sigma$ we have

$$\lambda_{\mathbf{A}}(\sigma, \tau) \leq \lambda_{(B,d)}(\Sigma) \,+\, \varepsilon \,+\, K, \tag{18}$$

which is to say

$$\lambda_{\mathbf{A}}(\Sigma) \leq \lambda_{(B,d)}(\Sigma) + \varepsilon + K.$$

Since this estimate can be made true for every $\varepsilon > 0$, and since $\lambda_{(A,d)}(\Sigma)$ is the inf of all possible $\lambda_{\mathbf{A}}(\Sigma)$ values, we clearly have established Equation (16). ∎

Let A be an interval $[a, b]$, and let B be the subinterval $[a + K, b - K]$. Map $\psi : A \longrightarrow B$ via

$$\psi(x) = (a + K) \vee [x \wedge (b - K)].$$

Then $d(a, \psi(a)) \leq K$ for all $a \in A$, as was assumed for Theorem 5. Equation (16) fails, however, if we take an inconsistent Σ consisting of the single equation $x_0 \approx x_1$. In this case λ_A and λ_B evaluate to the diameters of the respective intervals.

3 Some estimates of $\lambda_A(\Sigma)$.

As we remarked at the end of §1.2, it is easy to compute $\lambda_A(\Sigma)$ when $A \models \Sigma$. In most other cases it seems to be difficult, but we are able to get some approximate results. We are able to compute exact values of $\lambda_A(\Sigma)$ in §3.1, §3.1.2, §3.2.1, §3.2.3, §3.2.4, §3.2.5, §3.3.4 and §3.4.1.

The topics of §3 are gathered mostly according to a rough classification of the Σ appearing, which is to say, according to the rows of the tables in Figures 1 and 2 of §1.2. (§3.2 refers to spheres, but is then itself subdivided according to Σ.) It would of course be possible to arrange the material according to the space A involved (columns of the tables), or according to method of proof. Perhaps the following remarks will be of some use to the reader who wishes to see some organization according to method.

For proving lower estimates, i.e. estimates of the form $\lambda_A(\Sigma) \geq K$, we have essentially two methods. These are, roughly

(i) Homotopy. In some contexts, such as S^n, two maps near to each other must be homotopic. If we already know that Σ cannot be satisfied on A within homotopy, this knowledge may be applied to yield an estimate on $\lambda_A(\Sigma)$. This method may be found throughout §3.2, and appears again in §3.3.3.

(ii) We start with some A-points P, Q, etc., that are a known distance apart, $d(P,Q) \geq d \ldots$ etc. Some auxiliary points $E, F, \ldots$ are then located, using both the operations and various topological properties, such as

   (a) The fixed-point theorem of Brouwer et al. (For an example, see §3.3.1.)

   (b) Certain homotopy classes of maps must be onto. (For an example, see §3.3.4.)

   (c) Compactness; in particular, the theorem (of Bolzano and Weierstrass) that every sequence in a compact metric space has a convergent susequence. (This is applied in §3.3.8, which has a corollary in §3.4.1—an estimate for lattice-ordered groups.)

   (d) The Intermediate Value Theorem. (For examples, see §3.3.6, §3.4.4, §3.5.1. (Lemma 6 of §3.4.5 below supplies a slight generalization of the IVT, which is applied to lattice theory in Lemma 7 of §3.4.5, and then to an estimate of λ in §3.4.6.)

   (e) The Theorem of Borsuk and Ulam. (For examples, see §3.5.2, §3.5.4, §3.6.3.)

   Then the approximate satisfaction of Σ within K would force some of the points P, Q, E, F, $\ldots$, to be close to one another, and the triangle inequality would yield e.g. $d(P,Q) < d$. This contradiction establishes that $\lambda_A(\Sigma) > K$.

For proving an upper estimate on $\lambda_A(\Sigma)$, we generally employ a direct, constructive method. We propose a topological algebra $\mathbf{A}$ based on A, and then directly estimate the supremum appearing in (4) from some knowledge of the geometry of A and some knowledge of the term-operations $\sigma^{\mathbf{A}}$ and $\tau^{\mathbf{A}}$. For some direct applications of this method, see §3.4.4, §3.4.8, §3.5.1, §3.5.2 and §3.5.3. We sometimes also apply this method indirectly through the use of Theorem 5: in our definition of the topological algebra $\mathbf{A}$ is accomplished by Equations (17), and the required estimates are accomplished by Equations (18). As mentioned earlier, such applications of Theorem 5 can be seen in §3.2.2, §3.3.4, §3.4.2, §3.4.7, §3.6.3 and §4.3.1.

In anticipation of §4, at some places in §3—see e.g. §3.2.2, §3.2.6, §3.4.2, §3.4.7, §3.5.3 and §3.6.3 (Theorem 12)—we will introduce an unfamiliar metric d for a topological space A and then estimate $\lambda_{(A,d)}(\Sigma)$ from above, by one of the methods mentioned here.

Throughout §3, we allow the use of $a \approx b$, for a, b in a metric space, to mean that $d(a, b) < \varepsilon$ (where ε will be understood in context). We have thus overloaded the symbol "$\approx$," using it to denote both formal equations and approximate equality. (The former usage applies only to terms; the latter applies only to elements of metric spaces.)

Although a lot of detailed information arises in §3, we remind the reader to consult the charts in Figures 1 and 2 of §1.2 for an overview.

3.1 Inconsistent Σ

3.1.1 Σ contains $x_i \approx x_j$ for $i \neq j$.

As we saw in §2.2, in this case $\lambda_A(\Sigma) = \mathrm{diam}(A)$.

3.1.2 Σ contains no $x_i \approx x_j$ for $i \neq j$.

As we saw in §2.2, in this case $\lambda_A(\Sigma) \leq \mathrm{radius}(A)$. Here in §3.1.2 we give an example of Σ and Σ' of this type, with $\Sigma' \subseteq \Sigma^\star$, and where

$$\lambda_A(\Sigma) \;=\; 0, \quad \text{and} \quad \lambda_A(\Sigma') \;=\; \mathrm{diam}(A)/2 \;=\; \mathrm{radius}(A).$$

Take (A, d) to be a simplex of finite dimension, where, for now, d is any metric that defines the usual topology. We will think of A as equal to $[0, 1]^k$ for some k. If K is a closed subset of A^n for some n, and $\overline{F} : K \longrightarrow A$ is a continuous map, then $\overline{F}$ may be extended to a continuous map $\overline{F} : A^n \longrightarrow A$. (To see this, apply Tietze's Theorem to each coordinate map $\pi_j \circ \overline{F} : K \longrightarrow [0, 1]$, where π_j is the j-th co-ordinate projection $[0, 1]^k \longrightarrow [0, 1]$.)

For a single ternary operation symbol F and two constants a and b, we take Σ to comprise the equations

$$\begin{aligned} F(a, x_0, x_1) &\approx x_0 \\ F(b, x_0, x_1) &\approx x_1 \\ a &\approx b. \end{aligned}$$

This Σ is obviously inconsistent, hence not compatible with A. On the other hand, we shall easily see that $\lambda_A(\Sigma) = 0$. Moreover, for the sake of future reference (§4.2.1), we point out that our calculation for Σ applies to any metric that defines the given topology on A.

It will suffice, given $\varepsilon > 0$, to define $\overline{F}$, $\overline{a}$ and $\overline{b}$ on A, so that $\lambda_{\mathbf{A}}(\Sigma) < \varepsilon$ for $\mathbf{A} = (A; \overline{F}, \overline{a}, \overline{b})$. Since A is not discrete, there exist $\overline{a}, \overline{b} \in A$ with $0 < d(\overline{a}, \overline{b}) < \varepsilon$. The third equation on Σ now holds within ε; to complete the job we shall define a continuous ternary operation $\overline{F}$ for which the first two equations hold *exactly.* In this way we shall have all three equations holding within ε, as desired.

The first two equations amount to a definition of $\overline{F}$ on a closed subset of A^3. By the corollary of Tietze's Theorem that is described above, $\overline{F}$ may be extended to a continuous map with domain A^3, and the construction is complete. This completes our derivation of $\lambda_A(\Sigma) = 0$.

Let us now enlarge Σ as follows:

$$\Sigma' \;=\; \Sigma \cup \{F(a, x_0, x_1) \;\approx\; x_1\}.$$

Notice that Σ' differs from Σ only in including one logical consequence of Σ (derived from Σ by substitution). Nevertheless, as was mentioned in §2.2, the value of λ_A may increase under such an enlargement of Σ. In this case, as will be apparent from Lemma 2, $\lambda_A(\Sigma') > 0$ for any appropriate choice of the metric d.

For a precise calculation of $\lambda_A(\Sigma')$, we will now take d to be the usual Euclidean metric on A^k. Under this assumption, we shall now prove that $\lambda_A(\Sigma') = \mathrm{diam}(A)/2$.

By definition, there exist a constant $\overline{a}$ and a ternary operation $\overline{F}$ on A so that

$$\begin{aligned} d(\overline{F}(\overline{a}, a_1, a_2), a_1) \;&\leq\; \lambda_A(\Sigma') \\ d(\overline{F}(\overline{a}, a_1, a_2), a_2) \;&\leq\; \lambda_A(\Sigma'), \end{aligned}$$

for all $a_1, a_2 \in A$, which implies $d(a_1, a_2) \;\leq\; 2\lambda_A(\Sigma')$ or $\lambda_A(\Sigma') \;\geq\; \mathrm{diam}(A)/2$. On the other hand if we define $\overline{F}(a_1, a_2, a_3)$ to be the midpoint of the segment joining a_2 and a_3, then the resulting $\overline{F}$ is clearly continuous, and the equations of Σ' are clearly satisfied within $\mathrm{diam}(A)/2$; hence $\lambda_A(\Sigma') \;\leq\; \mathrm{diam}(A)/2$.

3.2 Some exact formulas for spheres.

An algebra $\mathbf{A}$ is *trite* iff every operation of $\mathbf{A}$ is either a projection map or constant. An equational theory Σ is *easily satisfied* or *undemanding* iff it has a trite model of more than one element—and hence has trite models of every cardinality. All other theories are *demanding.* (These definitions come from

[30].) It is easily seen that if Σ is undemanding and A is any space, then $A \models \Sigma$, and so $\lambda_A(\Sigma) = 0$. Hence every non-trivial calculation or estimation of $\lambda_A(\Sigma)$ in this paper will concern a demanding theory Σ.

Obviously Σ and $\Sigma^\star$ have the same models, and hence Σ is demanding if and only if $\Sigma^\star$ is demanding. Thus any evaluation of $\lambda_A(\Sigma)$ that is based on the demanding property, such as (19) below, holds equally well for $\lambda_A(\Sigma^\star)$.

We showed in [30] that *certain spaces A are compatible only with the undemanding theories.*[10] (These include the spaces A mentioned in §3.2.1 and §3.2.6 below.) In other words, if Σ is demanding then $A \not\models \Sigma$. In §3.2.1 we will have the stronger conclusion that $\lambda_A(\Sigma) > 0$ for certain such A and Σ. Indeed, in certain cases of interest for §3.2 the function λ_A takes a particularly simple form[11], namely

$$\lambda_A(\Sigma) \ = \ \begin{cases} 0 & \text{if } \Sigma \text{ is undemanding} \\ 1 & \text{if } \Sigma \text{ is demanding.} \end{cases} \tag{19}$$

As we shall see in §3.2.1, Equation (19) holds for $A = S^n$, a diameter-1 sphere of dimension n, where $n \neq 1, 3, 7$. Moreover the proof of (19) will be almost immediate from the main theorem of [30] (which is based on serious results of algebraic topology). Notice also that Equation (19) implies invariance under deduction (since "demanding" depends only on the class of models of Σ) – and hence the difficulties of §2.2 do not arise. The same method will also yield some values of $\lambda_A(\Sigma)$ for $A = S^1$ (see §3.2.3), for $A = S^3$ (see §3.2.4) for $A = S^7$ (see §3.2.5), and for various Σ.

3.2.1 Σ demanding, $A = S^n$ ($n \neq 1, 3, 7$).

S^n is the usual n-dimensional sphere, which may be explicitly realized as the set of points in $\mathbb{R}^{n+1}$ having Euclidean distance $1/2$ from the origin. Equation (19) actually holds for any diameter-1 metric d that has the following property: the diameter of (A, d) is realized by each pair of antipodal points, and by no other pairs of points. Scaled geodesic distance has this property, as does, more simply, scaled Euclidean distance as measured in the ambient space $\mathbb{R}^{n+1}$. (Given such a metric, we re-scale it so that the diameter is 1.)

[10] In [30] we trace one case of this back to work of J. F. Adams [1]. In [33] A. D. Wallace attributes it to E. Cartan in the special case of Σ = groups.

[11] By the estimates found in §2.1, this λ_A is as large as possible among all metric spaces A—such as spheres—that have radius and diameter both equal to 1.

The first alternative of (19) is immediate: if Σ is undemanding, then $\lambda_{S^n}(\Sigma) = 0$.

For the second alternative, we prove the contrapositive: given Σ with $\lambda_{S^n}(\Sigma) \neq 1$, we shall prove that Σ is undemanding. It is immediate from §2.1 that $\lambda_{S^n}(\Sigma) < 1$. By the definitions in §1.1, there exist a constant $K < 1$ and a topological algebra $\mathbf{A}$ based on $A = S^n$, such that $\lambda_{\mathbf{A}}(\sigma, \tau) \leq K$ for each equation $\sigma \approx \tau$ of Σ. According to Equation (4), for each $\sigma \approx \tau$ in Σ, and for each $\mathbf{a} \in (S^n)^\omega$, we have

$$d(\sigma^{\mathbf{A}}(\mathbf{a}), \tau^{\mathbf{A}}(\mathbf{a})) < K. \tag{20}$$

We now define a homotopy

$$\Gamma : (S^n)^\omega \times [0,1] \longrightarrow S^n$$

between $\sigma^{\mathbf{A}}$ and $\tau^{\mathbf{A}}$, as follows. Let us be given $\mathbf{a} \in (S^n)^\omega$ and $t_0 \in [0,1]$. By Equation (20), $\sigma^{\mathbf{A}}(\mathbf{a})$ and $\tau^{\mathbf{A}}(\mathbf{a})$ are not antipodes; hence there is a unique shortest geodesic $\gamma(t)$, whose parameter t is proportional to arc length along γ, and with $\gamma(0) = \sigma^{\mathbf{A}}(\mathbf{a})$ and $\gamma(1) = \tau^{\mathbf{A}}(\mathbf{a})$. We then define

$$\Gamma(\mathbf{a}, t_0) \;=\; \gamma(t_0).$$

Clearly Γ is continuous and is a homotopy between $\sigma^{\mathbf{A}}$ and $\tau^{\mathbf{A}}$.

It is now clear that $\mathbf{A}$ satisfies Σ up to homotopy; in other words, S^n is compatible with Σ up to homotopy. By Theorem 1 of [30], Σ is undemanding. This concludes our proof of the contrapositive of $\lambda_{S^n}(\text{demanding}) = 1$.

3.2.2 $\lambda_{S^2}(\Sigma)$ in a different metric, for Σ = H-spaces.

By Lemma 2 of §2.4, and by §3.2.1 just above, we clearly have

$$\lambda_{(S^n,d)}(\Sigma) \begin{cases} = \ 0 & \text{if } \Sigma \text{ is undemanding} \\ > \ 0 & \text{if } \Sigma \text{ is demanding,} \end{cases} \tag{21}$$

for *any metric* d on the sphere S^n ($n \neq 1, 3, 7$). Here we will see that the non-zero value associated to a demanding theory can, in some cases, be arbitrarily small, even when the diameter is constrained to be 1. (Similar remarks are made for S^1 in §3.4.2.)

In §3.2.2 we take Σ to be the theory of H-spaces, otherwise known as the theory of a two-sided unit element. It consists of the two equations

$$F(e,x) \approx x, \qquad F(x,e) \approx x, \tag{22}$$

for a binary operation F and a nullary operation e. One easily checks that this Σ is demanding.

Given real $\varepsilon > 0$, we shall place a non-standard metric d (of diameter 1) on the ordinary sphere S^2 and calculate that $\lambda_{(S^2,d)}(\Sigma^\star) \leq \varepsilon$. To define the metric d, we consider the realization of S^2 as the prolate ellipsoid that is the locus of

$$4\varepsilon^2 x^2 + 4y^2 + 4z^2 = \varepsilon^2$$

in ordinary 3-space. Then d is defined to be the ordinary Euclidean distance of $\mathbb{R}^3$, restricted to the ellipsoid. One easily checks that (S^2, d) has diameter 1.

We define a continuous function $\psi : S^2 \longrightarrow S^2$ via

$$\psi(x,y,z) = (x, \sqrt{y^2+z^2}, 0).$$

One easily checks that, for all $\mathbf{x} = (x,y,z) \in S^2$, the points $\mathbf{x}$ and $\psi(\mathbf{x})$ lie on a circle of radius $\leq \varepsilon/2$ (in a plane perpendicular to the x-axis); hence

$$d(\mathbf{x}, \psi(\mathbf{x})) \leq \varepsilon. \tag{23}$$

The image of ψ is the semi-ellipse E in the x,y-plane that is bijectively parametrized by

$$e(t) = (\frac{1}{2} \sin \frac{\pi t}{2}, \frac{\varepsilon}{2} \cos \frac{\pi t}{2}, 0)$$

for $-1 \leq t \leq 1$. One easily sees that

$$\psi \restriction E = \text{identity}. \tag{24}$$

The reader may check that we now have the hypotheses of Theorem 5 of §2.6 (with S^2 for A, E for B, and ε for K). In this context, the conclusion of the theorem is

$$\lambda_{(S^2,d)}(\Sigma^\star) \leq \lambda_{(E,d)}(\Sigma^\star) + \varepsilon.$$

Moreover E is topologically a closed segment, which can be made into an H-space (in several interesting ways); hence $\lambda_{(E,d)}(\Sigma^\star) = 0$. We now have the desired conclusion that $\lambda_{(S^2,d)}(\Sigma^\star) < \varepsilon$.

3.2.3 Σ not Abelian, $A = S^1$.

We call a set Σ of equations *Abelian* iff it is interpretable (in the sense of [12]) in the equational theory of Abelian groups (or $\mathbb{Z}$-modules). Equivalently, Σ is Abelian if and only if it has a model based on $\mathbb{Z}$ with operations of the form

$$\overline{F}(x_1, \cdots, x_n) \;=\; m_1x_1 + \cdots + m_nx_n, \tag{25}$$

where each $m_i \in \mathbb{Z}$.

It was proved in Theorem 41 on page 234 of [30] that if Σ is compatible with S^1, even if compatible only within homotopy, then Σ is Abelian, and conversely if Σ is Abelian, then Σ is compatible with S^1. Therefore, $\lambda_{S^1}(\Sigma) = 0$ for every Abelian Σ. In fact, we have the stronger result that

$$\lambda_{S^1}(\Sigma) \;=\; \begin{cases} 0 & \text{if } \Sigma \text{ is Abelian} \\ 1 & \text{otherwise.} \end{cases} \tag{26}$$

We sketch the proof. We have already established the first alternative of (26). To prove the second alternative through its contrapositive, we begin with $\lambda_{S^1}(\Sigma) \neq 1$. Proceeding as in §3.2.1, we get $\sigma^{\mathbf{A}}$ homotopic to $\tau^{\mathbf{A}}$ for each equation $\sigma \approx \tau$ of Σ. It is immediate from Theorem 41 on page 234 of [30] that Σ is Abelian. This completes the contrapositive proof of the second alternative.

If Σ is Abelian, then Σ is modeled by an algebra $\mathbf{Z} \cong (\mathbb{Z}, \cdots)$ of the aforementioned type; clearly $\mathbf{Z}$ also models $\Sigma^\star$; hence $\Sigma^\star$ is also Abelian. Thus (26) yields the same values for $\lambda_{S^2}(\Sigma^\star)$ and for $\lambda_{S^2}(\Sigma)$.

For some specific instances of non-Abelian equational theories, the reader is referred to §3.4.1.

3.2.4 Σ = Abelian groups, $A = S^3$.

S^3 differs from the general S^n of §3.2.1, and from S^7, in that S^3 is compatible with Γ, the equational theory of groups, and hence $\lambda_{S^3}(\Gamma) = 0$. In §3.2.4 we shall also prove that $\lambda_{S^3}(\Gamma_0) = 1$, where Γ_0 stands for any set of equations axiomatizing Abelian groups.

The proof follows that of §3.2.1 and §3.2.3, and proceeds by contradiction. If $\lambda_{S^3}(\Gamma_0) < 1$, then there is a topological algebra $\mathbf{A}$ on S^3 that substantiates $\lambda < 1$. As in the earlier proofs, we obtain $\sigma^{\mathbf{A}}$ homotopic to $\tau^{\mathbf{A}}$ for each equation $\sigma \approx \tau$ of Γ_0. In other words $\mathbf{A}$ is an Abelian group on S^3, up to homotopy. A 1953 theorem of R. Bott [5] says that no such $\mathbf{A}$ exists.

3.2.5 $\Sigma =$ Monoids, $A = S^7$.

S^7 differs from the general S^n of §3.2.1 in that S^7 is compatible with K, the equational theory of H-spaces, as defined in Equation (22). (Multiplication of unit Cayley numbers provides the requisite topological algebra.) Hence $\lambda_{S^7}(K) = 0$. In §3.2.5 we shall also see that $\lambda_{S^7}(M) = 1$, where M stands for any set of equations axiomatizing *monoids* (otherwise known as *associative H-spaces*). One possible form of M comprises the two equations of (22) together with the associativity law for F. The proof (details omitted) is like the proof in §3.2.4, except that it uses I. M. James' 1957 result [14] that there do not exist $\overline{e} \in S^7$ and a continuous binary operation $\overline{F}$ on S^7 that satisfy M within homotopy.

3.2.6 $\lambda_\infty(K)$ for K = H-spaces.

We briefly consider values of $\lambda_\infty(\Sigma)$, where ∞ stands for a union of two topological circles S_1, S_2 joined at a single point P. (For example a lemniscate is such a space.) In particular we shall examine values of $\lambda_\infty(H)$, where H denotes the theory of H-spaces, defined by Equations (22) of §3.2.2. We will examine the variation in the value of $\lambda_{(\infty,d)}(H)$ that occurs among diameter-1 metrics d that define the usual topology on ∞.

For the moment we work only on making $\lambda_\infty(H)$ small by a suitable choice of the metric d. Consider the ellipses $E_1(\varepsilon)$ and $E_2(\varepsilon)$ in 3-space, parametrized respectively by

$$\begin{aligned} \gamma_1(t) &= \frac{1}{2}\,(\cos t,\, \sin t,\, \varepsilon(1+\cos t)) \qquad \text{and} \\ \gamma_2(t) &= \frac{1}{2}\,(\cos t,\, \sin t,\, 0). \end{aligned}$$

The union $E = E_1(\varepsilon) \cup E_2(\varepsilon)$ is homeomorphic to ∞, with $(-1/2, 0, 0)$ corresponding to the point P common to S_1 and S_2, and so we give $E = \infty$ the Euclidean metric of $\mathbb{R}^3$ in which it is embedded. In this d, the diameter of ∞ is $\sqrt{1+\varepsilon^2}$, approximately $1 + \frac{1}{2}\varepsilon^2$, with a limit of diameter 1 as $\varepsilon \to 0$. (The reader may re-scale to diameter 1 if desired.)

To estimate $\lambda_{(\infty,d)}(H)$, we define a continuous function $\psi : E \longrightarrow E$ via

$$\psi(x, y, z) \;=\; (x, y, 0).$$

It is immediate from the geometry that, for all $\mathbf{x} \in E$,

$$d(\mathbf{x}, \psi(\mathbf{x})) \;\leq\; \varepsilon. \tag{27}$$

The image of ψ is the ellipse E_2, and one easily sees that

$$\psi \restriction E_2 \ = \ \text{identity}. \tag{28}$$

The reader may check that we now have the hypotheses of Theorem 5 of §2.6 (with $E = \infty$ for A, E_2 for B, and ε for K). In this context, the conclusion of the theorem is

$$\lambda_{(\infty,d)}(H^\star) \ \leq \ \lambda_{(E_2,d)}(H^\star) \ + \ \varepsilon.$$

Moreover E_2 is topologically a circle, which can be made into an H-space; hence $\lambda_{(E_2,d)}(H^\star) = 0$. We now have the desired conclusion that $\lambda_{(\infty,d)}(H^\star) < \varepsilon$ (which obviously remains true after rescaling to diameter 1).

3.3 Σ related to group theory

3.3.1 Group theory on spaces with the fixed-point property.

Let Σ be a consistent set of equations, in unary $-$ and binary $+$, containing

$$x_1 \ \approx \ \{x_0 + [((-x_0) + x_1) + x_2]\} + (-x_2) \tag{29}$$

$$x_0 \ \approx \ \{x_0 + x_1\} + (-x_1). \tag{30}$$

For instance, Σ could be a conservative extension of group theory. Notice that whatever we can prove about such a Σ will also be true for $\Sigma^\star$; hence §3.3.1 applies also to $\Sigma^\star$.

We now take A to be any space that has the *fixed-point property*; i.e., we shall assume that for any continuous function $F : A \longrightarrow A$, there exists $e \in A$ with $F(e) = e$. (By the Theorem of Brouwer, every finite-dimensional closed simplex has this property. More generally, the class of such spaces is closed under retraction, and every compact convex subset of a locally convex topological vector space has this property [13, Theorem 15.4].) We will show that[12] $\lambda_A(\Sigma) \geq \operatorname{diam}(A)/2$.

We will describe what happens for $\operatorname{diam}(A) < \infty$; the other case can be left to the reader. Let us fix an $\varepsilon > 0$. By definition of λ, there exist continuous operations $+$ and $-$ that satisfy our equations within $\lambda_A(\Sigma) + \varepsilon$. By definition of the diameter, there exist $a, b \in A$ with $d(a, b) > \operatorname{diam}(A) - \varepsilon$. Let e be a fixed point to the continuous function

$$x \ \longmapsto ((-a) + b) + x.$$

[12] In this context, we define $\infty/2$ to be ∞.

Then we have

$$b \;\approx\; \{a + [((-a) + b) + e]\} + (-e) \;=\; \{a + e\} + (-e) \;\approx\; a,$$

from which it follows that

$$d(a, b) \;\leq\; 2(\lambda_A(\Sigma) + \varepsilon).$$

Thus we have

$$\begin{aligned} \operatorname{diam}(A) - \varepsilon \;<\; d(a,b) \;&\leq\; 2\lambda_A(\Sigma) \;+\; 2\varepsilon \\ 2\lambda_A(\Sigma) \;&\geq\; \operatorname{diam}(A) \;-\; 3\varepsilon. \end{aligned}$$

Letting ε approach zero, we obtain $\lambda_A(\Sigma) \geq \operatorname{diam}(A)/2$.

If A is a space whose radius is half its diameter (e.g. $[0,1]^k$ for finite k), then by §2.1 we have $\lambda_A(\Sigma) = \operatorname{radius}(A)$.

3.3.2 Group theory on spheres S^n for various n.

Here we recapitulate the results of §3.2 that have to do with group theory. Recall that our spheres all have diameter 1. If $n \neq 1, 3, 7$, then $\lambda_{S^n}(\Sigma) = 1$, for every demanding theory Σ. Thus λ_{S^n} takes the value 1, for even the simplest interesting generalizations of group theory, such as H-spaces.

As for the exceptional values $1, 3, 7$, λ_{S^1} takes the value 0 on Abelian groups. λ_{S^3} takes the value 1 on Abelian groups, but 0 on groups. λ_{S^7} takes the value 1 on groups, even on monoids, but takes the value 0 on H-spaces.

3.3.3 Group theory on $S^1 \times Y$.

For §3.3.3 we take Γ to be any set of equations in a binary operation $+$ and a unary $-$ that contains the three equations

$$x + ((-y) + y) \;\approx\; x \tag{31}$$

$$x + ((-x) + y) \;\approx\; y \tag{32}$$

$$(x + y) + (-y) \;\approx\; x. \tag{33}$$

(These three are not the same as Equations (36–37) below, and neither are they the same as (29–30) above, but all three sets are consequences of any version of group theory based on $+$ and $-$.)

Let S^1 be the unit circle with distance defined by arc length, scaled (as in §3.2) to give S^1 a diameter of 1. Let Y be the triode that is also discussed in §3.4.6 below. More precisely, Y is the union of three unit-length segments in the plane, meeting at one point and at 120-degree angles. Y is given the metric inherited from the plane. We equip their product with the L^1-metric

$$d(\,(a,b),\,(c,d)\,) \;=\; d_{S^1}(a,c) \,+\, d_Y(b,d).$$

Let e denote the center point of Y—the meeting point where three angles occur. The set $S^1 \times \{e\}$ will be denoted C, and called the *central ring* of $S^1 \times Y$. If S is one of the three segments forming Y, then $S^1 \times S$ is called a *flange* of $S^1 \times Y$. Each two of the three flanges have C as their intersection. The reader who desires to do so may visualize $S^1 \times Y$ as a cylinder with an added flange (since two of the flanges make a cylinder). Locally —that is, if one looks at $U \times Y$ for U a segment in S^1—it may be seen as three rectangles in space (each of the form $(U \times Y) \cap S$), concurrent along the single segment $(U \times Y) \cap C$ (thereby forming three dihedral angles of 120 degrees[13]). Clearly $S^1 \times Y$ is non-homogeneous as a topological space, and hence incompatible with group theory. Here we will prove the sharper result that $\lambda_{S^1 \times Y}(\Gamma) \,\geq\, 0.1$.

For a contradiction, we assume that $\lambda_{S^1 \times Y}(\Gamma) \,<\, 0.1$. Thus there are continuous operations, $\boxplus$ binary and $\boxminus$ unary, satisfying (31–32) within 0.1 on $S^1 \times Y$.

For $a \in A$, we consider the continuous translation $\tau_a : A \longrightarrow A$ given by $\tau_a(x) = x \boxplus a$. We first prove that τ_a *is one-one up to* 0.2, by which we mean that

$$\text{if} \quad \tau_a(x) = \tau_a(y), \quad \text{then} \quad d(x,y) < 0.2. \tag{34}$$

Well, given $\tau_a(x) = \tau_a(y)$, we may calculate

$$x \;\approx\; (x \boxplus a) \boxplus (\boxminus a) \;=\; (y \boxplus a) \boxplus (\boxminus a) \;\approx\; y,$$

by two applications of (33). Now (34) follows by the triangle inequality.

Let S_i $(1 \leq i \leq 3)$ denote the three flanges of $S^1 \times Y$; for $i = 1,2$ choose $a_i \in S_i$ with $d(a_i, C) > 0.8$ for each i. Let us take a to be $(\boxminus a_1) \boxplus a_2$, and consider τ_a as defined above. By (32) we have

$$\tau_a(a_1) \;=\; a_1 \boxplus ((\boxminus a_1) \boxplus a_2) \;\approx\; a_2;$$

[13] Of course the dihedral angle measure is unimportant topologically; it is, however, important for the precise metric, and perhaps it helps for making a mental picture.

in other words, $d(a_2, \tau_a(a_1)) < 0.1$, and so by the triangle inequality we have $d(\tau_a(a_1), C) > 0.7$.

Let R_{12} denote the cylinder $S_1 \cup S_2$. (It is isometric to $S^1 \times [-1, 1]$.) Let $\gamma(t)$ be the parametrized straight path in R_{12} that goes in constant speed from $\gamma(0) = a_2$ to $\gamma(1) = a_1$. Consider now the homotopy

$$\theta_t = \tau_{(\boxminus\gamma(t))\boxplus a_2} : S^1 \times Y \longrightarrow S^1 \times Y.$$

Clearly $\theta_1 = \tau_a$, and by (31) we have θ_0 within 0.1 of the identity function.

Let c be the midpoint of segment $[a_1, a_2]$ in the cylinder R_{12}. Clearly c lies on the central ring C. Moreover $\gamma(0.5) = c$, so that $\theta_{0.5}$ moves c to within 0.1 of a_2 (reasoning as above). Following the trajectory of the ring C under θ_t as t goes from 0 to 1, we see one point c moves into S_2 at 0.5. Then $\theta_t(C)$ is constrained to the far side of $\theta_t(a_1)$ in S_2, by (34). Thus *part of the curve* $\theta_1(C) = \tau_a(C)$ *is farther than* 0.7 *from* C.

By continuity, some of $\tau_a(S_3)$ must lie in S_2 farther than 0.7 from C. Let Q be the outer edge of S_3. In considering a line from $\tau_a(c)$ to $\tau_a(a_1)$, let us consider whether we meet a point $\tau_a(q)$ for $q \in Q$. There are actually three cases here.

Case 1. $\tau_a(b_1) = \tau_a(q)$ for some $b_1 \in S_1$ and some $q \in Q$. This clearly contradicts (34), and so the proof is complete.

Case 2. $\tau_a[a_1, c] \subseteq \tau_a[S_3]$. In this case $\tau_a(a_1) = \tau_a(b_3)$ for some $b_3 \in S_3$. Again this contradicts (34).

Case 3. $\tau_a[a_1, c] \cap \tau_a[S_3] = \emptyset$. In considering a line from $\tau_a(c)$ to $\tau_a(a_2)$, let us consider whether we meet a point $\tau_a(q)$ for $q \in Q$. Here there are two cases analogous to Cases 1 and 2 above; details omitted. This completes the proof that $\lambda_{S^1 \times Y}(\Gamma) \geq 0.1$.

These λ-values—more precisely, the fact that they are non-zero—will be needed for Theorem 22 of §5.2.

3.3.4 Group theory on a thickening of S^1.

It is occasionally possible to identify $\lambda_A(\Sigma)$ as the size of some geometric feature of the metric space A. In the example that follows, A_α is a certain metric subspace of the right circular cylinder of §3.3.3, with 2α its height in the axial direction. We shall describe this space more thoroughly, and then prove that $\lambda_{A_\alpha}(\Gamma) = \alpha$, where Γ is given by the group-theoretic equations appearing in (36–37) below.

For α any real number with $0 < \alpha < 1$, we define

$$A_\alpha \;=\; \{\, (x, y, z) \,:\, x^2 + y^2 = 1 \;\& \\ \qquad\qquad (-\alpha x \leq z \leq \alpha x \;\text{ or }\; z = 0)\,\} \;\subseteq\; \mathbb{R}^3.$$

This space may easily be sketched as a subset of a cylinder in $\mathbb{R}^3$—such as was defined in §3.3.3. We give it the rectangular or taxicab metric in that space: $d(\mathbf{x}, \mathbf{y}) = \sum |x_i - y_i|$. (Notice that the spaces A_α are all homeomorphic one to another, but the homeomorphisms are not isometries.)

Notice that for $(x, y, z) \in A_\alpha$ with $x < 0$, the definition yields $z = 0$ as the only possible value for z. Thus A_α contains the circle

$$C \;=\; \{\, (x, y, 0) \,:\, x^2 + y^2 = 1 \,\},$$

and for negative x, these are the only points in A_α. For positive x, there are other points (x, y, z). The farthest of these from the circle C are $(1, 0, \alpha)$ and $(1, 0, -\alpha)$. Thus α is a measure of how far A_α extends away from the circle C.

The map $(x, y, z) \longmapsto (x, y, 0)$ is a retraction that moves no point farther than α. Its image is C, which is homeomorphic to S^1, and thus compatible with group theory, and so by Theorem 5 of §2.6, if Γ is any consistent set of equations holding in group theory (regardless of the precise similarity type used to express group theory, and clearly regardless of any distinction between Γ and $\Gamma^\star$), then

$$\lambda_{A_\alpha}(\Gamma) \;\leq\; \alpha. \tag{35}$$

For the reverse inequality, we restrict Γ to be a set of equations in the operations $+$ and $-$ that contains the three equations

$$(x + y) + (-y) \;\approx\; x \tag{36}$$

$$x + 0 \;\approx\; x; \qquad 0 + x \;\approx\; x. \tag{37}$$

We shall prove that for such a Γ,

$$\lambda_{A_\alpha}(\Gamma) \;\geq\; \alpha. \tag{38}$$

Now if $\lambda_{A_\alpha}(\Gamma) \geq 1$, then (38) holds by our assumption on α. Hence we will assume from now on that $\lambda_{A_\alpha}(\Gamma) < 1$. We will consider an arbitrary ε with $0 < \varepsilon < 1$, and will prove that $\lambda_{A_\alpha}(\Gamma) + \varepsilon > \alpha$.

We define a closed curve f in A_α (for $0 \le t \le 2\pi$), as follows:

$$f(t) \;=\; \begin{cases} (\cos t,\ \sin t,\ -\alpha\cos t) & \text{if } \cos t \ge 0 \\ (\cos t,\ \sin t,\ 0) & \text{if } \cos t \le 0. \end{cases}$$

(f maps, so to speak, to the lower periphery of A_α.) Concerning the point $A = (1, 0, \alpha) \in A_\alpha$, let us check that A has distance at least 2α from every point $B = (\cos t, \cdots)$ in the image of f. If $\cos t < 0$, then

$$d(A, B) \;=\; (1 - \cos t) \,+\, \cdots \,+\, (\alpha - 0) \;>\; 1 \,+\, \alpha \;>\; 2\alpha.$$

On the other hand, if $\cos t \ge 0$, then

$$\begin{aligned} d(A, B) \;&=\; |1 - \cos t| \,+\, |\sin t| \,+\, |\alpha + \alpha\cos t| \\ &\ge\; |1 - \cos t| \,+\, |\alpha + \alpha\cos t| \\ &=\; 1 - \cos t \,+\, \alpha + \alpha\cos t \\ &=\; 2\alpha \,+\, (1 - \cos t)(1 - \alpha) \;\ge\; 2\alpha, \end{aligned}$$

where the final inequality comes from our assumption that $\alpha < 1$.

For this proof, when we refer to a closed curve, we mean a continuous map with domain S^1. We view our $f(t)$ as such a closed curve, by representing S^1 as $\mathbb{R}/2\pi$, and relying on the periodicity of the trigonometric functions. The same applies to curves constructed from $f(t)$, such as in (39) and (40) below. Finally when we say that closed curves $g_0(t)$ and $g_1(t)$ in A are homotopic, we mean that there exists a map $G\colon S^1 \times [0, 1] \longrightarrow A$ such that $G(t, i) = g_i(t)$ for $t \in S^1$ and $i \in \{0, 1\}$.

By definition of λ, there exist continuous operations $\boxplus$ and $\boxminus$ satisfying Γ within $\lambda_{A_\alpha}(\Gamma) + \varepsilon$ on A_α. As a consequence of (35), these operations satisfy Γ within $1 + \varepsilon < 2$. We consider the closed curve

$$t \longmapsto 0 \boxplus f(t) \tag{39}$$

By the second equation of (37) it must stay within $\lambda_{A_\alpha}(\Gamma) + \varepsilon$ of $f(t)$, and hence within < 2 of $f(t)$. Therefore the curve (39) is homotopic to $f(t)$, by the homotopy that was introduced in §3.2.1. Since there is a path connecting 0 to A, the path

$$t \longmapsto A \boxplus f(t) \tag{40}$$

is also homotopic to $f(t)$. Thus (40) maps onto $\{f(t) : \pi/2 \leq t \leq 3\pi/2\}$, and in particular, there exists t_0 such that

$$A \boxplus f(t_0) \;=\; (-1, 0, 0).$$

By reasoning similar to that for (40), except using the first equation of (37), the map

$$s \longmapsto f(s) \boxplus f(t_0)$$

is homotopic to $f(s)$, and so there exists s_0 with

$$f(s_0) \boxplus f(t_0) \;=\; (-1, 0, 0).$$

Thus in particular we have

$$f(s_0) \boxplus f(t_0) \;=\; A \boxplus f(t_0).$$

We now calculate, using (36):

$$A \;\approx\; (A \boxplus f(t_0)) \boxplus (\boxminus f(t_0)) \;=\; (f(s_0) \boxplus f(t_0)) \boxplus (\boxminus f(t_0)) \;\approx\; f(s_0),$$

where $\approx$ refers to approximate equality within $\lambda_{A_\alpha}(\Gamma) + \varepsilon$. We proved above that $d(A, f(s)) \geq 2\alpha$. Thus the triangle inequality now yields

$$\begin{aligned} 2\alpha \;\leq\; d(A, f(s)) \;&\leq\; 2(\lambda_{A_\alpha}(\Gamma) + \varepsilon) \\ &=\; 2\lambda_{A_\alpha}(\Gamma) \,+\, 2\varepsilon. \end{aligned}$$

Since ε can be taken arbitrarily small and positive, we now have the estimate (38).

Combining (35) with (38), we see that if Γ is a set of equations in $+$ and $-$ that contains Equations (36–37), and if each equation of Γ holds in group theory, then $\lambda_{A_\alpha}(\Gamma) = \alpha$.

Also note that A_α is compatible with H-space theory. (The proof is left to the reader.)

3.3.5 Boolean algebra

In 1947, I. Kaplansky proved [15] that if **A** is a compact topological Boolean ring (equivalently, a compact topological Boolean algebra), then **A** is isomorphic (as a topological algebra) to a power of the two-element discrete

Boolean algebra. In particular, this result says that if A is a compact metrizable space that is not homeomorphic to a power of the two-element discrete space, then A is not compatible with Boolean rings.

Kaplansky's proof is probably the most sophisticated proof of incompatibility on record: it relies on the result that if G is a locally compact Abelian group and $g \in G$ with g not the identity element, then there exists a group character $f : G \longrightarrow \mathbb{C}$ with $f(g) \neq 1$. The present author has no clue how such a method could be extended to yield a lower estimation for λ in this context. The trouble is that when the group axioms are relaxed so as to hold only approximately, the entire apparatus of group duality becomes unavailable. (Some approximate version of it might be possible, but that would appear to be a difficult prospect indeed, requiring serious theory-building.)

Therefore, it seems unlikely that we shall soon see a theorem of the form

$$\lambda_A(BA) > K \tag{41}$$

for BA taken to be Boolean ring theory. There is indeed another reason (41) is unlikely: for most spaces A it would probably be more practical and more meaningful to obtain an estimate of $\lambda_A(\Gamma)$ or of $\lambda_A(\Lambda)$ (group theory or lattice theory), since the exclusion of either of these will exclude a Boolean ring structure. Therefore, an inequality like (41) would not have a significant role unless A is a compact metric space that is compatible with group theory and with lattice theory, but is not a Cantor space. Such spaces seem rare.

In 1969, T. H. Choe gave a simpler proof [7] of Kaplansky's result, but only for spaces of finite dimension. It is not clear how Choe's proof could be turned into an estimate of $\lambda_A(BA)$ for some space A.

3.3.6 $A = \mathbb{R}$; $\Sigma =$ groups of exponent 2.

Here we suppose that Σ is a finite set of equations in $+$ and 0 that includes

$$\begin{aligned} x_0 + x_0 &\approx 0 \\ 0 + x_0 &\approx x_0 \\ x_0 + (x_0 + x_1) &\approx x_1. \end{aligned}$$

For instance, Σ could be an axiomatization of the theory of groups of exponent 2. Here we shall prove that, in any metric for $\mathbb{R}$, $\lambda_{\mathbb{R}}(\Sigma) \geq \text{radius}(\mathbb{R})/2$. (We mean this to include the assertion that $\lambda_{\mathbb{R}}(\Sigma) = \infty$ whenever $\mathbb{R}$ is metrized with infinite radius.)

For a contradiction, let us suppose that $\lambda_{\mathbb{R}}(\Sigma) = K < \text{radius}(\mathbb{R})/2$. Therefore, there exist a constant $\overline{0} \in \mathbb{R}$ and a continuous binary operation $\boxplus$ on $\mathbb{R}$ that satisfy Σ within K. Since $2K < \text{radius}(\mathbb{R})$, the $2K$-ball centered at $\overline{0}$ is not all of $\mathbb{R}$. In other words, there exists $a \in \mathbb{R}$ with $d(a, \overline{0}) > 2K$. We may assume, without loss of generality, that $\overline{0} < a$; in this case, we of course have

$$\overline{0} + K < a - K. \tag{42}$$

We consider the continuous real-valued function

$$x \longmapsto \phi(x) \;=\; x \boxplus a.$$

It follows readily from Σ that

$$\begin{aligned} \phi(\overline{0}) \;&=\; \overline{0} \boxplus a \;\approx\; a, \qquad\qquad \text{and} \\ \phi(a) \;&=\; a \boxplus a \;\approx\; \overline{0}. \end{aligned}$$

From these estimates, and (42), we deduce

$$\begin{aligned} \phi(\overline{0}) \;&>\; a - K \;>\; \overline{0} + K \;>\; \overline{0} \\ \phi(a) \;&<\; \overline{0} + K \;<\; a - K \;<\; a \end{aligned}$$

These last inequalities display a sign change for the function $\phi(x) - x$; hence, by the IVT, ϕ has a fixed point, i.e. $c = \phi(c) = c \boxplus a$ for some $c \in \mathbb{R}$. From Σ, we have

$$a \;\approx\; c \boxplus (c \boxplus a) = c \boxplus c \;\approx\; \overline{0}.$$

Thus $d(a, \overline{0}) < 2K$, in contradiction to our specification of a. This contradiction completes the proof that $\lambda_{\mathbb{R}}(\Sigma) \geq \text{radius}(\mathbb{R})/2$.

3.3.7 $A = \mathbb{R}^2$; $\Sigma =$ groups of exponent 2.

In §3.3.7 we take Γ_2 to be a set of axioms for groups of exponent 2. (Or, one might wish to consider the weaker equations of §3.3.6. One may also extend the problem to $\mathbb{R}^n$ for all $n \geq 2$.)

Problem $\lambda_{\mathbb{R}^2}(\Gamma_2) \;=\; \infty\,?$

We will sketch a proof that Γ_2 is not compatible with $\mathbb{R}^2$, and then comment on a (still unknown) strengthening of that proof that might establish that $\lambda_{\mathbb{R}^2}(\Gamma_2) > 0$ (and hence $\quad = \infty$).

Suppose $(\mathbb{R}^2, \overline{F}, \overline{e})$ is a group of exponent 2, with multiplication $\overline{F}$ and unit element $\overline{e}$. Choose $a \in \mathbb{R}^2$ with $a \neq \overline{e}$. The map $\phi : x \longmapsto F(x, a)$ is a continuous involution of $\mathbb{R}^2$, i.e. a map satisfying $\phi(\phi(x)) = x$. According to a 1934 theorem of P. A. Smith (see J. Dugundji and A. Granas [10, Theorem 5.3, page 79]), ϕ has a fixed point b. In other words, we have $F(b, a) = b$, with $a \neq \overline{e}$. This contradiction to the laws of group theory establishes that there can be no such continuous group operation $\overline{F}$ of exponent 2.

Now in order to extend this proof to approximate models of Γ_2, we need some way of finding an approximate fixed point for an approximate involution of $\mathbb{R}^2$, in other words, an approximated version of Smith's theorem mentioned above. As far as the author is aware, no such result is available.

3.3.8 Two auxiliary theories.

In §3.3.8 we consider two infinite theories, which we shall (in this section and the next) denote Σ_1 and Σ_2. These are the only infinite theories that we shall consider as such. (The two estimates derived in §3.3.8 both require infinitely many equations; they—and the corollary estimate in §3.4.1—are the only such estimates in the paper.) For $i = 1, 2$, and for any compact metric space A, we shall have that $\lambda_A(\Sigma_i) \geq \text{diameter}(A)/4$. The result for Σ_2 will be applied in §3.4.1 to make a similar estimate for the (more naturally occurring) variety of lattice-ordered groups.

Our methods for estimating $\lambda_A(\Sigma_i)$ $(i = 1, 2)$ are similar, but different enough that we will present both in some detail. In both cases we consider points $a, b \in A$ whose distance is the diameter of A, and apply the triangle inequality to a certain triangle $\triangle abe$. In one case, e will be the value of a term-function $e = \overline{K}(a, b)$; in the other case it will be the limit of a sequence of such values.

Σ_1 will be the following infinite set of equations

$$F(\phi^k(x), x, y) \approx x \tag{43}$$

$$F(x, x, y) \approx y, \tag{44}$$

for $k \in \omega$, $k \geq 1$. We introduced this theory in 1986—see [29, §3.18, page 35]—and proved that it is incompatible with every compact Hausdorff space.

Here we shall prove the stronger result that *if A is a compact metric space, then*

$$\lambda_A(\Sigma_1) \;\geq\; \text{diameter}(A)/4. \tag{45}$$

It is easiest to prove this inequality by contradiction. If (45) fails, then there is a topological algebra $\mathbf{A} = \langle A, \overline{\phi}, \overline{F} \rangle$ with $\lambda_{\mathbf{A}}(\Sigma_1) \;<\; \text{diameter}(A)/4$. Therefore there is a positive real number ε such that

$$\lambda_{\mathbf{A}}(\Sigma_1) \;<\; \text{diameter}(A)/4 - \varepsilon.$$

Let a and b be points of A with $d(a,b)$ equal to the diameter of A. Consider the sequence $\overline{\phi}^i(a)$; by compactness it has a convergent subsequence:

$$\lim_{i \longrightarrow \infty} \overline{\phi}^{n(i)}(a) \;=\; c \in A.$$

By the triangle inequality, we have either $d(a,c) \geq \text{diameter}(A)/2$ or $d(b,c) \geq \text{diameter}(A)/2$. Without loss of generality, we will assume that

$$d(b,c) \geq \text{diameter}(A)/2. \tag{46}$$

We next consider the infinite sequence in A,

$$\alpha_i \;=\; \overline{F}(\overline{\phi}^{n(i+1)}(a),\, \overline{\phi}^{n(i)}(a),\, b).$$

By our choice of the subsequence $\overline{\phi}^{n(i)}(a)$, and by the continuity of $\overline{F}$, there exists i_0 such that

$$d(\alpha_{i_0},\, \overline{F}(c,c,b)) \;<\; \varepsilon \quad \text{and} \tag{47}$$
$$d(\overline{\phi}^{n(i_0)}(a),\, c) \;<\; \varepsilon. \tag{48}$$

Now by the approximate satisfaction of (43–44) we have

$$d(\overline{F}(c,c,b),\, b) \;<\; \text{diameter}(A)/4 - \varepsilon$$
$$d(\alpha_{i_0},\, \overline{\phi}^{n(i_0)}(a)) \;<\; \text{diameter}(A)/4 - \varepsilon.$$

Combining these two inequalities with (47–48), via the triangle inequality, yields $d(b,c) \;<\; \text{diameter}(A)/2$. This contradiction to (46) completes the proof of (45).

Now let us take Σ_2 to be the following (doubly infinite) set of equations:

$$G(\psi_{m+k}(x,y),\, \psi_m(x,y),\, x,\, y) \;\approx\; x \tag{49}$$
$$K(x,y) \;\approx\; G(u,u,x,y) \;\approx\; K(y,x), \tag{50}$$

for $m, k \in \omega$, with $k \geq 1$. We shall again establish that if A is compact metric space, then

$$\lambda_A(\Sigma_2) \;\geq\; \text{diameter}(A)/4. \tag{51}$$

It is easiest to prove this inequality by contradiction. If (51) fails, then there is a topological algebra $\mathbf{A} = \langle A, \overline{\psi}, \overline{G}, \overline{K}\rangle$ with $\lambda_{\mathbf{A}}(\Sigma_2) \;<\; \text{diameter}(A)/4$. Therefore there is a positive real number ε such that

$$\lambda_{\mathbf{A}}(\Sigma_2) \;<\; \text{diameter}(A)/4 - \varepsilon.$$

Let a and b be points of A with $d(a,b)$ equal to the diameter of A. By the triangle inequality, we have either $d(a, \overline{K}(a,b) \geq \text{diameter}(A)/2$ or $d(b, \overline{K}(a,b) \geq \text{diameter}(A)/2$. Without loss of generality, we shall assume that

$$d(b, \overline{K}(a,b)) \;\geq\; \text{diameter}(A)/2 \tag{52}$$

Consider the sequence $\overline{\psi}_i(b,a)$; by compactness it has a convergent subsequence:

$$\lim_{i \longrightarrow \infty} \overline{\psi}_{n(i)}(b,a) \;=\; c \in A.$$

We next consider the infinite sequence in A

$$\beta_i \;=\; \overline{G}(\overline{\psi}_{n(i+1)}(b,a),\, \overline{\psi}_{n(i)}(b,a),\, b,\, a).$$

By the continuity of $\overline{G}$, there exists i_0 such that

$$d(\overline{G}(c,c,b,a),\, \beta_{i_0}) \;<\; 2\varepsilon. \tag{53}$$

Now by the approximate satisfaction of (49–50) we have

$$d(\overline{G}(c,c,b,a),\, b) \;<\; \text{diameter}(A)/4 - \varepsilon;$$
$$d(\beta_{i_0},\, \overline{K}(a,b)) \;<\; \text{diameter}(A)/4 - \varepsilon.$$

Combining these two inequalities with (53), via the triangle inequality, yields $d(b, \overline{K}(a,b)) < \text{diameter}(A)/2$. This contradiction to (52) completes the proof of (51).

It may be noted that in the proof for Σ_1 we used only the continuity of $\overline{F}$, and in the proof for Σ_2 we used only the continuity of $\overline{G}$. We never invoked the continuity of $\overline{K}$, $\overline{\phi}$ or the $\overline{\psi}_m$'s. Similar occurrences will be noted below, in §3.5.1 and in Theorem 13 of §3.6.3.

3.3.9 Lattice-ordered groups.

We will let $\Lambda\Gamma$ stand for a certain set of equations (see (55–56) below) that is satisfied by the variety of lattice-ordered groups. That variety has a well-known finite axiomatization, which we paraphrase below, but do not state formally. Our $\Lambda\Gamma$ is an infinite set of equations true in LO-groups.[14]

Like a Boolean algebra, a lattice-ordered group has both lattice operations $\wedge, \vee$ and group operations $+, -, 0$. Therefore any incompatibilities for these two strong theories (see §3.3 and §3.4 *passim*) will be inherited by $\Lambda\Sigma$. But LO-groups indeed form a theory that is stronger than the join of groups and lattices. In addition to the usual axioms of group theory and lattice theory, it is assumed that addition distributes over meet and join: $x + (y \wedge z) \approx (x + y) \wedge (x + z)$, and dually. The resulting theory is quite strong. As was proved by M. Ja. Antonovskiĭ and A. V. Mironov [2] in 1967 *no compact Hausdorff space is compatible with lattice-ordered groups.* (Thus, for example, the Cantor set is a space that is compatible with Boolean algebra, but not with LO-groups.) Here we will prove, for any compact metric space A, the stronger result that

$$\lambda_A(\Lambda\Gamma) \;\geq\; \frac{1}{4}\,\text{diameter}(A). \tag{54}$$

Our method for proving (54) is very simple: in §3.3.8 we proved the corresponding estimate (51) for the special theory Σ_2. To prove (54), we need only give a careful definition of $\Lambda\Gamma$, prove that Σ_2 is interpretable (§2.3) in $\Lambda\Gamma$, and then invoke Theorem 1 of §2.3 (monotonicity of λ_A with respect to the interpretability relation).

[14] $\Lambda\Gamma$ is obviously weaker than the theory of LO-groups, since it does not mention $\vee$. See Equations (55–56) below.

$\Lambda\Gamma$ is defined to be the following (doubly infinite) set of equations:

$$x \approx x \wedge [(z_{m+k} - z_m) + (x \wedge y)] \tag{55}$$
$$x \wedge y \approx x \wedge [(u - u) + (x \wedge y)] \approx y \wedge x, \tag{56}$$

where z_n $(n \in \omega)$ are terms defined recursively as follows:

$$z_0 = 0; \qquad z_{n+1} = (z_n + (x - (x \wedge y))).$$

Since $x \geq x \wedge y$ in LO-groups, it is easy to see that for $r \leq s$, LO-groups satisfy $z_r \leq z_s$. To see that LO-groups satisfy (55), we calculate

$$\begin{aligned} x &\approx [z_m + (x - x \wedge y)] - z_m + x \wedge y \\ &\approx z_{m+1} - z_m + x \wedge y \leq z_{m+k} - z_m + x \wedge y, \end{aligned}$$

for $m, k \in \omega$ with $k \geq 1$. The validity of (56) in LO-groups is evident. Thus $\Lambda\Gamma$ is a subset of the equations holding in LO-groups.

Finally, we need to prove that $\Sigma_2 \leq \Lambda\Gamma$, where Σ_2 is as defined in §3.3.8 above (Equations (49–50)). In the theory of LO-groups we define the following terms:

$$G(u, v, x, y) = x \wedge [(u - v) + (x \wedge y)] \tag{57}$$
$$K(x, y) = x \wedge y \tag{58}$$
$$\psi_m(x, y) = z_m. \tag{59}$$

These equations define $G(u, v, x, y)$ (resp. $K(x, y)$, resp. $\psi_m(x, y)$) as the interpreting term for the operation G (resp. K, resp. ψ_m). Clearly Equation (49) interprets as (55) and Equations (50) interpret as (56). Since Equations (55–56) define $\Lambda\Gamma$, the interpretation is valid; we do have $\Sigma_2 \leq \Lambda\Gamma$.

3.4 Σ related to lattice theory

3.4.1 $A = S^1$ (circle); $\Sigma =$ semilattice theory.

As in §3.2, we let S^n stand for the n-sphere of diameter 1, embedded as a metric subspace of $\mathbb{R}^{n+1}$, in the usual way. In §3.2.3 we proved that $\lambda_{S^1}(\Sigma) = 1$ if Σ is not Abelian. It is not hard to see that the theory of semilattices is not Abelian. We will establish this fact for the weaker theory of an *idempotent commutative binary operation*:

$$F(x, y) \approx F(y, x); \qquad F(x, x) \approx x. \tag{60}$$

Suppose we had $(\mathbb{Z}, \overline{F})$ modeling Equations (60), where $\overline{F}(x,y) = ax + by$ for some integers a and b. The first equation of (60) forces $a = b$, and the second forces $a + b = 1$. These two equations in a and b have no solution in $\mathbb{Z}$; hence Equations (60) do not form an Abelian set. Thus, in particular,

$$\lambda_{S^1}(\text{idempotent commutative}) \;=\; 1.$$

The reader may take a similar path to discover that $\lambda_{S^1}(\Sigma) \;=\; 1$, where Σ is taken to be the equations of *multiplication with zero and one*:

$$F(0,x) \;\approx\; 0; \qquad F(1,x) \;\approx\; x. \tag{61}$$

In [29, §3.6, page 28], we proved that if A is arcwise-connected and not contractible, then A is not compatible with the equations (61). Nevertheless, for these equations we have a calculation of a non-zero value of λ only for the one space S^1.

The reader may also wish to consider the equations of *median algebras*:

$$m(x,y,z) \;\approx\; m(x,z,y) \;\approx\; m(y,z,x) \tag{62}$$

$$m(x,x,z) \;\approx\; x \tag{63}$$

$$m(m(x,y,z),u,v) \;\approx\; m(x, m(y,u,v), m(z,u,v)). \tag{64}$$

(See e.g. Bandelt and Hedlíková [3].) The reader may check that the proof given here in §3.4.1 for an idempotent commutative algebra may be applied also for Equations (62–64). (In fact Equations (62–63) suffice for a proof.) Therefore this theory also has λ_{S^1} equal to 1.

3.4.2 S^1 with a different metric.

In §3.4.2 we will let Σ be any theory compatible with $[0,1]$—such as, for example, the theories mentioned in §3.4.1 (semilattices, multiplication with zero and one, median algebra). Here we shall see that, for each $\varepsilon > 0$, there is a metric d on S^1 such that $\lambda_{(S^1,d)}(\Sigma^\star) < \varepsilon$. The method is like that of §3.2.2.

We begin with the realization of S^1 as the ellipse that is the locus of

$$4\varepsilon^2 x^2 \;+\; 4y^2 \;=\; \varepsilon^2$$

in ordinary 2-space. Then d is defined to be the ordinary Euclidean distance of $\mathbb{R}^2$, restricted to the ellipsoid. One easily checks that (S^1, d) has diameter

1. If we define $\psi : S^1 \longrightarrow S^1$ via $\psi(x, y) = (x, |y|)$, then we clearly have the hypotheses of Theorem 5 of §2.6, with the image of ψ homeomorphic to $[0, 1]$. Therefore

$$\lambda_{(S^1,d)}(\Sigma^\star) \;\leq\; \lambda_{[0,1]}(\Sigma^\star) \,+\, \varepsilon \;=\; \varepsilon.$$

3.4.3 An extension of §3.4.1

Here we consider an arbitrary path-connected and locally path-connected metric space A whose fundamental group is isomorphic to $\mathbb{Z}$ (under ordinary addition), and which has the following property: *there exists* $K > 0$ *such that if* $\gamma(t), \delta(t)$ *are loops beginning and ending at the same point* $a \in A$, *and if* $d(\,\gamma(t),\, \delta(t)\,) < K$ *for all* t, *then* γ *and* δ *are homotopic as loops.*

Let Σ be an idempotent set of equations that is not Abelian (as defined in §3.2.3). We shall in fact require[15] that each term appearing in Σ shall be declared as idempotent by an equation in Σ. (For example, if Σ contains $x + (y + z) \approx (x + y) + z$, then Σ must also contain $x + (x + x) \approx x$ and $(x + x) + x \approx x$.) We shall prove that $\lambda_A(\Sigma) \geq K$. (In §3.4.1 we had these assumptions true for $K = 1$, and the conclusion was $\lambda_A(\Sigma) = 1$.)

For a contradiction, let us suppose that $\lambda_A(\Sigma) < K$. This means that $A \models_\varepsilon \Sigma$ for some $\varepsilon < K$. (See (3) of §0.2.) Thus there are operations $\overline{F}_t$ corresponding to the symbols F_t appearing in Σ, such that for each equation $\sigma \approx \tau$ in Σ, the terms σ and τ evaluate to functions $\overline{\sigma}$ and $\overline{\tau}$ that are within ε of each other. Our contradiction will be that Σ is Abelian; for this we need to satisfy Σ in an algebra $(\mathbb{Z}, F_t^\star)_{t\in T}$, where each operation $F_t^\star(x_1, x_2, \ldots)$ has the form $\sum m_i x_i$ in $\mathbb{Z}$.

We pick a base point $a_0 \in A$ and consider homotopy classes $[\alpha]$ of loops in A at a_0. We define $[\beta] = F_t^\star([\alpha_1], [\alpha_2], \ldots)$ to be the homotopy class of the loop defined by $\overline{F}_t(\alpha_1(t), \alpha_2(t), \ldots)$. Since $\overline{F}_t$ is idempotent within ε, this loop starts and stops at a point a' near a_0. Any two paths that connect a_0 to a' that stay within ε of a_0 are homotopic. Hence we may (up to homotopy) unambiguously append such a path to the beginning, and its reverse to the end, of $[\beta]$; the modified $[\beta]$ is what we take for $[F_t^\star([\alpha_1], [\alpha_2], \ldots)$. Now this loop may be continuously deformed, without moving the endpoints, to one that begins with $\overline{F}_t(\alpha_1(t), a_0, a_0, \ldots)$, followed by $\overline{F}_t(a_0, \alpha_2(t), a_0, \ldots)$, and

[15] This requirement would be used in the proof of (65), which we skipped. It is used to get each loop to start and stop at a_0, as we do just below in the definition of $F^\star$.

so on. From this we see that $F^\star([\alpha_1],[\alpha_2],\ldots)$ has the required linear form $\sum m_i[\alpha_1]$.

For the satisfaction of Σ in $(\mathbb{Z}, F_t^\star)_{t\in T}$, we consider an equation $\sigma \approx \tau$ of Σ. We skip the proof (inductive over complexity of σ) that the derived operation $\sigma^\star$ corresponding to σ in $(\mathbb{Z}, F_t^\star)_{t\in T}$ satisfies

$$\sigma^\star([\alpha_1],[\alpha_2],\ldots) \;\approx\; [\sigma'], \tag{65}$$

where

$$\sigma'(t) \;=\; \overline{\sigma}(\alpha_1(t),\alpha_2(t),\ldots). \tag{66}$$

Now, if $\sigma \approx \tau$ is in Σ, then by assumption the equation $\overline{\sigma}(x_1,x_2,\ldots) = \overline{\tau}(x_1,x_2,\ldots)$ holds within ε on A. By (66) we have $\sigma'(t) = \tau'(t)$, within ε, for all t. By our assumptions on the metric space A, we have σ' homotopic to τ'; in other words that $[\sigma'] = [\tau']$. It is immediate from (65) that $\sigma^\star = \tau^\star$, as operations on $\mathbb{Z}$.

One special case of the result of §3.4.3 occurs when Σ is the theory of an idempotent commutative operation, as presented in (60), and A has the form $S^1 \times Y$, where Y has trivial fundamental group. This configuration will be of interest in §5.2 and §8 below.

3.4.4 $A = \mathbb{R}$; $\Sigma =$ join-semilattice theory with zero.

We let Σ consist of any axioms for semilattice theory, expressed in terms of a join operation $\vee$, together with a zero operation for $\vee$. More precisely, the equations that we shall require in Σ are these somewhat weaker equations:

$$\begin{gathered}(x \vee y) \vee y \;\approx\; x \vee y, \quad y \vee (y \vee x) \;\approx\; y \vee x,\\ 0 \vee x \;\approx\; x \;\approx\; x \vee 0,\\ x \vee x \;\approx\; x.\end{gathered}$$

Here we shall show that each of the equations

$$\lambda_{(A,d)}(\Sigma) \;=\; 0 \tag{67}$$

$$\lambda_{(A,d)}(\Sigma) \;=\; \infty \tag{68}$$

holds for an appropriate choice of a metric d inducing the usual topology on a space A homeomorphic to $\mathbb{R}$. (Moreover, (67) provides us with a further example of A not compatible with Σ but for which $\lambda_A(\Sigma) = 0$. It differs from the example in §3.1 in that this Σ is consistent. For an example with A compact, see §3.4.8 below.)

For (67), we take $A = (0, 1)$ (homeomorphic to $\mathbb{R}$), with d the usual metric on $(0, 1)$. To establish (67), it will suffice to find continuous operations that satisfy Σ within ε, for every $\varepsilon > 0$. This is easily accomplished: we let $\overline{\vee}$ be the usual semilattice operation, and take $\overline{0}$ (the element of our algebra that is denoted by the constant symbol 0) to be ε. The detailed verification is left to the reader.

For (68), we take $A = \mathbb{R}$, with d the usual metric. For a contradiction, suppose that $\lambda_{(A,d)}(\Sigma) = K < \infty$. By definition, there are a constant $\overline{0}$ and a continuous operation $\overline{\vee}$ on $\mathbb{R}$ satisfying Σ within K. Take $a < b \in \mathbb{R}$, with $\overline{0}$ between a and b and with $d(a, \overline{0}) = d(b, \overline{0}) = 3K$.

Let us suppose, without loss of generality, that $a \,\overline{\vee}\, b \geq 0$. Now consider the continuous real-valued function

$$x \longmapsto a \,\overline{\vee}\, x.$$

It follows from Σ that $a \,\overline{\vee}\, a < \overline{0}$. Thus our function maps a and b to $a \,\overline{\vee}\, a$ and $a \,\overline{\vee}\, b$, which lie on opposite sides of $\overline{0}$. By the IVT, there exists $e \in \mathbb{R}$ such that $a \,\overline{\vee}\, e = \overline{0}$. Using Σ, we have

$$\overline{0} \;=\; a \,\overline{\vee}\, e \;\approx\; a \,\overline{\vee}\, (a \,\overline{\vee}\, e) \;=\; a \,\overline{\vee}\, \overline{0}.$$

Therefore $d(\overline{0}, a \,\overline{\vee}\, \overline{0}) \leq K$; again invoking Σ, we have $d(a \,\overline{\vee}\, \overline{0}, a) \leq K$ We now use the triangle inequality to compute

$$\begin{aligned} d(\overline{0}, a) \;&\leq\; d(\overline{0},\, a \,\overline{\vee}\, \overline{0}) \;+\; d(a \,\overline{\vee}\, \overline{0},\, a) \\ &\leq\; K \;+\; K \;=\; 2K. \end{aligned}$$

This contradiction completes the proof of (68).

3.4.5 A lemma on approximate satisfaction of lattice equations.

In Lemma 7 which follows, Y is the space of a connected acyclic one-dimensional simplicial complex, of which $U = [u_0, u_1]$ is a designated edge. Each edge of Y has a known length, and then distance $d(A, B)$ is measured by accumulating these lengths along the unique injective path from A to B (with distance pro-rated along an incomplete edge).

If we arbitrarily choose one ordering of the vertices of U, say $u_0 < u_1$, then all of Y can be ordered, as follows. If there is no injective path containing A, B, u_0 and u_1, then A and B are not $<$-related. Otherwise a unique such path exists, and we can list the four vertices in the order they appear along the path, making sure to list u_0 before u_1. For instance, we might have u_0ABu_1, or Bu_0u_1A, or a number of other possibilities. Then we simply take $A \leq B$ (resp. $A \geq B$) to mean that A appears before (resp. after) B in this listing. The notation $[A, B]$ denotes, in the usual way, a closed interval under this ordering.

It is easy to see that *if* $A \in U$ *and* $B \in Y$*, then trichotomy holds:* $A < B$ *or* $A = B$ *or* $A > B$.

Lemma 6 *If* K *is a connected subset of* Y*, if* $r, s, t \in Y$ *with* $r \leq s \leq t$*, and if* $r, t \in K$ *and* $s \in U$*, then* $s \in K$*.*

Proof. By contradiction. If $s \notin K$, then the two sets

$$K_0 = \{ s \in K : x < s \} \quad \text{and} \quad K_1 = \{ s \in K : x > s \}$$

are disjoint non-empty open sets in K with union K. ∎

In the context of operation symbols $\wedge$ and $\vee$, the *dual* of any lexical object is formed by interchanging these two symbols.

Commutative rearrangement is the smallest equivalence relation on $(\wedge, \vee)$-terms that satisfies the following condition: if τ_i is a commutative rearrangement of σ_i $(i = 1, 2)$, then $\tau_2 \vee \tau_1$ is a commutative rearrangement of $\sigma_1 \vee \sigma_2$ (and similarly for $\wedge$). A *commutative rearrangement* of an equation $\sigma_1 \approx \sigma_2$ is any equation $\tau_1 \approx \tau_2$ where τ_i is a commutative rearrangement of σ_i $(i = 1, 2)$.

In the following lemma, Y and U continue to be as specified at the start of §3.4.5.

Lemma 7 *Let* $\varepsilon > 0$*. Suppose that a topological algebra* $\mathbf{Y} = (Y, \overline{\wedge}, \overline{\vee})$ *is given, and that the following equation-set, when closed under duals and commutative rearrangements, holds within* ε *for all* $x_0, x_1 \in U$*:*

$$\begin{aligned} x_0 \vee x_0 &\approx x_0 \\ x_0 \vee (x_0 \vee x_1) &\approx x_0 \vee x_1 \\ x_0 \vee (x_1 \wedge x_0) &\approx x_0. \end{aligned}$$

Suppose that $a, b \in U$*, with* $d(a, b) > \varepsilon$*. Then* *(i) Either $a \overline{\vee} b \in [a,b]$, or $d(a \overline{\vee} b, a) \leq \varepsilon$, or $d(a \overline{\vee} b, b) \leq \varepsilon$. The same disjunction holds for $a \overline{\wedge} b$.*

(ii) Either $a \overline{\vee} b$ is within 2ε of b, or $a \overline{\wedge} b$ is within 2ε of b.

Proof. Proof of (i). Without loss of generality, we will assume that $a < b$. We need prove (i) only for $a \overline{\vee} b$. (The dual of our proof for $a \overline{\vee} b$ is a proof for $a \overline{\wedge} b$.) If the first alternative holds, we are done; hence we will assume that it fails, i.e. that $a \overline{\vee} b \notin [a,b]$. By trichotomy, this means that $a \overline{\vee} b > b$ or $a \overline{\vee} b < a$. We consider the first of these possibilities; the second is similar (subject to some commutative rearrangement), and will be left to the reader.

We are assuming that $d(a, a \overline{\vee} a) < \varepsilon$, but $d(a,b) > \varepsilon$. Therefore $a \overline{\vee} a < b$. On the other hand we are in the case where $a \overline{\vee} b > b$. Let us take K to be the image of the function $x \longmapsto a \overline{\vee} x$ over the interval $[a,b]$. Clearly K is connected and contains $a \overline{\vee} a$ and $a \overline{\vee} b$ with $a \overline{\vee} a \ < \ b \ < \ a \overline{\vee} b$. We shall apply Lemma[16] 6 with $r = a \overline{\vee} a$, $s = b$ and $t = a \overline{\vee} b$.

By the Lemma, we have $b \in K$. In other words, there exists $e \in U$ for which $a \overline{\vee} e = b$. By the approximate satisfaction of our equations, we have

$$a \overline{\vee} b \ = \ a \overline{\vee} (a \overline{\vee} e) \ \approx \ a \overline{\vee} e \ = \ b,$$

and hence $d(a \overline{\vee} b, b) < \varepsilon$. This completes the proof of Part (i).

For Part (ii), we shall assume, by way of contradiction, that

$$d(a \overline{\vee} b, b) \ \geq \ 2\varepsilon; \qquad d(a \overline{\wedge} b, b) \ \geq \ 2\varepsilon. \tag{69}$$

Therefore, for both $a \overline{\vee} b$ and $a \overline{\wedge} b$, the first or second clause of (i) must hold. In particular, $a \overline{\vee} b$ is either in $[a,b]$ or $< a$, and the same may be said of $a \overline{\wedge} b$. These alternatives divide the rest of the proof into three cases.

Case 1. $a \overline{\vee} b < a$. We know that $b \overline{\vee} b \approx b$; which implies that $a < b \overline{\vee} b$ (since we are given that $d(a,b) > \varepsilon$). Therefore, we may apply Lemma 6 to see the existence of e with $e \overline{\vee} b = a$. By our equations, we now have

$$b \ \approx \ b \overline{\wedge} (e \overline{\vee} b) = b \overline{\wedge} a,$$

[16] Here Lemma 6 replaces the Intermediate Value Theorem. If we knew that $K \subseteq U$, or even that K lies in some subset of Y homeomorphic to $\mathbb{R}$, then we could use the IVT directly.

which is one alternative of the desired conclusion of Part (ii).

Case 2. $a \,\overline{\wedge}\, b < a$. The proof is dual to that of Case 1.

Case 3. $a \,\overline{\vee}\, b$ and $a \,\overline{\wedge}\, b$ both lie in $[a, b]$. In this case, we may apply trichotomy and observe that without loss of generality, we have

$$a \,\overline{\wedge}\, b \;\leq\; a \,\overline{\vee}\, b \;<\; b. \tag{70}$$

By one of our given approximate identities, we have $d(b \,\overline{\wedge}\, b, b) < \varepsilon$; combining this with (69) and (70), we see that

$$a \,\overline{\wedge}\, b \;\leq\; a \,\overline{\vee}\, b \;\leq\; b \,\overline{\wedge}\, b.$$

Now we have $a \,\overline{\vee}\, b$ between two values of the continuous function $x \longmapsto x \,\overline{\wedge}\, b$. Moreover, $[a, b] \subseteq U$, and hence $a \,\overline{\vee}\, b \in U$. Another application of Lemma 6 yields $e \in U$ such that $e \,\overline{\wedge}\, b \;=\; a \,\overline{\vee}\, b$.

Using the approximate satisfaction of the given equations, we calculate

$$b \;\approx\; b \,\overline{\vee}\, (e \,\overline{\wedge}\, b) \;=\; b \,\overline{\vee}\, (a \,\overline{\vee}\, b) \;\approx\; a \,\overline{\vee}\, b.$$

Thus $d(b, a \,\overline{\vee}\, b) \;<\; 2\varepsilon$, and Part (ii) is proved. ∎

The following corollary was first proved by A. D. Wallace in the nineteen-fifties. See [11, 16, 17, 33].

Corollary 8 *If $([0, 1], \overline{\wedge}, \overline{\vee})$ is a topological lattice, then either $\overline{\wedge}$ and $\overline{\vee}$ are ordinary meet and join of real numbers (i.e. $x \,\overline{\wedge}\, y$ is the smaller of x and y, and $x \,\overline{\vee}\, y$ is the larger), or dually.*

Proof. We apply Lemma 7 in the special case where Y has only one simplex of dimension 1; in other words, in the case where $Y = U = [0, 1]$. Considering arbitrary $a, b \in [0, 1]$, Part (ii) of the lemma tells us that $a \,\overline{\vee}\, b$ is a limit point of the set $\{a, b\}$. Hence $a \,\overline{\vee}\, b \in \{a, b\}$. Similarly $a \,\overline{\wedge}\, b \in \{a, b\}$.

We may assume, without loss of generality, that $0 \,\overline{\vee}\, 1 \;=\; 1$. We now consider two subsets of $(0, 1]$, namely

$$K_0 \;=\; \{\, x \in (0, 1] \;:\; 0 \,\overline{\vee}\, x = 0 \,\}; \quad K_1 \;=\; \{\, x \in (0, 1] \;:\; 0 \,\overline{\vee}\, x = x \,\}.$$

It follows readily from the continuity of $0 \,\overline{\vee}\, x$ that K_i is open in $(0, 1]$ ($i = 0, 1$), and from what we have already proved that $(0, 1]$ is the disjoint union of these two sets. By connectedness, one of the two sets is empty. Clearly $1 \in K_1$, so K_0 is empty. We have now established that $0 \mathbin{\overline{\vee}} x = x$ for all x.

We next consider $b > 0$, and define the two sets

$$K_0 \;=\; \{\, x \in [0, b) \,:\, b \mathbin{\overline{\vee}} x = b \,\}; \quad K_1 \;=\; \{\, x \in [0, b) \,:\, b \mathbin{\overline{\vee}} x = x \,\}.$$

As before, the two sets form an open partition of $[0, b)$. By what we have done before, $0 \in K_0$, so $K_0 = [0, b)$. In other words we have proved that if $a < b$, then $a \mathbin{\overline{\vee}} b = b$. Finally, for $a \leq b$, we use lattice theory to calculate that $a \mathbin{\overline{\wedge}} b = a \mathbin{\overline{\wedge}} (a \mathbin{\overline{\vee}} b) = a$. Thus $\overline{\wedge}$ and $\overline{\vee}$ agree with the usual lattice operations on $[0, 1]$. ▮

3.4.6 $A = Y$, the triode; $\Sigma =$ lattice theory.

Let B, C, D, E be four non-collinear points in the Euclidean plane, with E in the interior of $\triangle BCD$. Our space Y is defined to be the union of the three (closed) segments BE, CE and DE, called *legs*, with the topology inherited from the plane. In fact, in order to give Y a definite metric d, we will further require that $\triangle BCD$ be equilateral with E at its center, and that each leg have unit length. We then let d be the metric of the plane, as inherited by Y.

We let Σ consist of any axioms for lattice theory (expressed in terms of $\wedge$ and $\vee$), which includes the equations of Lemma 7 on page 49 of §3.4.5, and also the commutative law $x_0 \wedge x_1 \approx x_1 \wedge x_0$, and its dual. It was proved by A. D. Wallace in the mid-1950's (see [33, *Alphabet Theorem*] for a statement of the result) that Y is not compatible with lattice theory. Here we shall prove the sharper result that $\lambda_Y(\Sigma) \geq 0.25$.

For a contradiction, suppose that $\lambda_Y(\Sigma) < 0.25$. By definition, there are continuous operations $\overline{\wedge}$ and $\overline{\vee}$ on Y satisfying Σ within $\lambda_Y(\Sigma)$, which is < 0.25.

We shall now apply Lemma 7 of §3.4.5 to this Y, and to U taken to be, first BE, then CE, then DE. Conclusion (ii) of that Lemma yields, respectively,

$$\begin{aligned}
&\big(d(B, B \mathbin{\overline{\wedge}} E) < 0.5\big) \quad \text{or} \quad \big(d(B, B \mathbin{\overline{\vee}} E) < 0.5\big);\\
&\big(d(C, C \mathbin{\overline{\wedge}} E) < 0.5\big) \quad \text{or} \quad \big(d(C, C \mathbin{\overline{\vee}} E) < 0.5\big);\\
&\big(d(D, D \mathbin{\overline{\wedge}} E) < 0.5\big) \quad \text{or} \quad \big(d(D, D \mathbin{\overline{\vee}} E) < 0.5\big).
\end{aligned}$$

Clearly either the $\overline{\wedge}$-alternative holds for at least two of these three propositions, or the $\overline{\vee}$-alternative holds at least twice. We will assume, without loss of generality, that the $\overline{\wedge}$-alternative holds in at least two of the propositions, and in particular that it holds for legs BE and CE. Thus we have

$$\big(d(B, B\,\overline{\wedge}\,E) < 0.5\big) \quad \text{and} \quad \big(d(C, C\,\overline{\wedge}\,E) < 0.5\big). \tag{71}$$

The proof now divides into two cases, depending on the location of the point $B\,\overline{\wedge}\,C$. Clearly it must lie in one of the three legs.

Case 1. $B\,\overline{\wedge}\,C \in DE$ **or** $B\,\overline{\wedge}\,C \in BE$**.** Consider the continuous map

$$z \longmapsto z\,\overline{\wedge}\,C,$$

defined for $z \in Y$. Clearly the image of C is $C\,\overline{\wedge}\,C$, which lies within 0.25 of C (by the approximate satisfaction of Σ), and hence is a point of leg CE. On the other hand, the image of B is $B\,\overline{\wedge}\,C$, which we have assumed to lie either on leg DE or on leg BE. Thus the image of this map is a connected subset of Y that contains points on at least two legs. Therefore, the image of our map must contain E, which is to say that

$$P\,\overline{\wedge}\,C \;=\; E$$

for some $P \in Y$.

Using the equations of Σ, we calculate

$$E \;=\; P\,\overline{\wedge}\,C \;\approx\; (P\,\overline{\wedge}\,C)\,\overline{\wedge}\,C \;=\; E\,\overline{\wedge}\,C \;\approx\; C\,\overline{\wedge}\,E,$$

which implies that

$$d(E, C\,\overline{\wedge}\,E) \;<\; 0.5.$$

Combined with (71), this last inequality yields $d(E, C) < 1$, in contradiction to our specification of CE as a segment of length 1. This contradiction completes the proof for Case 1.

Case 2. $B\,\overline{\wedge}\,C \in CE$**.** This situation is symmetric to $B\,\overline{\wedge}\,C \in BE$, which was considered in Case 1. (Interchanging the letters B and C in that proof, while sending z to $B\,\overline{\wedge}\,z$, yields a proof here.) This completes the proof that $\lambda_Y(\Sigma) \geq 0.25$.

3.4.7 $A = Y$ with a new metric.

Let Σ be the axiom-set (expressing part of lattice theory) that was presented in §3.4.6, and let Y be the space defined there (a union of three segments). We proved that for a certain metric of diameter $\sqrt{3}$ the space Y satisfies $\lambda_Y(\Sigma) \geq 0.25$. In other words, $\lambda_Y(\Sigma) \geq 0.25/\sqrt{3} = 0.14433\ldots$ for a certain natural metric of diameter 1.

Here we shall show that for each real $\varepsilon > 0$, there exists a diameter-1 metric d for the topology of Y with $\lambda_{(Y,d)}(\Sigma^\star) < \varepsilon$.

The metric d is obtained very simply by expressing Y—as before—as the union of unit-length segments EB, EC and ED in the plane; this time with B, C and D chosen to lie within ε of one another, and with no two of them collinear with E. The metric d is then simply the restriction to Y of the metric in the plane. Clearly, for $\varepsilon < 1$, the metric space (Y, d) has diameter 1, and so it will be enough to prove that $\lambda_{(Y,d)} < \varepsilon$.

We first define

$$\psi : Y \longrightarrow Y$$

as follows: for any point $X \in Y$, $\psi(X)$ is the unique point on segment EB that satisfies $d(E, \psi(X)) = d(E, X)$. Clearly ψ is continuous and ψ is the identity function on EB.

The reader may check that we now have the hypotheses of Theorem 5 of §2.6 (with Y for A, EB for B, and ε for K). In this context, the conclusion of the theorem is

$$\lambda_{(Y,d)}(\Lambda) \;\leq\; \lambda_{(EB,d)}(\Lambda) \,+\, \varepsilon.$$

Moreover EB is topologically a closed segment, which is compatible with lattice theory; hence $\lambda_{(EB,d)}(\Lambda) = 0$. We now have the desired conclusion that $\lambda_{(Y,d)}(\Sigma) < \varepsilon$.

3.4.8 $A = [0,1]$, Σ_c an extension of lattice theory

Let Σ_c be the following set of equations:

$$\begin{array}{llll} \text{axioms of lattice theory in } \wedge, \vee & & & \\ 0 \wedge x \approx 0 & & & \\ 1 \vee x \approx 1 & & & \\ F(0) \approx 0 & & F(1) \approx 1 & \\ G(F(x)) \approx 1 & & G(a) \approx 0. & \end{array}$$

We will see that Σ_c is consistent, that Σ_c is not compatible with $[0,1]$, and that $\lambda_{[0,1]}(\Sigma_c) = 0$.

Let us first see that Σ_c is not compatible with the compact interval $[0,1]$. Looking for a contradiction, we will suppose that $([0,1], \overline{\wedge}, \overline{\vee}, \overline{0}, \overline{1}, \overline{F}, \overline{G}, \overline{a}) \models \Sigma_c$, with $\overline{\wedge}$, $\overline{\vee}$, $\overline{F}$ and $\overline{G}$ continuous. Applying Corollary 8, we may assume, without loss of generality, that $\overline{\wedge}$ and $\overline{\vee}$ are ordinary meet and join on the interval $[0,1]$. Obviously then, $\overline{0} = 0$ and $\overline{1} = 1$. By the continuity of $\overline{F}$. its range is connected, and clearly the range contains 0 and 1. Therefore, $\overline{F}$ maps $[0,1]$ onto itself. Thus the penultimate equation, $G(F(x)) \approx 1$, tells us that $\overline{G}$ is a constant function with value 1. The final equation, however, yields $\overline{G}(\overline{a}) = 0$; hence $\overline{G}$ is not a constant function. This contradiction concludes the proof that $[0,1]$ is not compatible with Σ_c.

We next prove that

$$\lambda_{([0,1],d)}(\Sigma_c) = 0, \tag{72}$$

where d is the usual metric on $[0,1]$. In order to prove (72), for each $\varepsilon > 0$ we shall construct continuous $[0,1]$-operations modeling Σ_c to within ε.

We let $\overline{\vee}$ and $\overline{\wedge}$ be ordinary join and meet on $[0,1]$. These are well known to be continuous, and certainly they satisfy the equations of lattice theory. We define $\overline{0}$ to be a point of $[0,1]$ other than 0 (the usual zero of $\mathbb{R}$), such that $d(\overline{0}, 0) < \varepsilon$. The definition of $\overline{1}$ is dual to that of $\overline{0}$. Let us check the equation $0 \wedge x \approx 0$. Its ε-interpretation in this model is that $d(\overline{0} \,\overline{\wedge}\, x, \overline{0}) < \varepsilon$ for all $x \in [0,1]$. For $x \in [\overline{0}, 1]$, we in fact have $\overline{0} \,\overline{\wedge}\, x = \overline{0}$, so we need only consider $x \in [0, \overline{0}]$. Since $d(\overline{0}, 0) < \varepsilon$, we clearly have

$$d(\overline{0} \,\overline{\wedge}\, x, \overline{0}) \;=\; d(x, \overline{0}) \;\leq\; \varepsilon,$$

as desired. A similar calculation yields $d(\overline{1} \,\overline{\vee}\, x, \overline{1}) < \varepsilon$.

We next select a continuous function $\overline{F}\colon[0,1] \longrightarrow [\overline{0},\overline{1}]$, such that $\overline{F}(\overline{0}) = \overline{0}$ and $\overline{F}(\overline{1}) = \overline{1}$. Now all equations are true within ε, except those in the last line, involving G. We define $\overline{a}$ to be 0 (i.e. the real zero of $\mathbb{R}$), and define $\overline{G}\colon[0,1] \longrightarrow [0,1]$ to be a continuous function (piecewise linear will do) such that $\overline{G}$ is constantly $\overline{1}$ on $[\overline{0},\overline{1}]$, and $\overline{G}(0) = 0$. The equations in the last line now hold exactly.

This completes the proof of (72). In fact, it is now evident from Lemma 2 that (72) holds for any metric defining the usual topology on $[0,1]$. (We revisit this fact in §4.2.2 below.)

To see that Σ_c is consistent, take any bounded lattice L of three or more elements, and an $\overline{F} : L \longrightarrow L$ that is not onto and satisfies $\overline{F}(\overline{0}) = \overline{0}$ and $\overline{F}(\overline{1}) = \overline{1}$. Select $\overline{a}$ from the complement of the range of $\overline{F}$, and then define $\overline{G}$ in the obvious way. Thus we have a model of Σ_c with more than one element.

3.5 The theory of an injective binary operation.

3.5.1 $A = [0,1]$; $\Sigma =$ "injective binary."

Consider Σ consisting of the two equations

$$\begin{aligned} F_0(G(x_0,x_1)) &\approx x_0 \\ F_1(G(x_0,x_1)) &\approx x_1. \end{aligned}$$

They imply, among other things, that in any topological model $\overline{G}$ must be a one-one continuous binary operation. Euclidean spaces of non-zero finite dimension do not have such operations, hence are not compatible with Σ. In §3.5.1 we will be concerned with $A = [0,1]$. Spaces A of higher dimension will be considered in §3.5.2 and §3.5.3.

For $A = [0,1]$ we will show that $\lambda_{(A,d)}(\Sigma) \geq 0.5 \cdot \mathrm{diam}(A)$, for any metric d that defines the usual topology on A. (And of course, then $\lambda_{(A,d)}(\Sigma^\star) \geq 0.5 \cdot \mathrm{diam}(A)$.) It thus follows from §2.1 that if ρ is the usual Euclidean metric, then $\lambda_{(A,\rho)}(\Sigma) = \lambda_{(A,\rho)}(\Sigma^\star) = 0.5 \cdot \mathrm{diam}(A)$.

For the proof, we let d be an appropriate metric, and we consider continuous operations $\overline{F}_0$, $\overline{F}_1$ and $\overline{G}$ that satisfy Σ within K. We shall prove that $K \geq 0.5 \cdot \mathrm{diam}(A)$.

Since $[0,1]$ is compact, there exist $a_0, a_1 \in A$ with $d(a_0,a_1) = \mathrm{diam}(A)$. For flexibility of notation, we take two such pairs: $d(a_0,a_1) = d(b_0,b_1) =$

diam(A). Considering the four real numbers

$$\overline{G}(a_0, b_0),\ \ \overline{G}(a_1, b_0),\ \ \overline{G}(a_0, b_1),\ \ \overline{G}(a_1, b_1),$$

we may assume, without loss of generality, that the smallest among them is $\overline{G}(a_0, b_0)$. Again without loss of generality, we may assume that $\overline{G}(a_1, b_0)) \leq \overline{G}(a_0, b_1)$. In other words, we have

$$\overline{G}(a_0, b_0) \ \leq\ \overline{G}(a_1, b_0) \ \leq\ \overline{G}(a_0, b_1).$$

Thus, along the segment $\overline{(a_0, b_0)(a_0, b_1)}$ in the square $[0, 1]^2$, the continuous function $\overline{G}$ takes values that are above and below the value $\overline{G}(a_1, b_0)$. By the IVT, there exists $e \in [0, 1]$ such that

$$\overline{G}(a_0, e) \ =\ \overline{G}(a_1, b_0).$$

From Σ we now calculate

$$a_0 \ \approx\ \overline{F}_0(\overline{G}(a_0, e)) \ =\ \overline{F}_0(\overline{G}(a_1, b_0)) \ \approx\ a_1.$$

From this we immediately see the desired conclusion that diam$(A) \leq 2K$.

Notice that this argument does not require that the operations $\overline{F}_i$ be continuous.

3.5.2 $A = [0, 1]^2$; $\Sigma =$ "injective binary."

So, we consider the same F_0, F_1 and G, as they might exist on the square $B = [0, 1]^2$ (with the usual Euclidean metric).

Let Σ be as in §3.5.1. We will show that $\lambda_B(\Sigma) = 0.5$. We first show that it is ≤ 0.5. To this end, consider the following definitions of operations on $B = [0, 1]^2$:

$$\begin{aligned}
\overline{G}((a_0, a_1), (b_0, b_1)) &= (a_0, b_1) \\
\overline{F}_0((a_0, a_1)) &= (a_0, 0.5) \\
\overline{F}_1((a_0, a_1)) &= (0.5, a_1).
\end{aligned}$$

Now $\overline{F}_0(\overline{G}((a_0, a_1), (b_0, b_1))) = (a_0, 0.5)$, so its distance from (a_0, a_1) is $d(a_1, 0.5) \leq 0.5$. A similar calculation applies to $\overline{F}_1(\overline{G}((a_0, a_1), (b_0, b_1)))$, and so both equations of Σ are seen to hold within 0.5.

We next show that $\lambda_B(\Sigma) \geq 0.5$. To this end, suppose that its true value is K, and now consider any operations $\overline{G}, \overline{F}_0, \overline{F}_1$ that satisfy Σ within K on B, and with $\overline{G}$ continuous. Consider the action of $\overline{G}$ on the boundary of B^2, which is homeomorphic to $[0,1]^4$. The boundary of this space is a three-sphere S^3. By the Borsuk-Ulam Theorem[17], $\overline{G}$ takes on the same value at two antipodal points. Without loss of generality, two antipodal points have the form $((0,x_1)(y_0,y_1))$ and $((1,u_1)(v_0,v_1))$. So now, by the triangle inequality,

$$\begin{aligned}1 \;&=\; d((0,0),(1,0)) \;\leq\; d((0,x_1),(1,u_1)) \\ &\leq\; d((0,x_1),\overline{F}_0\overline{G}((0,x_1),(y_0,y_1))) \;+\; d((1,u_1),\overline{F}_0\overline{G}((1,u_1)(v_0,v_1))) \\ &\leq\; K \;+\; K.\end{aligned}$$

We remark that if the metric is scaled to make the diameter equal to 1, then $\lambda_B(\Sigma) = 0.5/\sqrt{2} \approx .3505$.

3.5.3 $A = [0,1]^2$, with a new metric.

Let Σ be defined as in §3.5.1, and take $B = [0,1]^2$, in its usual topology. Here we will show how to define a unit-diameter metric d for the usual topology on B such that $\lambda_{(B,d)}(\Sigma)$ is arbitrarily small. More precisely, given real $\varepsilon > 0$, we construct a unit-diameter metric d for the topology of B, such that $\lambda_{(B,d)}(\Sigma) \leq \varepsilon$.

Let us replace B by the set $[0, \sqrt{1-\varepsilon^2}] \times [0,\varepsilon]$, while taking d to be the Euclidean metric, restricted to this set. Clearly this B is homeomorphic to our original B, and moreover B has unit diameter. For an upper estimate on $\lambda_{(B,d)}(\Sigma)$, we define these three operations on B:

$$\begin{aligned}\overline{G}((a_0,a_1),(b_0,b_1)) \;&=\; (a_0,b_0) \\ \overline{F}_0(a_0,a_1) \;&=\; (a_0,0) \\ \overline{F}_1(a_0,a_1) \;&=\; (a_1,0).\end{aligned}$$

We now calculate

$$\begin{aligned}d((a_0,a_1),\overline{F}_0(\overline{G}((a_0,a_1),(b_0,b_1)))) \;&=\; d((a_0,a_1),(a_0,0)) \\ &\leq\; \varepsilon.\end{aligned}$$

[17]Proved by K. Borsuk in 1933—see [4]. See Steinlein [23] or Matoušek [18] for a thorough discussion of this theorem and its applications in modern mathematics.

and

$$\begin{aligned} d((b_0, b_1), \overline{F}_1(\overline{G}((a_0, a_1), (b_0, b_1)))) &= d((b_0, b_1), (b_0, 0)) \\ &\le \varepsilon. \end{aligned}$$

Thus the equations of Σ hold within ε, and our estimate on $\lambda_{(B,d)}(\Sigma)$ is established.

3.5.4 Generalization of §3.5.1 and §3.5.3

For $m > n$, let $\Sigma_{m,n}$ denote this set of equational axioms:

$$\begin{aligned} F_1(G_1(x_1, \cdots, x_m), \cdots, G_n(x_1, \cdots, x_m)) &\approx x_1 \\ &\vdots \\ F_m(G_1(x_1, \cdots, x_m), \cdots, G_n(x_1, \cdots, x_m)) &\approx x_m. \end{aligned} \tag{73}$$

If we let F and G stand for the appropriate tuples of function symbols, and let x stand for a tuple of variables x_i, then the equations may be symbolically abbreviated as

$$F(G(x)) \approx x.$$

To model $\Sigma_{m,n}$ on a space A is to find an n-tuple of continuous functions $\overline{G}_i : A^m \longrightarrow A$, in other words a continuous function $\overline{G} : A^m \longrightarrow A^n$, and also a continuous function $\overline{F} : A^n \longrightarrow A^m$, such that the composite mapping

$$A^m \xrightarrow{\overline{G}} A^n \xrightarrow{\overline{F}} A^m \tag{74}$$

is the identity map on A^m. This is of course generally impossible for $m > n$ and for a space A of dimension d with $0 < d < \infty$; in other words, for such spaces A, the compatibility relation $A \models \Sigma_{m,n}$ does not hold. Nevertheless, the associated metrical invariant λ may or may not be close to zero, as we see in Theorem 9 just below. The theorem was already proved for $m = 2$, $n = 1$ and $k = 1, 2$, in §3.5.1, §3.5.2 and §3.5.3 above.

Theorem 9 *Let $B = [0, 1]^k$, with $k \in \{1, 2, 3, \ldots\}$. Then*

(i) $\lambda_{(B,d)}(\Sigma_{m,n}) \le 0.5$ for d the Euclidean metric (scaled to diameter 1).

(ii) If $m \ge nk + 1$, then $\lambda_{(B,d)}(\Sigma_{m,n}) \ge 0.5$ for d any metric of diameter 1.

(iii) *If $m \leq nk$, then for every $\varepsilon > 0$ there is a diameter-1 metric d on B such that $\lambda_{(B,d)}(\Sigma_{m,n}) \leq \varepsilon$.*

Proof. We first observe that, in the Euclidean metric, $\lambda_B(\Sigma_{m,n}) \leq 0.5$, for any $m, n, k \geq 1$. As was pointed out in §2.1, we may define all operations to be constant at the centroid, thereby obtaining the radius (in this case half the diameter) as an upper estimate for λ_B. This observation takes care of point (i).

For (ii), we suppose that d is a metric for B with diameter 1, and that $\overline{F}$ and $\overline{G}$ are continuous functions as in (74) (with B in place of A). There exist $a, b \in B$, with $d(a, b) = 1$. By arc connectedness, there is an arc I in B with endpoints a and b. Let $S^{m-1} \subseteq B^m$ be the subset of B^m that is the boundary of I^m. It is a sphere of dimension $m-1$. We now have a restriction of $\overline{G}$ which maps as follows:

$$S^{m-1} \;\subseteq\; B^m \;\xrightarrow{\overline{G}}\; B^n \;\cong\; [0,1]^{nk}.$$

Now since $m-1 \geq nk$, the Borsuk-Ulam Theorem applies, and so there exist antipodal points $x, y \in S^{m-1}$ with $\overline{G}(x) = \overline{G}(y)$. Without loss of generality, $x = (a, x_2, \ldots)$ and $y = (b, x_2, \ldots)$. Thus

$$1 \;=\; d(a,b) \;\leq\; d(a, \overline{F}_1(\overline{G}(x))) \;+\; d(b, \overline{F}_1(\overline{G}(y))).$$

Thus one of these two summands must be ≥ 0.5; without loss of generality,

$$d(x_1, \overline{F}_1(\overline{G}(x))) \;=\; d(a, \overline{F}_1(\overline{G}(x))) \;\geq\; 0.5.$$

In other words, the two sides of Equation (73) must be distant by 0.5. This establishes our claim that $\lambda_{(B,d)}(\Sigma_{m,n}) \geq 0.5$.

Finally, we consider assertion (iii). Given arbitrary $\varepsilon > 0$, we give $B = [0,1]^k$ the metric

$$d(\mathbf{x}, \mathbf{y}) \;=\; \sup(|x_1 - y_1|,\, \varepsilon|x_2 - y_2|,\, \cdots,\, \varepsilon|x_k - y_k|).$$

Clearly B has diameter 1 in this metric, and retains its original topology.

Further, define $\overline{F}$ and $\overline{G}$ (as in (74)) via

$$\overline{G}(a_1, \cdots, a_m) \;=\; ((a_1^1, \cdots, a_k^1), (a_{k+1}^1, \cdots, a_{2k}^1),\; \cdots) \;\in\; ([0,1]^k)^n \tag{75}$$
$$\overline{F}(b_1, \cdots, b_n) \;=\; ((b_1^1, 0, \cdots, 0), (b_1^2, 0, \cdots, 0),\; \cdots) \;\in\; ([0,1]^k)^m \tag{76}$$

(Our notation is that b_u^v is the v-th component of an element b_u of B^m (or B^n, as needed).) For Equation (75), if $m = nk$ the right-hand is well defined. If $m < nk$, the formula calls for a_j^1 with $m < j \le nk$, which is not defined. In this case, we simply use $a_j^1 = 0$.

As for Equation (76), the intent is to cycle through all co-ordinates of all the b_i. In other words,

$$\begin{aligned} \overline{F}(b_1, \cdots, b_n) &= (c_1, \cdots, c_m) \in B^m, \quad \text{where} \\ c_{k(u-1)+v} &= (b_u^v, 0, \cdots, 0) \in [0,1]^k = B, \end{aligned}$$

for $u = 1, \ldots n$ and $v = 1, \ldots, k$.

The continuity of the operations $\overline{G}$ and $\overline{F}$ is evident. To check the approximate satisfaction of the equations $\Sigma_{m,n}$, let us evaluate $\overline{F}_j(\overline{G}(a_1, \cdots, a_m))$, where $j = k(u-1) + v$, for $u = 1, \ldots n$ and $v = 1, \ldots, k$. (Since $m \le nk$, every j is representable in this form.) Now

$$\overline{F}_j(\overline{G}(a_1, \cdots, a_m)) = \big(\big[\overline{G}(a_1, \cdots, a_m)\big]_u^v, 0, \cdots, 0\big),$$

where the subscript u indicates the u^{th} vector in the vector of vectors that appears on the right-hand side of Equation (75), namely $(a_{k(u-1)+1}^1, a_{k(u-1)+2}^1, \cdots)$. Moreover the superscript v refers to the v^{th} component of that vector, namely $a_{k(u-1)+2}^1$. In other words,

$$\overline{F}_j(\overline{G}(a_1, \cdots, a_m)) = \big(a_{k(u-1)+v}^1, 0, \cdots, 0\big) = \big(a_j^1, 0, \cdots, 0\big).$$

As for approximate satisfaction, if we let ε_i denote 1 for $i = 1$ and ε for $i > 1$, we now have

$$\begin{aligned} d(a_j, \overline{F}_j(\overline{G}(a_1, \cdots, a_m))) &= \sup_{i=1\cdots m} \varepsilon_i \left|a_j^i - \overline{F}_j(\overline{G}(a_1, \cdots, a_m))^i\right| \\ &= \sup\big(|a_j^1 - a_j^1|, \varepsilon|a_j^2 - 0|, \cdots, \varepsilon|a_j^k - 0|\big) \\ &= \sup\big(0, \varepsilon|a_j^2|, \cdots, \varepsilon|a_j^k|\big) \\ &\le \varepsilon. \end{aligned}$$

∎

In case $k = 1$, the above proof, while of course correct, supplies more. In that case, the operations $\overline{F}$ and $\overline{G}$ defined in (75) and (76) model $\Sigma_{m,n}$ exactly on $B = I$.

3.6 $[n]$-th power varieties.

3.6.1 The definition of $\Sigma^{[n]}$.

In §3.6 we begin with a similarity type (i.e. list of operation symbols) that does not contain d or g. Thus an equation-set Σ under consideration does not mention d or g, and its models have no operations $\overline{d}$ or $\overline{g}$. The construction of $\Sigma^{[n]}$ presented below enlarges Σ by adding some new equations involving the given operation symbols and new operations symbols d and g. Throughout §3.6 we adopt the convention that d and g appear only in the guise of having been expressly added in the formation of $\Sigma^{[n]}$. Obviously, this convention imposes no essential restriction on the generality of what is written here.

So let Σ be a set of such equations (where the type is implicitly determined by the operation symbols appearing in Σ). By the *$[n]$-th power equations of Σ* we mean the set $\Sigma^{[n]}$ that is formed by adjoining the following equations to Σ:

$$\begin{aligned}
g^n(x) &\approx x && (77)\\
d(x,\cdots,x) &\approx x && (78)\\
d(g(x_1),\cdots,g(x_n)) &\approx g(d(x_2,\cdots,x_n,x_1)) && (79)\\
d(d(x_{11},\cdots,x_{1n}),\cdots,d(x_{n1},\cdots,x_{nn}))& && \\
&\approx d(x_{11},\cdots,x_{nn}) && (80)\\
F(g(x_1),g(x_2),\cdots) &\approx g(F(x_1,x_2,\cdots)) && (81)\\
F(d(x_{11},\cdots,x_{1n}),d(x_{21},\cdots,x_{2n}),\cdots)& && \\
&\approx d(F(x_{11},x_{21},\cdots),\cdots,F(x_{1n},x_{2n},\cdots)), && (82)
\end{aligned}$$

where (81) and (82) are really schemes of equations, one for each operation F of Σ. The construction of $\Sigma^{[n]}$ first appears in McKenzie [19]. Further exposition and application of the theory appears in Taylor [28, 30] and Garcia and Taylor [12].

The essential fact about $\Sigma^{[n]}$ [*op. cit.*] is that, within isomorphism, eacb of its (topological) models is the n-th power of a (topological) model of Σ, where the new operations $\overline{d}$ and $\overline{g}$ operate on the n-th power by the shuffling of co-ordinates:

$$\begin{aligned}
\overline{d}(\langle\alpha_{11},\cdots,\alpha_{1n}\rangle,\cdots,\langle\alpha_{n1},\cdots,\alpha_{nn}\rangle) &= \langle\alpha_{11},\cdots,\alpha_{nn}\rangle && (83)\\
\overline{g}(\langle\alpha_1,\cdots,\alpha_n\rangle) &= \langle\alpha_n,\alpha_1,\cdots,\alpha_{n-1}\rangle. && (84)
\end{aligned}$$

If the original Σ-algebra is denoted $\mathbf{A}$, then the n-th power model so described is denoted $\mathbf{A}^{[n]}$.

3.6.2 $\mathbf{Sets}^{[n]}$.

Here we consider Equations (77–82) in the special case where Σ has no operation symbols and no equations (the so-called variety of sets). This essentially means that Equations (81–82) do not appear, and so we will be considering only Equations (77–80). In this context, a (topological) model of $\Sigma^{[n]}$ is completely determined by Equations (83–84). It follows from §3.6.1 that

Theorem 10 *$A \models Sets^n$ iff there exists a space B such that A is homeomorphic to B^n.* ∎

We skip the proof of Corollary 11, which involves a study of when $[0,1]^m$ is homeomorphic to an n-th power. In §3.6.3 immediately below, we shall implicitly include a proof for the case $m = n + 1$.

Corollary 11 $[0,1]^m \models$ *$Sets^{[n]}$ iff $n|m$.* ∎

3.6.3 $A = [0,1]^m; \Sigma = \mathbf{Sets}^{[n]}$.

The proof that follows is valid for all $k \geq n$, but the statement of the theorem ignores a sharper conclusion that can be drawn when $k = n$, or indeed when k is any multiple of n. (By Corollary 11, $B \models \Sigma$ for any such k.)

Theorem 12 *If $B = [0,1]^k$ and $\Sigma = Sets^{[n]}$, with $k \geq n$, then for each $\varepsilon > 0$ there exists a diameter-1 metric d for B with $\lambda_{(B,d)}(\Sigma^\star) \leq \varepsilon$.*

Proof. We may assume that $\varepsilon \leq 1$. Let d be defined by

$$d(\mathbf{x}, \mathbf{y}) \;=\; \sup(|x_1 - y_1|, \cdots, |x_n - y_n|, \varepsilon|x_{n+1} - y_{n+1}|, \cdots, \varepsilon|x_k - y_k|).$$

Now define $\psi : B \longrightarrow B$ via

$$\psi(\mathbf{x}) \;=\; (x_1, \cdots, x_n, 0. \cdots, 0).$$

We omit the easy proofs that the image of ψ is a subset E homeomorphic to $[0,1]^n$, that $\psi \restriction E =$ identity, and that $d(\mathbf{x}, \psi(\mathbf{x})) \leq \varepsilon$ for all $\mathbf{x}$.

We are now in a position to apply Theorem 5 of §2.6 (with B for A, E for B, and ε for K). The conclusion is that

$$\lambda_{(B,d)}(\Sigma^\star) \;\leq\; \lambda_{(E,d)}(\Sigma^\star) \,+\, \varepsilon.$$

But $(E,d) \models \Sigma^\star$, by Corollary 11, and hence we have the desired result that $\lambda_{(B,d)}(\Sigma^\star) \leq \varepsilon$. ∎

To state a result in the opposite direction (a non-zero lower estimate on certain values of λ for $n = 2$ and $k = 1$), we need to modify Σ slightly, to include those consequences of Equations (77–80) (for $n = 2$) that we will be using in the proof of Theorem 13. (Recall from §2.2 that λ is not independent of equational deductions. Here we know the result only for these consequences of Σ, and not for the original Σ.) We consider the equations

$$x \;\approx\; g(g(x,u), g(v, g(w,x))) \tag{85}$$

$$x \;\approx\; g(g(x,u), d(g(g(d(x),w),v))). \tag{86}$$

If σ is a term in a single binary operation g, then its *opposite* is the term obtained by recursively replacing $g(x,y)$ by $g(y,x)$. The opposite of an equation $\sigma \approx \tau$ simply pairs the opposite of σ with the opposite of τ.

Theorem 13 *Let $B = [0,1]$ and let $\Sigma =$ be defined by Equations (85–86) and their opposite equations. (Σ is thus a subset of the theory of Sets[2].) In the Euclidean metric (scaled to diameter 1) $\lambda_B(\Sigma) = 0.5$, and moreover $\lambda_B(\Sigma) \geq 0.5$ in every diameter-1 metric.*

Proof. The inequality $\lambda_B(\Sigma^+) \leq 0.5$ in the Euclidean case follows as at the start of the proof of Theorem 9. For $\lambda_B(\Sigma^+) \geq 0.5$, we suppose that ρ is a metric ρ for B with diameter 1, and that $\overline{g}$ and $\overline{d}$ are continuous operations on B, binary and unary, respectively. For the desired inequality, we will assume that $(B, \overline{g}, \overline{d})$ satisfies Equations (85–86) and their opposites within $K \in \mathbb{R}$, and then prove that $K \geq 0.5$.

By compactness, there exist $a, b \in B$ with $\rho(a,b) = 1$. Let I be an arc in B joining a to b. Let S^1 denote the boundary of $I^2 \subseteq B^2$. By the Borsuk-Ulam Theorem, $\overline{g}$ takes on equal values at antipodal points of S^1. Without loss of generality, $\overline{g}(a,c) = \overline{g}(b,d)$ for some $c, d \in B$.

We now wish to establish the existence of $e, f \in B$ such that either

$$\overline{g}(e, \overline{g}(\overline{d}(a), a)) \;=\; \overline{g}(f, \overline{g}(\overline{d}(b), b)) \quad \text{or} \quad \overline{g}(\overline{g}(\overline{d}(a), a), e)) \;=\; \overline{g}(\overline{g}(\overline{d}(b), b), f). \tag{87}$$

Clearly such e, f exist if $\overline{g}(\overline{d}(a), a) = \overline{g}(\overline{d}(b), b)$; thus we may assume that these are distinct elements of B. Thus there is an arc J joining $\overline{g}(\overline{d}(a), a)$ and $\overline{g}(\overline{d}(b), b)$. As before, two antipodal boundary points of J^2 take equal values under $\overline{g}$; one easily sees that this observation is tantamount to (87). To complete the proof of Theorem 13, we consider two cases, which correspond to the two alternatives of (87):

Case 1. $\overline{g}(a, c) = \overline{g}(b, d)$ and $\overline{g}(e, \overline{g}(\overline{d}(a), a)) = \overline{g}(f, \overline{g}(\overline{d}(b), b))$. In B, we may define

$$\begin{aligned} r &= \overline{g}(\overline{g}(a, c), \overline{g}(e, \overline{g}(\overline{d}(a), a))) \\ &= \overline{g}(\overline{g}(b, d), \overline{g}(f, \overline{g}(\overline{d}(b), b))) \end{aligned}$$

(where the second equality clearly comes from the assumptions of Case 1). If now we substitute into Equation (85)—taking a for x, c for u, e for v, and $d(a)$ for w—clearly the right-hand side of (85) takes the value r. Approximate satisfaction now yields $\rho(a, r) < K$. A similar calculation—substituting b for x, d for u, f for v, and $d(b)$ for w—yields $\rho(b, r) < K$. Finally, by the triangle inequality, we have

$$1 = \rho(a, b) \leq \rho(a, r) + \rho(r, b) < K + K = 2K,$$

and so $K \geq 0.5$, as desired.

Case 2. $\overline{g}(a, c) = \overline{g}(b, d)$ and $\overline{g}(\overline{g}(\overline{d}(a), a), e)) = \overline{g}(\overline{g}(\overline{d}(b), b), f)$. In B, we may define

$$\begin{aligned} s &= \overline{g}(\overline{g}(a, c), \overline{d}(\overline{g}(\overline{g}(\overline{d}(a), a), e))) \\ &= \overline{g}(\overline{g}(b, d), \overline{d}(\overline{g}(\overline{g}(\overline{d}(b), b), f))) \end{aligned}$$

(where the second equality clearly comes from the assumptions of Case 2). Now if we substitute into Equation (86)—taking a for x, c for u, e for v and a for w—the right-hand side of (86) takes the value a. Approximate satisfaction yields $\rho(a, s) < K$. Continuing as in Case 1, we again obtain $K \geq 0.5$. ∎

Remark on the proof of Theorem 13. The proof never invoked the continuity of $\overline{d}$. Thus, even if we relax the notion of compatibility to allow non-continuous $\overline{d}$ (along with continuous $\overline{g}$), we still cannot satisfy (85–86) any closer than half the diameter of $[0, 1]$. (Similar remarks hold in §3.3.8 above.)

4 Two topological invariants.

As we have seen, in §3.4.4, §3.4.7, §3.5.3 and elsewhere, λ_A depends on the choice of the metric d (among those metrics d that yield the given topology on A), and hence is not a topological invariant. For a space A of finite diameter, one may narrow the choice of d by insisting that $\mathrm{diam}(A, d) = 1$, but λ_A is still not invariant.

On the other hand, we can obviously obtain a topological invariant by considering the extreme values that occur for $\lambda_{(A,d)}(\Sigma)$ when d is allowed to range over all appropriate metrics.

$$\begin{aligned} \delta_A^-(\Sigma) &= \inf \left\{ \lambda_{(A,d)}(\Sigma) \,:\, \mathrm{diam}(A,d) \geq 1 \right\} \\ \delta_A^+(\Sigma) &= \sup \left\{ \lambda_{(A,d)}(\Sigma) \,:\, \mathrm{diam}(A,d) \leq 1 \right\} \end{aligned}$$

where

$$\mathrm{diam}(A,d) \;=\; \sup\left\{ d(x,y) \,:\, x, y \in A \right\}.$$

Obviously, the values $\delta_A^-(\Sigma)$ and $\delta_A^+(\Sigma)$ are equal, respectively, to the inf and sup of the single set

$$\left\{ \lambda_{(A,d)}(\Sigma) \,:\, \mathrm{diam}(A,d) \;=\; 1 \right\}.$$

Short versions:

$$\begin{aligned} \delta_A^-(\Sigma) &= \inf_d \inf_{\mathbf{A}} \sup_{\substack{(\sigma,\tau) \\ \mathbf{a}}} d(\sigma^{\mathbf{A}}(\mathbf{a}), \tau^{\mathbf{A}}(\mathbf{a})) \\ \leq\; \delta_A^+(\Sigma) &= \sup_d \inf_{\mathbf{A}} \sup_{\substack{(\sigma,\tau) \\ \mathbf{a}}} d(\sigma^{\mathbf{A}}(\mathbf{a}), \tau^{\mathbf{A}}(\mathbf{a})). \end{aligned}$$

4.1 Connection with radius and diameter.

If Σ contains no equation $x_i \approx x_j$ for $i \neq j$, then by §2.1, $\lambda_A(\Sigma) \leq \mathrm{radius}(A)$. It follows immediately that

$$\delta_A^-(\Sigma) \;\leq\; \frac{\mathrm{radius}(A)}{\mathrm{diameter}(A)} \,.$$

In particular, if A can be metrized with diameter$(A) = 2 \cdot$ radius(A)—as can any simplex of finite dimension—then the above inequality yields

$$\delta^-_A(\Sigma) \;\leq\; 0.5.$$

The value 0.5 is realized for A a simplex in §4.3.7 and §4.3.9 below.

4.2 Some estimates of $\delta^+_A(\Sigma)$.

Theorem 14 *$\delta^+_A(\Sigma) = 0$ or 1. If A is compact, then $\delta^+_A(\Sigma) = 0$ iff $\lambda_{(A,d)}(\Sigma) = 0$ for all metrics d defining the topology of A. If A is compact, then $\delta^+_A(\Sigma) = 1$ iff $\lambda_{(A,d)}(\Sigma) > 0$ for all metrics d defining the topology of A.*

Proof. If δ is a metric for the topology of A, then so also is $f \circ \delta$, where $f(x) = 1 \wedge (2x)$. Thus $\delta^+_A(\Sigma)$ is the sup of an f-closed subset of $[0,1]$. The last two sentences are immediate from Lemma 2. ∎

Corollary 15 *If $\delta^-_A(\Sigma) > 0$, then $\delta^+_A(\Sigma) = 1$.*

4.2.1 An inconsistent example where $\delta^+ = 0$.

Let A be any non-discrete metric space where Tietze's theorem is applicable. Let Γ be the (inconsistent) equations

$$\begin{aligned} F(a, x_0, x_1) &\;\approx\; x_0 \\ F(b, x_0, x_1) &\;\approx\; x_1 \\ a &\;\approx\; b \end{aligned}$$

(which were first mentioned in §3.1.2). It follows from §3.1.2 that $\delta^+_A(\Gamma) = 0$.

4.2.2 A consistent incompatible example where $\delta^+ = 0$.

In §3.4.8 we considered a consistent expansion Σ_c of lattice theory that is not compatible with $[0,1]$. We saw there that $\lambda_{([0,1],d)}(\Sigma_c) = 0$ for any metric d that generates the usual topology on $[0,1]$. Therefore

$$\delta^+_{[0,1]}(\Sigma_c) \;=\; \delta^-_{[0,1]}(\Sigma_c) \;=\; 0.$$

Notice that $\delta^- = 0$ does not distinguish the situation of this Σ_c—that all appropriate λ-values are zero—from the situation that is seen in §4.3.2, §4.3.6, §4.3.8, §4.3.9 and §4.3.10, etc., below. In these latter situations, λ is positive but approaches zero (by suitable choice of metrics); hence δ^- is again zero. Nevertheless δ^+ does distinguish these two situations.

4.3 Some estimates of $\delta^-_A(\Sigma)$.

For an initial exploration of the possible values of $\delta^-_A(\Sigma)$, we revisit the examples of §3. All these examples will have $\delta^+_A(\Sigma) = 1$. In some cases we will be able to state $\delta^-_A(\Sigma^\star) = 0$; obviously this assertion entails $\delta^-_A(\Sigma) = 0$ (which we will not mention).

4.3.1 $A = S^1$; $\Sigma =$ semilattice theory.

From §3.4.2, we see that $\delta^-_{S^1}(\Sigma^\star) = 0$. This method also applies to any theory Σ that is compatible with $[0,1]$.

More generally, let us suppose that Σ is not compatible with $[0,1]$, but that $\delta^-_{[0,1]}(\Sigma) = 0$. We shall see that we still have $\delta^-_{S^1}(\Sigma) = 0$. Let us be given $\varepsilon > 0$. By our hypothesis, there is a metric d on $[0,1]$ such that $\lambda_{([0,1],d)} \leq \varepsilon$. We give $[0,1] \times \mathbb{R}$ the metric

$$d'((x_0,y_0),(x_1,y_1)) = d(x_0,y_0) + |x_1 - y_1|.$$

We define ψ as §3.4.2; clearly things go as before, and one may apply Theorem 5 of §2.6. Much as before, we have

$$\lambda_{(S^1,d')}(\Sigma) \leq \lambda_{([0,1],d)}(\Sigma) + \varepsilon \leq \varepsilon + \varepsilon = 2\varepsilon.$$

In fact, our proof easily extends to

$$\delta^-_{S^n}(\Sigma) \leq \delta^-_{[0,1]}(\Sigma).$$

4.3.2 $A = S^2$; $\Sigma =$ H-spaces.

From §3.2.2, we see that $\delta^-_{S^2}(\Sigma^\star) = 0$. This method also applies to any theory Σ that is compatible with $[0,1]$.

If Σ is not compatible with $[0,1]$, but $\delta^-_{[0,1]}(\Sigma) = 0$, then we still have $\delta^-_{S^2}(\Sigma) = 0$. (The proof is like that in §4.3.1.)

In the same way, some of the positive λ-values of §3.2 can be seen to correspond to zero-values for δ^-.

4.3.3 Group theory on spaces with the fixed-point property.

It is immediate from §3.3.1 that $\delta_A^-(\Sigma^\star) \geq 0.5$. For a number of spaces, such as $[0,1]$ and its direct powers (cubes), §4.1 yields $\delta_A^-(\Sigma^\star) = 0.5$

4.3.4 Groups of exponent 2 on $\mathbb{R}$.

It is immediate from §3.3.6 and §2.1 that (if Σ is as described in §3.3.6, then)

$$\delta_{\mathbb{R}}^-(\Sigma) \ \geq \ \frac{\text{radius}(\mathbb{R})}{2 \cdot \text{diameter}(\mathbb{R})} \ \geq \ 0.25.$$

4.3.5 $A = \mathbb{R}$; $\Sigma =$ lattices with zero.

It is immediate from §3.4.4 that $\delta_{\mathbb{R}}^-(\Sigma^\star) = 0$.

4.3.6 The triode Y and $\Lambda =$ lattice theory, revisited.

From §3.4.7 we have $\delta_Y^-(\Lambda) = 0$.

4.3.7 No injective binary on interval, revisited.

Let Σ be defined as above in §3.5.1, and take $A = [0,1]$, in its usual topology. In §3.5.1 it was proved that $\lambda_{(A,d)}(\Sigma^\star) \geq \lambda_{(A,d)}(\Sigma) \geq 0.5$ for any metric d of diameter 1, and moreover that for one such metric (namely the usual metric), we have $\lambda_{(A,d)}(\Sigma^\star) = \lambda_{(A,d)}(\Sigma) = 0.5$. Therefore, clearly

$$\delta_{[0,1]}^-(\Sigma) \ = \ \delta_{[0,1]}^-(\Sigma^\star) \ = \ 0.5.$$

4.3.8 No injective binary on a square, revisited.

From §3.5.3, $\delta_B^-(\Sigma) = 0$.

4.3.9 Generalizations of §4.3.7 and §4.3.8.

§3.5.4 revisited. $\Sigma_{m,n}$ as before.

Theorem 16 *Let $B = [0,1]^k$, with $k \in \{1,2,3,\ldots\}$. Then*

$$\delta_B^-(\Sigma_{m,n}) \ = \ \begin{cases} 0 & \textit{if } m \leq nk \\ 0.5 & \textit{if } m \geq nk+1. \end{cases}$$

4.3.10 n-th power varieties.

(Refer to §3.6 for definitions.) By Theorem 12 of §3.6.3, if $B = [0,1]^k$ and $\Sigma = \mathrm{Sets}^{[n]}$, with $k \geq n$, then $\delta_B^-(\Sigma^\star) = 0$.

5 Product varieties

For any sets Γ and Δ of equations, (finite or infinite, deductively closed or not), one may construct a new theory $\Gamma \times \Delta$, which has the following properties (among others):

(i) if Γ and Δ are finite, then $\Gamma \times \Delta$ is also finite;

(ii) $\Gamma^\star \times \Delta^\star \subseteq (\Gamma \times \Delta)^\star$;

(iii) if $[\Gamma]$ denotes the class of Γ in the lattice $\mathcal{L}$ of varieties ordered by interpretability, then $[\Gamma \times \Delta]$ is the meet of Γ and Δ in $\mathcal{L}$—i.e. $[\Gamma \times \Delta] = [\Gamma] \wedge [\Delta]$;

(iv) if A is a topological space, then $A \models \Gamma \times \Delta$ iff there are spaces C and D such that A is homeomorphic to $C \times D$, $C \models \Gamma$ and $D \models \Delta$;

(v) consequently, if A is a space that cannot be factored non-trivially, and if $A \models \Gamma \times \Delta$, then either $A \models \Gamma$ or $A \models \Delta$.

For an explication of the lattice $\mathcal{L}$, and for a detailed explication of these definitions and results, the reader is referred to [12]. We include a definition of $\Gamma \times \Delta$ immediately below. In §5 we shall investigate to what extent there are analogs of the last two points for approximate satisfaction.

In defining the product $\Gamma \times \Delta$ of finite equation-sets Γ and Δ, we will assume that the operations of Γ are G_i $(i \in I)$, and the operations of Δ are D_j $(j \in J)$, for sets I and J with $I \cap J = \emptyset$. We augment Γ with a new operation D_j and a new equation

$$D_j(x_1, x_2, \cdots) \;\approx\; x_1, \tag{88}$$

for each $j \in J$. Similarly, we augment Δ with a new operation G_i and a new equation

$$G_i(x_1, x_2, \cdots) \;\approx\; x_1, \tag{89}$$

for each $i \in I$. Clearly the augmented Γ defines a variety that is definitionally equivalent to the one defined by the original Γ, and similarly for Δ. Now when we refer to Γ and Δ, we mean the augmented equation-sets.

The operations of $\Gamma \times \Delta$ will be those of Γ and Δ, together with a new binary operation denoted p. To define the equation-set $\Gamma \times \Delta$, we first need one more piece of notation. Let y be a variable not in the set $\{x_1, x_2, \cdots\}$. For any operation symbol $F \in \Gamma \cup \Delta \cup \{p\}$, we define F^R and F^L to be the terms $p(F(x_1, x_2, \cdots), y)$ and $p(y, F(x_1, x_2, \cdots))$, respectively. Let τ be any term whose operations symbols lie in $\Gamma \cup \Delta \cup \{p\}$, and whose variables are among $x_1, x_2, \ldots$. We now recursively define τ^R as follows.

- If $\tau = x_i$, then $\tau^R = p(x_i, y)$ and $\tau^L = p(y, x_i)$.
- If $\tau = F(\tau_1, \cdots, \tau_n)$, then

$$\begin{aligned} \tau^R &= p(F(\tau_1^R, \tau_2^R, \cdots), y) \\ \tau^L &= p(y, F(\tau_1^L, \tau_2^L, \cdots)). \end{aligned} \tag{90}$$

The equations of $\Gamma \times \Delta$ may now be declared as Equations (91–96) that follow. One easily sees that if Γ and Δ are both finite, then $\Gamma \times \Delta$ is finite.

$$p(x, x) \approx x \tag{91}$$

$$p(p(x, y), p(u, v)) \approx p(x, v) \tag{92}$$

$$\tau^R \approx p(\tau, y); \qquad \tau^L \approx p(y, \tau) \tag{93}$$

$$\tau(p(x_1, y_1), p(x_2, y_2), \cdots) \approx p(\tau(x_1, x_2, \cdots), \tau(y_1, y_2, \cdots)) \tag{94}$$

$$(\tau \text{ any } \Gamma\text{-term or any } \Delta\text{-term})$$

$$p(\sigma, x) \approx p(\tau, x) \qquad (\sigma \approx \tau \in \Gamma) \tag{95}$$

$$p(x, \alpha) \approx p(x, \beta) \qquad (\alpha \approx \beta \in \Delta). \tag{96}$$

In equational logic, i.e. the logic of exact equality, Equations (93) may be deduced from (91–92) and(94), and thus are redundant from that point of view. We shall need them, however, as approximate identities, and in that context they cannot be deduced.

5.1 Product identities on product-indecomposable spaces

Here we consider a possible analog, for approximate satisfaction, of the result, mentioned above, that if A is product-indecomposable and if $A \models \Gamma \times \Delta$, then $A \models \Gamma$ or $A \models \Delta$. We know of no analog that holds generally for product-indecomposable spaces, but rather we must find examples case by case. See Theorems 19 and 20. The first lemma is easy:

Lemma 17 *If A is a metric space, and if $\lambda_A(\Gamma) < \varepsilon$, then $\lambda_A(\Gamma \times \Delta) < \varepsilon$.*

Proof. Suppose we have operations that satisfy Γ within ε. We obtain $(\Gamma \times \Delta)$-operations as follows. The Γ-operations are retained unchanged. The Δ-operations, as well as $\overline{p}$, are taken to be first co-ordinate projection. It is straightforward to verify that Equations (91–96) hold within ε. ▮

It is obvious that the conclusion of Lemma 17 in fact holds if we are given either $\lambda_A(\Gamma) < \varepsilon$ or $\lambda_A(\Delta) < \varepsilon$. Therefore an appropriate converse would, when true, deduce this disjunction (or something slightly weaker) from $\lambda_A(\Gamma \times \Delta) < \varepsilon$. Such a converse generally requires a re-adjustment of ε. Even so, the converse tends to be false for decomposable spaces A—a specific example will be given as Theorem 22 in §5.2 below. We now turn to two product-indecomposable spaces ($[0, 1]$ and Y), where some version of the converse holds—see Theorems 19 and 20 below. First, a lemma that applies to both of these spaces:

Lemma 18 *Suppose that $\Gamma \times \Delta$ is satisfiable within ε on a metric space A, by continuous operations that include $\overline{p}$ (realizing the p that appears in (91–96)). For $a \in A$, let R_a and L_a be the images of right and left $\overline{p}$-multiplication by a:*

$$R_a \;=\; \{\,\overline{p}(x, a) \,:\, x \in A\}; \qquad\qquad L_a \;=\; \{\,\overline{p}(a, x) \,:\, x \in A\}.$$

Let us endow R_a and L_a each with the metric obtained by restricting the given metric on A. Then Γ is satisfiable within 3ε by continuous operations on R_a, and Δ is is satisfiable within 3ε by continuous operations on L_a.

Proof. It will be enough to prove the part about Γ and R_a. So let us suppose that continuous operations $\overline{G}_i$ $(i \in I)$ and $\overline{D}_j$ $(j \in J)$ on A form, along with $\overline{p}$, a topological algebra $\mathcal{A}$ that satisfies $\Gamma \times \Delta$ to within ε.

We define a topological algebra $\mathcal{R}_a$ based on the subspace R_a as follows. For each operation symbol G_i of Γ, we define

$$\overline{G}^a_i(x_1, x_2, \cdots) \;=\; \overline{p}(\overline{G}_i(x_1, x_2, \cdots), a).$$

It is clear that each $\overline{G}_i^a$ is continuous and maps into R_a. We thus have a topological algebra $\mathcal{R}_a = (R_a, \overline{G}_i^a)_{i \in I}$. Our task is to prove that $\mathcal{R}_a$ satisfies Γ within 3ε.

So let us first consider an arbitrary term τ in the language of Γ. Let $\overline{\tau}$ denote the realization of τ in the original algebra $\mathcal{A}$ (it is a continuous function $A^\omega \longrightarrow A$, which depends on only finitely many variables). Further, let $\overline{\tau}^a$ denote the realization of τ in the algebra $\mathcal{R}_a$. We propose to prove, by induction, that

$$\overline{\tau}^a(a_1, a_2, \cdots) \;=\; \overline{\tau^R}(a_1, a_2, \cdots, a) \tag{97}$$

wherever both sides are defined, which is to say for $a_1, a_2, \cdots \in R_a$, where τ^R is as defined in (90) (and where the a has been substituted for the variable y of τ^R). Now (97) obviously holds for τ a variable. For τ a composite term, we may write $\tau = G_i(\tau_1, \tau_2, \cdots)$. By induction, we have

$$\begin{aligned}
\overline{\tau}^a(a_1, a_2, \cdots) \;&=\; \overline{G}_i^a(\overline{\tau}_1^a(a_1, a_2, \cdots), \overline{\tau}_2^a(a_1, a_2, \cdots), \cdots) \\
&=\; \overline{G}_i^a(\overline{\tau_1^R}(a_1, a_2, \cdots, a), \overline{\tau_2^R}(a_1, a_2, \cdots, a), \cdots) \\
&=\; \overline{p}(\overline{G}_i(\overline{\tau_1^R}(a_1, a_2, \cdots, a), \overline{\tau_2^R}(a_1, a_2, \cdots, a), \cdots), a) \\
&=\; \overline{\tau^R}(a_1, a_2, \cdots, a).
\end{aligned}$$

Finally, if $\sigma \approx \tau$ is an equation of Γ, then we may use equations (93), (95) and (97) to see

$$\begin{aligned}
\overline{\sigma}^a(a_1, a_2, \cdots) \;&=\; \overline{\sigma^R}(a_1, a_2, \cdots, a) \;\approx\; \overline{p}(\overline{\sigma}(a_1, a_2, \cdots), a) \\
&\approx\; \overline{p}(\overline{\tau}(a_1, a_2, \cdots), a) \;\approx\; \overline{\tau^R}(a_1, a_2, \cdots, a) \\
&=\; \overline{\tau}^a(a_1, a_2, \cdots).
\end{aligned}$$

In other words, the realization of σ in $\mathcal{R}_a$ is equal within 3ε to the realization of τ in $\mathcal{R}_a$. ∎

The following theorem will be applied in Theorem 41 of §8, to show that the condition $\lambda_{[0,1]}(\Sigma) > 0$ defines a filter of theories.

Theorem 19 *Throughout this theorem,* $[0, 1]$ *denotes the unit interval with its usual metric. Suppose* $0 < \varepsilon < 1/16$*. If* $\lambda_{[0,1]}(\Gamma \times \Delta) < \varepsilon$*, then either* $\lambda_{[0,1]}(\Gamma) < 4\varepsilon$ *or* $\lambda_{[0,1]}(\Delta) < 4\varepsilon$*.*

Proof. Suppose that we have continuous operations $\overline{p}$, $\overline{G}_i$ ($i \in I$) and $\overline{D}_j$ ($j \in J$) on $[0,1]$, forming a topological algebra $\mathcal{A}$, and satisfying equations $\Gamma \times \Delta$ within ε.

Let us take R_0, R_1, L_0 and L_1 as defined in the statement of Lemma 18. By continuity of $\overline{p}$, each of the four sets is a compact interval $\subseteq [0,1]$. Now consider R_0 and L_1: their intersection contains the point $\overline{p}(1,0)$; hence their union is again an interval. Now obviously this interval contains $\overline{p}(0,0) \approx 0$ and $\overline{p}(1,1) \approx 1$, and hence

$$R_0 \cup L_1 \;\supseteq\; [\overline{p}(0,0), \overline{p}(1,1)] \;\supseteq\; [\varepsilon, 1-\varepsilon],$$

where the second inclusion comes from (91). Therefore

$$\mu(R_0) \,+\, \mu(L_1) \;\geq\; 1 - 2\varepsilon, \tag{98}$$

where μ stands for Lebesgue measure (in this case, the length of the interval). Similar considerations yield

$$\mu(R_1) \,+\, \mu(L_0) \;\geq\; 1 - 2\varepsilon. \tag{99}$$

It is an easy consequence of (98) and (99) that either

$$\mu(R_0) \,+\, \mu(L_0) \;\geq\; 1 - 2\varepsilon \tag{100}$$

or

$$\mu(R_1) \,+\, \mu(L_1) \;\geq\; 1 - 2\varepsilon.$$

Without loss of generality, we shall assume that (100) holds.

The next part of the proof (through (101)) is borrowed from the example in §3.5.1. Let us suppose that the diameter of L_0 is attained by the two points $\overline{p}(0,a)$ and $\overline{p}(0,b)$ (in other words, they are its endpoints, in unspecified order); and further that $\overline{p}(c,0)$ and $\overline{p}(d,0)$ play the same role for R_0. Consider the four real numbers $\overline{p}(c,a)$, $\overline{p}(d,a)$, $\overline{p}(c,b)$, $\overline{p}(d,b)$. We may assume, without loss of generality, that the minimum of these four number is $\overline{p}(c,a)$, and then that of the two numbers $\overline{p}(d,a)$ and $\overline{p}(c,b)$, the latter is smaller (or equal). In other words, we may suppose that

$$\overline{p}(c,a) \;\leq\; \overline{p}(c,b) \;\leq\; \overline{p}(d,a).$$

Applying the Intermediate Value Theorem to the continuous function $\overline{p}(\cdot, a)$, we infer the existence of $e \in [0,1]$ such that

$$\overline{p}(e,a) \ = \ \overline{p}(c,b).$$

Now, by the approximate satisfaction of (92), we have

$$\overline{p}(0,a) \ \approx \ \overline{p}(\overline{p}(0,0), \overline{p}(e,a)) \ = \ \overline{p}(\overline{p}(0,0), \overline{p}(c,b)) \ \approx \ \overline{p}(0,b).$$

Thus the distance between the two endpoints of L_0 is $< 2\varepsilon$; in other words $\mu(L_0) < 2\varepsilon$. It now follows from (100) that

$$\mu(R_0) \ \geq \ 1 - 4\varepsilon. \tag{101}$$

By Lemma 18, there are continuous operations on R_0 that satisfy Γ within 3ε. If we expand this algebra by a factor of $1/\mu(R_0)$ (the reciprocal of its length), we now have a topological algebra based on a unit interval. In this expanded algebra, the laws of Γ are satisfied within

$$\frac{3\varepsilon}{\mu(R_0)} \ \leq \ \frac{3\varepsilon}{1 - 4\varepsilon} \ \leq \ 4\varepsilon,$$

where the final inequality is easily derived from our assumption that $0 < \varepsilon < 1/16$. ▮

Remark. It is immediate from (98) that either $\mu(R_0) \geq (1-2\varepsilon)/2$ or $\mu(L_1) \geq (1-2\varepsilon)/2$; without loss of generality, $\mu(R_0) \geq (1-2\varepsilon)/2$. For small ε, this estimate differs from (101) approximately by a factor of 2, yielding approximate satisfaction to within about 8ε. Thus the steps between (98) and (101) are unnecessary, unless one really cares about the 4 versus the 8.

Theorem 20 *Let Y denote the figure-Y space, as defined and metrized in* §3.4.6. *Suppose* $0 < \varepsilon < 1/28$. *If* $\lambda_Y(\Gamma \times \Delta) < \varepsilon$, *then either* $\lambda_Y(\Gamma) < 4\varepsilon$ *or* $\lambda_Y(\Delta) < 4\varepsilon$.

Proof. Suppose that we have continuous operations $\overline{p}$, $\overline{G}_i$ ($i \in I$) and $\overline{D}_j$ ($j \in J$) on Y, forming a topological algebra $\mathcal{Y}$, and satisfying equations $\Gamma \times \Delta$ within ε.

Retaining the notation of §3.4.6, we let E, B, C, D denote the center point of Y and its three endpoints (i.e., non-cutpoints). We refer to the segments BE, CE and DE as the *legs* of Y. Let us consider the sets R_B, L_B, R_C, L_C, R_D, L_D, as defined in the statement of Lemma 18.

We first prove that either all R_x have measure $< 2\varepsilon$ or all L_x have measure $< 2\varepsilon$ ($x \in \{A, B, C\}$). Suppose, to the contrary, that e.g. $\mu(L_A) > 2\varepsilon$ and $\mu(R_B) > 2\varepsilon$. Thus there exist $P, Q, R, S \in Y$ such that

$$d(\overline{p}(A,P), \overline{p}(A,Q)) \;>\; 2\varepsilon; \qquad\qquad d(\overline{p}(R,B), \overline{p}(S,B)) \;>\; 2\varepsilon. \tag{102}$$

We now arrange the four pairs (R,P), (R,Q), (S,P), (S,Q) in a graph, as follows

$$\begin{tikzcd} (S,Q) \arrow[r, dash] \arrow[d, dash] & (S,P) \arrow[d, dash] \\ (R,Q) \arrow[r, dash] & (R,P) \end{tikzcd}.$$

Of these four vertices, two must have $\overline{p}$-values that lie in a single leg of Y. Suppose e.g. that V_1 and V_2 are two of the four vertices, with $\overline{p}(V_1)$ and $\overline{p}(V_2)$ on a single leg, say EX where $X \in \{B, C, D\}$, and with

$$d(\overline{p}(V_1), X) \;\leq\; d(\overline{p}(V_2), X) \;\leq\; \text{the smaller of } \{d(\overline{p}(V_3), X), d(\overline{p}(V_4), X)\}$$

for V_3 and V_4 the other two vertices (labeled in any manner). Now either V_3 or V_4 is related to V_1 in the above graph; by choice of notation, we will assume that V_1 is related to V_3. From our choice of the relative positions of $\overline{p}(V_i)$ ($i = 1, 2, 3$), one easily sees that every connected subset of Y that contains $\overline{p}(V_1)$ and $\overline{p}(V_3)$ also contains $\overline{p}(V_2)$.

From here the proof depends slightly on whether the edge V_1V_3 is vertical or horizontal. We present only the vertical case; the horizontal case is similar. Without loss of generality, we may in fact assume these values:

$$V_1 \;=\; (S,Q); \qquad V_2 \;=\; (S,P); \qquad V_3 \;=\; (R,Q).$$

By continuity, the image of $\overline{p}(\cdot, Q)$ is a connected subset of Y that contains $\overline{p}(V_1)$ and $\overline{p}(V_3)$; by our remarks above, it also contains $\overline{p}(V_2)$. Thus there exists $T \in Y$ such that

$$\overline{p}(T,Q) \;=\; \overline{p}(S,P).$$

Finally

$$\overline{p}(A,P) \;\approx\; \overline{p}(\overline{p}(A,A), \overline{p}(S,P)) \;=\; \overline{p}(\overline{p}(A,A), \overline{p}(T,Q)) \;\approx\; \overline{p}(A,Q),$$

and so $d(\overline{p}(A,P),\overline{p}(A,Q)) < 2\varepsilon$, in contradiction to (102). This contradiction completes the proof of our claim that either all R_x have measure $< 2\varepsilon$ or all L_x have measure $< 2\varepsilon$ ($x \in \{A,B,C\}$). Without loss of generality, we may from now on suppose that all L_x have measure $< 2\varepsilon$.

Now let us consider R_B, L_C and L_D. Each is a connected subset of Y, the point $\overline{p}(C,B)$ lies in $R_B \cap L_C$, and the point $\overline{p}(D,B)$ lies in $R_B \cap L_D$. Therefore $R_B \cup L_C \cup L_D$ is connected. Moreover it contains the three points

$$\overline{p}(B,B), \quad \overline{p}(C,C), \quad \overline{p}(D,D),$$

which are within ε of B, C and D, respectively, by (91) The only such set is all of Y, minus three intervals, each of size $< \varepsilon$. Thus

$$\mu(R_B \cup L_C \cup L_D) \;\geq\; 3 - 3\varepsilon.$$

So finally

$$\begin{aligned} \mu(R_B) \;&\geq\; \mu(R_B \cup L_C \cup L_D) - \mu(L_C) - \mu(L_D) \\ &\geq (3-3\varepsilon) - 2\varepsilon - 2\varepsilon \;=\; 3 - 7\varepsilon. \end{aligned}$$

By Lemma 18, there are continuous operations making the space R_B into a topological algebra modeling Γ to within 3ε . Clearly R_B is homeomorphic to Y, and will become isometric to Y upon rescaling each leg. The maximum rescaling factor occurs if all the error is in a single leg, namely $1/1-7\varepsilon$. Therefore, after rescaling, the maximum error in the equations of Γ will be

$$\frac{3\varepsilon}{1-7\varepsilon} \;\leq\; 4\varepsilon,$$

where the final inequality is easily derived from our assumption that $0 < \varepsilon < 1/28$. ∎

There are results like Theorems 19 and 20, for example for λ_{S^1} and λ_{S^n} ($n \neq 1,3,7$), that hold simply because in these special cases, approximate satisfaction of Σ implies a strict structural property of Σ. Consider the case of λ_{S^n} for the indicated values of n. If $\lambda_{S^n}(\Gamma\times\Delta) < \varepsilon$ with $\varepsilon \leq 1$, then by §3.2.1, $\Gamma\times\Delta$ is undemanding. From there, it is not hard to see that either Γ or Δ must be undemanding. Hence $\lambda_{S^n}(\Gamma) = 0$ or $\lambda_{S^n}(\Delta) = 0$.

The case of λ_{S^1} is similar. If If $\lambda_{S^1}(\Gamma\times\Delta) < \varepsilon$ with $\varepsilon \leq 1$, then by §3.2.3 $\Gamma\times\Delta$ is Abelian. From there, it is not hard to see that either Γ or Δ must be Abelian. Hence $\lambda_{S^n}(\Gamma) = 0$ or $\lambda_{S^n}(\Delta) = 0$.

5.2 Product identities on a product of two spaces

In comparing λ_A and λ_B with $\lambda_{A\times B}$, one needs to have a metric on $A \times B$ that is related in some way to the metrics on A and B. For example, if $A = (A, d)$ and $B = (B, e)$ are metric spaces, one can use the Pythagorean metric on $A \times B$, namely ρ, where

$$\rho((a, b), (a', b')) \;=\; \sqrt{d(a, a')^2 + e(b, b')^2}.$$

Two further possible metrics for $A \times B$ are these:

$$\begin{aligned} \sigma((a, b), (a', b')) &= \max(d(a, a'), e(b, b')) \\ \tau((a, b), (a', b')) &= d(a, a') + e(b, b'). \end{aligned}$$

For §5.2 only, we will retain the notations ρ, σ and τ for these three metrics. In any case, the three are related by the obvious inequalities

$$\sigma(\alpha, \alpha') \;\le\; \rho(\alpha, \alpha') \;\le\; \tau(\alpha, \alpha') \;\le\; 2\sigma(\alpha, \alpha').$$

Thus in any case, switching between ρ, σ and τ will change $\lambda_{A\times B}$ at most by a factor of 2. In Theorem 22 below, we will use XXX as our metric on $A \times B$, simply because the calculations are easier with this metric.

The following theorem is easy; its proof will be omitted. Versions obviously exist for other metrics on $A \times B$, as well.

Theorem 21 *If $\lambda_{(A,d)(\Gamma)}(\Gamma) < \delta$ and $\lambda_{(B,e)}(\Delta) < \varepsilon$, then*

$$\begin{aligned} \lambda_{(A\times B,\rho)}(\Gamma \times \Delta) &< \sqrt{\delta^2 + \varepsilon^2} \\ \lambda_{(A\times B,\sigma)}(\Gamma \times \Delta) &< \max(\delta, \varepsilon) \\ \lambda_{(A\times B,\tau)}(\Gamma \times \Delta) &< \delta + \varepsilon. \blacksquare \end{aligned}$$

Theorem 21 makes it rather easy for a product of metric spaces to satisfy the product equations $\Gamma\times\Delta$ approximately. This realization will facilitate our seeing that the approximate-primality results of §5.1 (Theorems 19 and 20) do not extend to product-decomposable spaces. A specific counter-example is provided by Theorem 22 immediately below.

In Theorem 22, (S^1, d) is the one-dimensional sphere with the diameter-1 metric d that is proportional to arc length along shortest paths, and (Y, e) is the triode described in §3.4.6. $S^1 \times Y$ will be given the metric τ described above (addition of metrics in the two components). Γ is the theory of groups (a weaker version could be used instead—see §3.3.3), and Σ is semilattice theory. (Or one may use Σ = commutative idempotent algebras of type $\langle 2 \rangle$.) Topologically, it has long been known that $S^1 \times Y \models \Gamma \times \Sigma$, that $S^1 \times Y \not\models \Gamma$, and that $S^1 \times Y \not\models \Sigma$. The following metric theorem is sharper.

Theorem 22 *$S^1 \times Y \models \Gamma \times \Sigma$—and thus $\lambda_{S^1 \times Y}(\Gamma \times \Sigma) = 0$—while $\lambda_{S^1 \times Y}(\Gamma) \geq 0.1$ and $\lambda_{S^1 \times Y}(\Sigma) \geq 1.0$.*

Proof. Since $S^1 \models \Gamma$ and $Y \models \Sigma$, the first assertion is immediate from Theorem 21 (or from Point (iv) at the start of §5). The estimate for $\lambda_{S^1 \times Y}(\Gamma)$ comes from §3.3.3, and the estimate for $\lambda_{S^1 \times Y}(\Sigma) \geq 0.4$ comes from §3.4.3. ∎

6 Approximate satisfaction by piecewise linear (simplicial) operations.

In §6 we will deal with finite simplicial complexes, corresponding to compact spaces that can be triangulated. (Most of the compact spaces in this paper fall into this category.) Our objective will be to prove that continuous satisfaction within ε is equivalent to piecewise linear (i.e. simplicial) satisfaction within ε, in a manner that can be recursively enumerated (see results of §7.1).

6.1 Simplicial complexes and maps

For definiteness, we paraphrase the definitions and notation of Spanier [22, pages 108–128]. A *(simplicial) complex* is a family K of nonempty finite sets such that

(a) if $v \in \bigcup K$, then $\{v\} \in K$;

(b) if $\emptyset \neq s \subseteq t \in K$, then $s \in K$.

The elements of K are sometimes called the (abstract) *simplices* of K. The elements of $V = \bigcup K$ are called *vertices* of K. (Condition (a) provides a one-one correspondence between vertices of K and one-element simplices of K.) If K and L are complexes and $L \subseteq K$, then L is called a *subcomplex* of K. For each simplex s of K, we let $\overline{s}$ denote the set of all non-empty subsets of s, and $\dot{s}$ denote $\overline{s} \smallsetminus \{s\}$. One easily checks that $\overline{s}$ and $\dot{s}$ are both subcomplexes of K.

The connection with metric topology is this. The *standard geometric realization* of K is a metric space $|K|$ whose underlying set consists of all vectors $\alpha \in [0,1]^V$ such that

(c) The *carrier* of α, namely the set $\{v \in V : \alpha_v > 0\}$, is a simplex of K. (In particular, it is a finite subset of K.)

(d) $\sum_{v \in V} \alpha_v = 1$.

For finite K—for which V is also finite—the metric on $|K|$ is the restriction of the usual Euclidean metric on $[0,1]^V$. In other words,

$$d(\alpha, \beta) \; = \; \sqrt{\sum_{v \in V} (\alpha_v - \beta_v)^2}. \tag{103}$$

In fact, even for infinite K, the sum appearing in Equation (103) is finite, and hence (103) may be used as a distance formula for all K, finite or infinite. See Spanier [22, *loc. cit.*] for the fact that $|K|$ is compact if and only if K is finite. (In fact [*loc. cit.*] another topology is often used in the case of infinite K.) In §6 our interest is in compact spaces; hence we will mainly work with finite K.

For $w \in V$, define $w^\star \in |K|$ *via*

$$w^\star(v) \; = \; \begin{cases} 1 & \text{if } v = w; \\ 0 & \text{otherwise.} \end{cases}$$

In this notation, the α introduced in clause (c) above may also be denoted $\sum_{v \in s} \alpha_v v^\star$. It is then not hard to see that $w \longmapsto w^\star$ maps V injectively to $|K|$, and that for each simplex s, the convex hull of $\{w^\star : w \in s\}$ in $[0,1]^V$ is the set of points in $|K|$ with carrier $\subseteq s$. This convex hull is denoted $|s|$. Since $s \subseteq V$, there is a natural projection $[0,1]^V \longrightarrow [0,1]^s$; this projection restricts to a homeomorphism $|s| \longrightarrow |\overline{s}|$. The space $|\overline{s}|$—the geometric realization of an abstract simplex—is a *geometric n-simplex* for some n, i.e., the set of points $(\alpha_0, \cdots, \alpha_n) \in [0,1]^{n+1}$ satisfying $\sum \alpha_i = 1$, with the usual Euclidean metric.

For $s \in V$ we call $|s|$ the *closed geometric simplex of s*. The *open geometric simplex of s* is

$$\langle s \rangle \; = \; |s| \smallsetminus \bigcup \{\, |t| \, : \, t \subseteq s, \; t \neq s\}.$$

$\langle s\rangle$ is an open subset of $|s|$, homeomorphic to an open n-disk (for appropriate n). Each vector $\alpha \in |K|$ lies in $\langle s\rangle$, where s is the carrier of α, and in no other set of the form $\langle t\rangle$. Thus the sets $\langle s\rangle$ partition the space $|K|$.

A *simplicial map* $\phi : K_1 \longrightarrow K_2$ is a map between the corresponding vertex-sets, $\phi : V_1 \longrightarrow V_2$, such that if $s \in K_1$, then $\phi[s] \in K_2$. The *geometric realization of* ϕ is the map $|\phi| : |K_1| \longrightarrow |K_2|$ that is defined as follows: if

$$\alpha \;=\; \sum_{v\in s} \alpha_v \, v^\star,$$

where each $\alpha_v > 0$ and $\sum \alpha_v = 1$, then

$$|\phi|\,(\alpha) \;=\; \sum_{v\in s} \alpha_v \, (\phi(v))^\star.$$

It is not hard to check that the right-hand-side here lies in $|K_2$ and is well-defined, and that the resulting map between metric spaces is continuous.

6.2 The product complex.

Suppose that K_1 and K_2 are complexes, and that $<$ is a total order on $V_1 \cup V_2$. We will define a complex $K_1 \times_< K_2$ with vertex-set $V_1 \times V_2$.

Suppose that $s^1_m = \{v^1_0, \cdots, v^1_m\}$ is an m-simplex of K_1, with $v^1_0 < \cdots < v^1_m$, and that $s^2_n = \{v^2_0, \cdots, v^2_n\}$ is an n-simplex of K_2, with $v^2_0 < \cdots < v^2_n$. Now let

$$\lambda \;=\; \big((r_0, s_0),\; (r_1, s_1),\; \cdots\; (r_{m+n}, s_{m+n})\big) \tag{104}$$

be a finite sequence of pairs with $r_0 = s_0 = 0$, and such that, for each i, either $r_{i+1} = r_i + 1$ and $s_{i+1} = s_i$, or $r_{i+1} = r_i$ and $s_{i+1} = s_i + 1$ and such that the first alternative happens m times and the second alternative happens n times. (It of course follows that $r_{m+n} = m$ and $s_{m+n} = n$. There are clearly $\binom{m+n}{n}$ sequences (104).) We then define

$$s^\lambda_{m+n} \;=\; \left\{\, (v^0_{r_0}, v^1_{s_0}), \cdots, (v^0_{r_m}, v^1_{s_n}) \,\right\}$$

to be an $(m+n)$-simplex of $K_1 \times_< K_2$. We omit the (standard) proof that the set of all such s^λ_{m+n} (for all appropriate λ, for all m and n, and for s^1_m and s^2_n ranging over all simplices in K_1 and K_2, respectively), form a complex, which we will denote $K_1 \times_< K_2$.

It is not hard to check (using formula (104)) that each of the two coordinate projections $\pi_i : V_1 \times V_2 \longrightarrow V_i$ is a simplicial map. Therefore we have continuous maps

$$|\pi_i| : |K_1 \times_< K_2| \longrightarrow |K_i|$$

for $i = 1, 2$. We therefore have the continuous map

$$|\pi_1| \times |\pi_2| \,:\, |K_1 \times_< K_2| \;\longrightarrow\; |K_1| \times |K_2|. \tag{105}$$

We omit the (standard) proof that the map appearing in (105) is a homeomorphism; in other words, $K_1 \times_< K_2$ triangulates the product space $|K_1| \times |K_2|$.

In fact, there is a standard way to triangulate any finite product of geometric realizations, $|K_1| \times \cdots \times |K_n|$. One may give an analogous construction, or simply iterate the binary product that we already have. We omit the details.

6.3 Barycentric subdivision

Let K be a complex, with vertex-set V, and let $|K|$ be the standard geometric realization of K (§6.1). For each n, and each n-simplex $s \in K$, we define the *barycenter* of $s = \{w_0, \cdots, w_n\}$ to be the point in $|K|$ defined by

$$b(s) \;=\; \frac{1}{n+1} \sum_{w \in s} w^\star \;=\; \frac{1}{n+1} \sum_{j=0}^{n} w_j^\star\,. \tag{106}$$

In other words,

$$[b(s)]\,(v) \;=\; \begin{cases} \frac{1}{n+1} & \text{if } v \in s; \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to see that $b(s)$ is in the open simplex $\langle s \rangle$ of $|K|$; therefore there is a one-one correspondence between K and the set of all barycenters.

We define a complex K', called the *barycentric subdivision* of K, as follows. The vertex set V' for K' is the set of barycenters $b(s)$ for $s \in K$, and an n-simplex of K' is a finite set

$$\Gamma \;=\; \big\{\, b(s_0),\, \cdots,\, b(s_n) \,\big\}, \tag{107}$$

where

$$s_0 \;\subset\; s_1 \;\subset\; \cdots \;\subset\; s_n \;\in\; K. \tag{108}$$

Clearly K and K' have the same dimension. Conditions (a) and (b) of §6.1 are easily seen to hold for K', and hence K' is a complex. We will see that in fact $|K'|$ is homeomorphic to $|K|$.

Since V' is a set of elements of $|K|$, we have the identity inclusion $\iota : V' \longrightarrow |K|$. Let us suppose we have a K'-simplex Γ as in (107). Then Condition (108) assures us that each vertex $b(s_j)$ of Γ in fact lies in the realization $|s_n|$ of s_n (a geometric simplex $\subseteq |K_n|$). Thus ι has an affine-linear extension $|\iota|$ to the geometric realization $|K'|$ of K'. In particular, on the simplex Γ defined by (107) and (108), we have

$$|\iota| \left(\sum_{j=0}^{n} \alpha_j \, [b(s_j)]^{\star} \right) \; = \; \sum_{j=0}^{n} \alpha_j \, b(s_j) \; \in \; |K|$$

for all $\alpha_0, \cdots, \alpha_n$ with $\sum \alpha_j = 0$. For the proof of the following well-known result see [22, *loc. cit.*].

Lemma 23 *$|\iota|$ maps $|K'|$ homeomorphically to $|K|$.* ∎

Recall that we have metrized $|K|$ according to formula (103). We could of course metrize the geometric realization $|K'|$ according to (103), but in that case the homeomorphism $|\iota|$ of Lemma 23 would not be an isometry. We prefer to keep a single metric; in other words, we shall metrize $|K'|$ in the unique way that makes $|\iota|$ an isometry.

We can now iterate the process of subdivision, yielding a tower of isometries[18]

$$\cdots \xrightarrow{|\iota|} |K^{(m)}| \xrightarrow{|\iota|} \cdots \xrightarrow{|\iota|} |K''| \xrightarrow{|\iota|} |K'| \xrightarrow{|\iota|} |K|.$$

We saw in §6.1 that the open simplices $\langle s \rangle$ of $|K|$ partition $|K|$. This result applies, of course, to each iteration of the subdivision. Thus, for example, the open simplices of $|K'|$ partition $|K'|$. These can be carried by $|\iota|$ to sets in $|K|$, which of course partition the original $|K|$. It is not hard to check that, if Γ is a simplex of K' (see (107)) with largest vertex $s_n \in K$, then $|\iota| \, [\langle \Gamma \rangle] \subseteq \langle s_n \rangle$. In other words,

Lemma 24 *The partition of $|K|$ by sets of the form $|\iota| \, [\langle \Gamma \rangle]$ refines the partition by sets of the form $\langle s \rangle$.* ∎

[18] We call them all $|\iota|$, even though, strictly speaking, they are distinct maps.

The next lemma is almost obvious from what has come before. The notation continues from above.

Lemma 25 $|\iota|\,[\langle\Gamma\rangle]$ *is a subset of* $\langle s\rangle$ *that is defined by linear inequalities.* ∎

The partition of $|K|$ by sets of the form $\langle s\rangle$ will be called the *natural triangulation* of $|K|$. The partition of $|K|$ mentioned at the start of Lemma 24 will be called the *first subdivision* of the natural triangulation. Obviously one also has the second, third, ..., m^{th}, ... subdivisions of the natural triangulation. (The elements of the m^{th} subdivision are the $|\iota|^m$-images of sets in the natural partition of $|K^{(m)}|$.)

If A is a metric space and if $\mathcal{Z}$ is a family of subsets of A, the *mesh* of $\mathcal{Z}$ is

$$\sup_{B\in\mathcal{Z}}\ \sup_{b,c\in B}\ d(b,c).$$

Suppose that K is a complex of dimension n. It is clear from the definitions in §6.1 that the natural triangulation of $|K|$ has mesh $\leq \sqrt{2}$. The next lemma follows from some elementary considerations of metric geometry; see e.g. [22, *loc. cit.*].

Lemma 26 *Let K be a complex of dimension n. The m^{th} subdivision of the natural triangulation has mesh*

$$\sqrt{2}\,\left(\frac{n}{n+1}\right)^m. \;\blacksquare$$

In particular, the iterated subdivisions have mesh approaching zero.

6.4 Simplicial approximation of continuous operations

At the end of §6.1, we defined the notion of simplicial maps and their associated geometric realizations. Here we extend this idea as follows. Let K and L be complexes. An (M,N)-*simplicial map* from K to L is a simplicial map $\phi : K^{(M)} \longrightarrow L^{(N)}$, where $K^{(M)}$ and $L^{(N)}$ are as defined in §6.3. Its *ground-level geometric realization* is the (piecewise linear, continuous) composite map

$$|\phi|_0 \;=\; |K| \xrightarrow{(|\iota|^M)^{-1}} |K^{(M)}| \xrightarrow{|\phi|} |L^{(N)}| \xrightarrow{|\iota|^N} |L|\,. \qquad (109)$$

The virtues of such maps are two: (1) they are amenable to an algorithmic approach; (2) (allowing free choice of M and N) they can approximate any continuous function between spaces $|K|$ and $|L|$ (for K and L finite). As for (1), that will be the topic of §7.1 below. As for (2), we have the following version of the classical *Simplicial Approximation Theorem*.

Theorem 27 *For K finite and L an arbitrary complex, given real $\varepsilon > 0$ and continuous $f : |K| \longrightarrow |L|$, there exist positive integers M and N and a simplicial map $\phi : K^{(M)} \longrightarrow L^{(N)}$ such that $|\phi|_0$ approximates f within ε on $|K|$.*

Proof. The traditional version of this theorem (see e.g. Spanier [22, Theorem 8, page 128]) says this. Given f and N, there exist M and a simplicial map $\phi : K^{(M)} \longrightarrow L^{(N)}$ such that $|\phi|_0$ approximates f in the following sense: if $x \in |K|$ and $f(x) \in |\iota|^N[\langle s \rangle]$ for a simplex s of $L^{(N)}$, then $|\phi|_0(x) \in |\iota|^N[|s|]$. In order to obtain the conclusion of Theorem 27 as stated, we merely need to choose N here large enough that each geometric simplex $|s|$ of $L^{(N)}$ has diameter $< \varepsilon$. That this is possible is immediate from Lemma 26. ∎

Now, for definiteness, let us suppose that the ground-level complexes under discussion (i.e. K, L, etc., but not $K^{(M)}$, etc.) all have vertices in a fixed set V_1, which has a fixed strict total order $<$. If K is a simplicial complex and $n \in \mathbb{Z}^+$, then by K^n we mean the complex

$$(\cdots((K \times_< K) \times_< K) \times_< \cdots \times_< K),$$

having n factors K, that is formed by an iteration of the construction in §6.2. An iterated version of Equation (105) provides a natural homeomorphism

$$|\pi|^n \;=\; (\cdots(|\pi_1| \times |\pi_2|) \times \cdots \times |\pi_n|) \;:\; |K^n| \;\longrightarrow\; |K|^n. \qquad (110)$$

Again, this map is a homeomorphism, but not an isometry according to the way the metrics have been defined. We get around this by redefining the metric on $|K^n|$ so that $|\pi|^n$ becomes an isometry.

Theorem 28 *For K finite, $n \in \mathbb{Z}^+$, and L an arbitrary complex, given real $\varepsilon > 0$ and continuous $F : |K|^n \longrightarrow |L|$, there exist positive integers M and N and a simplicial map $\phi : (K^n)^{(M)} \longrightarrow L^{(N)}$ such that $|\phi|_0 \circ (|\pi|^n)^{-1}$ approximates F within ε on $|K|^n$.*

Proof. $F \circ |\pi|^n$ maps $|K^n|$ to $|L|$; hence by the Simplicial Approximation Theorem (27), $F \circ |\pi|^n$ is ε-approximated by $|\phi_0|$ for some simplicial map $\phi\colon (K^n)^{(M)} \longrightarrow L^{(N)}$. Therefore F is ε-approximated by $|\phi|_0 \circ (|\pi|^n)^{-1}$. ∎

6.5 Approximation of term operations

The considerations of §6.5 apply to operations on any compact metric space; triangulation is not required here.

We consider a finite similarity type $\langle F_i \rangle_{i<N}$, with each F_i of arity n_i. The *depth* of a term τ in this language is defined recursively as follows:

(i) If τ is a variable, then τ has depth 0;

(ii) if $\tau = F_i(\tau_1, \cdots, \tau_{n_i})$, then

$$\mathrm{depth}(\tau) \;=\; 1 \,+\, \max\{\, \mathrm{depth}(\tau_j) \;:\; 1 \le j \le n_i\}.$$

Suppose now that $\mathbf{A} = (A, F_i^{\mathbf{A}})_{i<N}$ is a topological algebra of this similarity type, with A a compact metric space. As is well known, the operations $F_i^{\mathbf{A}}$ are in fact *uniformly continuous*.

The aim in §6.5 is to see how appropriate approximations to the operations $F_i^{\mathbf{A}}$ (say by piecewise linear operations, or by differentiable operations) can lead to approximations of term operations $\tau^{\mathbf{A}}$. Thus suppose we are given a real $\varepsilon > 0$, and we would like to be able to approximate any depth-n term operation[19] $\tau^{\mathbf{A}}$ within ε. We recursively define[20] real numbers $\varepsilon_k > 0$ as follows:

(i) $\varepsilon_0 \,=\, \varepsilon$;

(ii) suppose that ε_k has already been defined. By uniform continuity, there exists[21] real $\varepsilon^\star > 0$ such that, for all $i < N$, and for all $x_j, y_j \in A$

[19] The precise meaning of "approximating a term-operation" is deferred until the statement of Lemma 29.

[20] In the forthcoming work [32] we shall say that the operation $\overline{F}$ is *constrained* by $(\varepsilon_1, \varepsilon_0/2)$, by $(\varepsilon_2, \varepsilon_1/2)$, and so on.

[21] In the unobstructed realm of all continuous operations, there is no effective value that can be assigned to $\varepsilon^\star$. This is why we need the enumerative approach in §7.1 below.

$(1 \leq j \leq n_i)$, we have:

$$\text{if } (d(x_j, y_j) < \varepsilon^\star \text{ for } 1 \leq j \leq n_i, \quad \text{then} \\ d(F_i^{\mathbf{A}}(x_1, \ldots, x_{n_i}), F_i^{\mathbf{A}}(y_1, \ldots, y_{n_i})) \; < \; \frac{\varepsilon_k}{2}. \tag{111}$$

We choose such an $\varepsilon^\star$ and define

$$\varepsilon_{k+1} \;=\; \min\{\, \varepsilon^\star, \; \frac{\varepsilon_k}{2} \,\}. \tag{112}$$

The next lemma says, in paraphrase, that if we approximate the operations $F_i^{\mathbf{A}}$ within ε_M, then for all terms τ of depth $\leq M$, we will also approximate $\tau^{\mathbf{A}}$ within ε. The proof is a simple recursion based on the triangle inequality.

Lemma 29 *Let us be given a topological algebra $\mathbf{A} = (A, F_i^{\mathbf{A}})_{i<N}$ (as above), an integer $M \geq 1$, and a real $\varepsilon > 0$. Let ε_M be as defined above in* (111–112)*. If $\mathbf{B} = (A, F_i^{\mathbf{B}})_{i<N}$ is a similar algebra defined on A (topological or not), and if the operations $F_i^{\mathbf{B}}$ satisfy*

$$d(F_i^{\mathbf{B}}(\mathbf{x}), F_i^{\mathbf{A}}(\mathbf{x})) \; < \; \varepsilon_{M-1} \tag{113}$$

for each $\mathbf{x} \in A^{n_i}$, then for each term τ of depth $\leq M$, we have

$$d(\tau^{\mathbf{B}}(\mathbf{x}), \tau^{\mathbf{A}}(\mathbf{x})) \; < \; \varepsilon$$

for each $\mathbf{x} \in A^\omega$.

Proof. We will prove, by induction on k, that if σ is a term (in this language) of depth $\leq k$, then

$$d(\sigma^{\mathbf{B}}(\mathbf{x}), \sigma^{\mathbf{A}}(\mathbf{x})) \; < \; \varepsilon_{M-k} \tag{114}$$

for each $\mathbf{x} \in A^\omega$. The lemma then follows from the truth of (114) for $k = M$.

For $k = 1$, the inductive assertion (114) is immediate from (113). We now suppose that the assertion holds for $k \geq 1$ and prove it for $k + 1$. So suppose that σ is a term of depth $\leq k + 1$. We may clearly assume that $\sigma = F_i(\sigma_1, \cdots, \sigma_{n_i})$ for some $i < N$ and some terms σ_j of depth $\leq k$. By an inductive appeal to (114), for each j we have

$$d(\sigma_j^{\mathbf{B}}(\mathbf{x}), \sigma_j^{\mathbf{A}}(\mathbf{x})) \; < \; \varepsilon_{M-k}$$

for each $\mathbf{x} \in A^\omega$. To calculate (114) for σ, we begin with the triangle inequality, obtaining

$$\begin{aligned} d(\sigma^{\mathbf{B}}(\mathbf{x}), \sigma^{\mathbf{A}}(\mathbf{x})) &= d(F^{\mathbf{B}}(\sigma_1^{\mathbf{B}}(\mathbf{x}), \ldots), F^{\mathbf{A}}(\sigma_1^{\mathbf{A}}(\mathbf{x}), \ldots)) \\ &\leq d(F^{\mathbf{B}}(\sigma_1^{\mathbf{B}}(\mathbf{x}), \ldots), F^{\mathbf{A}}(\sigma_1^{\mathbf{B}}(\mathbf{x}), \ldots)) \\ &\qquad + d(F^{\mathbf{A}}(\sigma_1^{\mathbf{B}}(\mathbf{x}), \ldots), F^{\mathbf{A}}(\sigma_1^{\mathbf{A}}(\mathbf{x}), \ldots)) \\ &\leq \varepsilon_{M-1} + \frac{\varepsilon_{M-k-1}}{2} \leq \frac{\varepsilon_{M-k-1}}{2} + \frac{\varepsilon_{M-k-1}}{2} = \varepsilon_{M-k-1}\,. \end{aligned}$$

(In the final line, the first term of the estimate (i.e. ε_{M-1}) comes from (113), and the second term (i.e. ε_{M-k-1}) comes from (111) and (114). The final inequality comes from (112). ▮

6.6 Approximate satisfaction by simplicial operations.

Suppose that $A = |K|$, the geometric realization of a complex K. We call an operation $G : A^n \longrightarrow A$ *simplicial* if there exist M, N and an (M, N)-simplicial map $\phi : (K^n)^{(M)} \longrightarrow L^{(N)}$ such that $G = |\phi|_0 \circ (|\pi|^n)^{-1}$. (For notation, see Theorem 28 on page 85, and material preceding Theorem 28.)

Corollary 30 *Let $A = |K|$, with K finite, and let $\mathbf{A} = (A, F_i^{\mathbf{A}})_{i<N}$ be a topological algebra based on A. For each $M \in \mathbb{Z}^+$ and each real $\varepsilon > 0$, there exist simplicial operations $F_i^{\mathbf{B}} : A^{n_i} \longrightarrow A$ such that each term τ of depth $\leq M$ in the operation symbols F_i satisfies*

$$d(\tau^{\mathbf{B}}(\mathbf{x}), \tau^{\mathbf{A}}(\mathbf{x})) < \varepsilon$$

for each $\mathbf{x} \in A^\omega$.

Proof. Clearly A is compact, and so we may appeal to Theorem 29. Let ε_{M-1} be as supplied by Theorem 29 (in other words, coming from Equations (111–112)). By Theorem 28, for each operation $F_i^{\mathbf{A}}$, there is a simplicial map ϕ_i such that $|\phi_i|_0 \circ (|\pi|^n)^{-1}$ approximates $F_i^{\mathbf{A}}$ within ε_{M-1}. In other words, if we define $F_i^{\mathbf{B}}$ to be the simplicial operation $|\phi_i|_0 \circ (|\pi|^n)^{-1}$, then we have Equation (113) holding for all $\mathbf{x} \in A^{n_i}$. The desired conclusion is now immediate from Theorem 29. ▮

Recall from §1.1 that

$$\begin{aligned}
\lambda_{\mathbf{A}}(\sigma,\tau) &= \sup\bigl\{\, d(\sigma^{\mathbf{A}}(\mathbf{a}),\tau^{\mathbf{A}}(\mathbf{a})) \,:\, \mathbf{a}\in A^{\omega}\bigr\} \;\in\; \mathbb{R}^{\geq 0}\cup\{\infty\};\\
\lambda_{\mathbf{A}}(\Sigma) &= \sup\bigl\{\lambda_{\mathbf{A}}(\sigma,\tau) \,:\, \sigma\approx\tau \,\in\Sigma\bigr\} \;\in\; \mathbb{R}^{\geq 0}\cup\{\infty\};\\
\lambda_{A}(\Sigma) &= \inf\bigl\{\lambda_{\mathbf{A}}(\Sigma) \,:\, \mathbf{A}=(A;\overline{F}_t)_{t\in T},\ \overline{F}_t \text{ any continuous operations}\bigr\}.
\end{aligned}$$

Lemma 31 *If A is a metric space, and $\mathbf{A}=(A,F_t^{\mathbf{A}})_{t\in T}$ and $\mathbf{B}=(A,F_i^{\mathbf{B}})_{i<N}$ are two similar algebras based on A, and if σ and τ are terms in the language of $\mathbf{A}$ satisfying*

$$d(\sigma^{\mathbf{B}}(\mathbf{x}),\sigma^{\mathbf{A}}(\mathbf{x})) \;<\; \varepsilon\,;\quad d(\tau^{\mathbf{B}}(\mathbf{x}),\tau^{\mathbf{A}}(\mathbf{x})) \;<\; \varepsilon$$

for all $\mathbf{x}\in A^{\omega}$, then $\lambda_{\mathbf{B}}(\sigma,\tau)\leq\lambda_{\mathbf{A}}(\sigma,\tau)\,+\,2\,\varepsilon$. ∎

Theorem 32 *Suppose that $A=|F|$, the geometric realization of a finite complex, and suppose that Σ is a finite set of equations of type $\langle F_t \,:\, t\in T\rangle$. For each topological algebra $\mathbf{A}=(A,F_t^{\mathbf{A}})_{t\in T}$ of this type based on A, and for each real $\varepsilon>0$, there exists an algebra $\mathbf{B}=(A,F_t^{\mathbf{B}})_{t\in T}$, each of whose operations is a simplicial operation, and such that*

$$\lambda_{\mathbf{B}}(\Sigma) \;\leq\; \lambda_{\mathbf{A}}(\Sigma)\,+\,\varepsilon.$$

Proof. Let M be the maximum depth of terms appearing in Σ. Take M and $\varepsilon/2$ to Corollary 30, and let $F_i^{\mathbf{B}}\colon A^{n_t}\longrightarrow A$ be the simplicial operations that it yields. Thus Corollary 30 yields

$$d(\sigma^{\mathbf{B}}(\mathbf{x}),\sigma^{\mathbf{A}}(\mathbf{x})) \;<\; \varepsilon/2\,;\quad d(\tau^{\mathbf{B}}(\mathbf{x}),\tau^{\mathbf{A}}(\mathbf{x})) \;<\; \varepsilon/2$$

for $\mathbf{x}\in A^{\omega}$ and for each equation $\sigma\approx\tau$ of Σ. Then Lemma 31 yields $\lambda_{\mathbf{B}}(\sigma,\tau)\;\leq\;\lambda_{\mathbf{A}}(\sigma,\tau)\,+\,\varepsilon$ for each equation $\sigma\approx\tau$ of Σ. Collecting the individual equations into one Σ yields the desired result. ∎

In performing the infimum that occurs in the definition of λ_A for a metric space A, we might wish to restrict our attention to simplicial operations. In other words, we may define

$$\lambda_A^{\mathrm{simp}}(\Sigma) \;=\; \inf\bigl\{\lambda_{\mathbf{A}}(\Sigma) \,:\, \mathbf{A}=(A;\overline{F}_t)_{t\in T},\ \overline{F}_t \text{ any simplicial operations}\bigr\}.$$

Clearly $\lambda_A^{\mathrm{simp}}(\Sigma)\;\geq\;\lambda_A(\Sigma)$ for any Σ and for A the geometric realization of any complex. In one special case we have equality:

Corollary 33 *Suppose that $A = |F|$, the geometric realization of a finite complex, and suppose that Σ is a finite set of equations. Then*

$$\lambda_A^{simp}(\Sigma) \;=\; \lambda_A(\Sigma). \blacksquare$$

7 Algorithmic considerations.

Recall from [31] that there is no algorithm to determine whether a finite Σ is compatible with $\mathbb{R}$. In fact, the set of finite Σ that are not $\mathbb{R}$-compatible fails to be recursively enumerable [31].

Our main tool will be an algorithm $\mathcal{T}$ that decides the truth in $\langle \mathbb{R}; +, \cdot, 0, 1\rangle$ of all first-order sentences in $+$, $\cdot$, 0, and 1. Such an algorithm was devised by Alfred Tarski in 1931 — see [24, 25, 26] — with improved versions devised later by G. E. Collins in 1975 (see [9]), and by several others in later years (see e.g. [6]). (Notice that $x \le y$ can be replaced in this context by $\exists z(y = x + z^2)$, so the algorithm $\mathcal{T}$ can also work with inequalities.)

7.1 The main simplicial algorithm

We shall assume that we have available an input language that accommodates the description of a finite sets of sets (e.g. a simplicial complex), a finite similarity type $(n_1, \cdots, n_k)$, and equations in operation symbols F_1 (n_1-ary),…,F_k (n_k-ary).

Theorem 34 *There exists an algorithm $\mathcal{A}$ with the following behavior. $\mathcal{A}$ accepts as input positive integers M, N, r and s, a finite complex K, and a finite equation-set Σ. $\mathcal{A}$ yields an answer to the question, do there exist M, N-simplicial maps on the appropriate powers of $|K|$ so that the equations of Σ are satisfied within $< r/s$?*

Proof. We consider only a single equation $\sigma \approx \tau$; the extension to a finite set offers no important complications.

- Adopt an algorithmic system for the handling of complexes, their products (§6.2), their subdivision (§6.3), and their geometric representations (§6.1). For use with the latter, we shall require real-vector calculations within any specified tolerance. In particular the system must be capable of a symbolic modeling of the computations seen in §6.4: the ground-level geometric realization $|\phi|_0$ defined in Equation (109), the function $|\pi|^n$ described in Equation (110), and composites involving these, as used in Theorem 28.

- Establish a list — F_1(n_1-ary),. . . ,F_k (n_k-ary) — of the operation symbols that appear either in σ or in τ, and their arities.

- Establish a representation in this system for the complex $K^{(N)}$, and for each complex $(K^{n_j})^{(M)}$ (for $1 \le j \le k$).

- Loop through all of the finitely many k-tuples $(\phi_1, \cdots, \phi_k)$, where each ϕ_j maps the vertices of $(K^{n_j})^{(M)}$ to the vertices of $K^{(N)}$. For each k-tuple, loop through these instructions:

  - Examine each ϕ_j to see whether it is a simplicial map. If all are simplicial, continue this loop; if one is not simplicial, jump to the next k-tuple of maps and return to this instruction.
  - Represent each $|\phi_j| \circ (|\pi|^{n_j})^{-1}$ — as in Theorem 28 — as a piecewise affine map with unknown real coefficients α_u^v. (With u ranging over all simplices of $(K^{n_j})^{(M)}$, and v ranging over $\{0, \cdots, \dim(u)\}$.)
  - Recursively realize each subterm of σ by a piecewise affine map with unknown coefficients. These unknown coefficients may be expressed as ring-theoretic combinations of the α_u^v. The same is to be done for τ. Now distances between $\sigma(\mathbf{x})$ and $\tau(\mathbf{x})$ may be computed on each simplex of the subdivision, in terms of the unknown coefficients α_u^v.
  - Finally we express a question of whether this distance can made less than r/s over each simplex. This question may be expressed as the existential closure of a conjunction of ring-theoretic inequalities in the unknowns α_u^v.
  - Using Tarski's algorithm $\mathcal{T}$, we determine the truth in $\mathbb{R}$ of this existential sentence. If the answer is affirmative, we terminate the algorithm, answering yes to the question of satisfiability within $< r/s$.

- If we reach this point, having never answered yes, we terminate the algorithm, answering no the question of satisfiability within $< r/s$.

It is obvious that the algorithm terminates at some point. The only way for it to terminate with a "yes" answer is if, in the penultimate instruction, Tarski's algorithm tells us that a certain family of piecewise-linear operations, namely $|\phi_j| \circ (|\pi|^{n_j})^{-1}$ — as in Theorem 28 — yields a topological algebra that satisfies $\sigma \approx \tau$ within $< r/s$. On the other hand, if a simplicial map exists allowing satisfaction within r/s, then it must have been considered, and Tarksi algorithm must have answered "yes." Hence a "yes" answer must have been obtained by our algorithm. This shows that the algorithm is correct. ∎

For the next corollary, let us fix a list of operation symbols F_i $(i \in \omega)$, which includes each arity infinitely often.

Corollary 35 *There exists an algorithm $\mathcal{B}$ that takes no input and whose output is an infinite sequence of sextuples $\langle K, M, N, r, s, \Sigma \rangle$. Each sextuple satisfies*

1. *K is a finite simplicial complex.*
2. *M, N, r and s are positive integers.*
3. *Σ is a finite set of equations, whose operation symbols are among the F_i.*
4. *There exist M, N-simplicial maps on the appropriate powers of $|K|$ so that the equations of Σ are satisfied within $< r/s$.*

Moreover every sextuple satisfying (1)-(4) is in the output of the algorithm $\mathcal{B}$.

Proof. It is well known that there exists an algorithm $\mathcal{C}$ that lists all sextuples satisfying 1, 2 and 3. To obtain the desired $\mathcal{B}$, we merely filter $\mathcal{C}$ using the algorithm $\mathcal{A}$ of Theorem 34. In other words $\mathcal{B}$ successively takes each output of $\mathcal{B}$ and passes it to $\mathcal{A}$, which returns an answer of whether there exist M, N-simplicial maps on the appropriate powers of $|K|$ so that the equations of Σ are satisfied within $< r/s$ (Point 4). If the answer is "no," $\mathcal{B}$ takes no further action at that stage; if the answer is "yes," then $\mathcal{B}$ outputs the sextuple in question. ∎

7.2 Recursive enumerability of $\lambda_{|K|}(\Sigma) < \alpha$

For *computable* real numbers, the reader is referred to [21]. All we shall require of computability is this: if α is computable, then there is an algorithm to decide $s\,\alpha > r$ for positive integers r and s. Clearly every rational is computable; hence the computable reals are dense in $\mathbb{R}$.

As in Corollary 35, we fix a list of operation symbols F_i $(i \in \omega)$, which includes each arity infinitely often. Clearly every finite set of equations is definitionally equivalent to a finite set involving only the operation symbols F_i.

Corollary 36 *Let K be a finite simplicial complex, with $|K|$ its geometric realization (as usual), and let $\alpha > 0$ be a computable real number. There is an algorithm $\mathcal{E}_{K,\alpha}$ whose output consists of those finite sets Σ of equations in F_i $(i \in \omega)$ for which $\lambda_{|K|}(\Sigma) < \alpha$.*

Proof. We run the algorithm $\mathcal{B}$ of Corollary 35, and filter the output as follows. When $\mathcal{B}$ outputs $\langle K, M, N, r, s, \Sigma\rangle$, the new algorithm $\mathcal{E}_{K,\alpha}$ either outputs Σ or rests. If $s\,\alpha > r$, then $\mathcal{E}_{K,\alpha}$ outputs Σ; if not, then $\mathcal{E}_{K,\alpha}$ has no output. It is immediate from Corollary 35 that $\mathcal{E}_{K,\alpha}$ has the desired output. ∎

Corollary 37 *Let K be a finite complex and $\alpha > 0$ a computable real number. Restricting attention to equations in F_i $(i \in \omega)$, the set of finite sets Σ with $\lambda_{|K|}(\Sigma) < \alpha$ is recursively enumerable.* ∎

Corollary 38 *Let $\alpha > 0$ be a computable real number. Restricting attention to equations in F_i $(i \in \omega)$, the set of pairs (K, Σ) where K is a finite complex, Σ is a finite set of equations, and where $\lambda_{|K|}(\Sigma) < \alpha$, is recursively enumerable.* ∎

7.3 The arithmetic character of $\lambda_{|K|}(\Sigma) = 0$.

As in Corollaries 35–37, we fix a list of operation symbols F_i $(i \in \omega)$, which includes each arity infinitely often. We also fix a syntax that describes nothing but complexes and describes each complex at least once up to isomorphism.

Corollary 39 *There is an algorithm $\mathcal{F}$ that takes one finite simplicial complex K as input, and whose output is an infinite stream of pairs (Σ, s), with each Σ a finite set of equations in the symbols F_i ($i \in \omega$) and with each $s \in \mathbb{Z}^+$, such that the following condition holds. Such a finite set Σ satisfies $\lambda_{|K|}(\Sigma) = 0$ iff (Σ, s) occurs in the output stream of $\mathcal{F}_K$ for arbitrarily large s.*

Proof. The algorithm $\mathcal{F}$ accepts K, and then runs the algorithm $\mathcal{B}$ of Corollary 35, while filtering the output as follows. Suppose that one piece of output from $\mathcal{B}$ is $\langle K', M, N, r, s, \Sigma\rangle$. If $K' \neq K$, or if $r \neq 1$, this output is filtered out completely. On the other hand, if $K' = K$ and $r = 1$, then $\mathcal{F}$ outputs the pair (Σ, s).

To see the equivalence of $\lambda_{|K|}(\Sigma) = 0$ with the algorithmic condition, we suppose first that Σ is a finite set of equations for which (Σ, s) appears with arbitrarily large s. By Corollary 35(4), for arbitrary large s there are simplicial operations on $|K|$ that cause the equations Σ to be satisfied within $1/s$. In other words $\lambda_{|K|}(\Sigma) \leq 1/s$ for arbitrarily large s, i.e. $\lambda_{|K|}(\Sigma) = 0$.

Conversely, if $\lambda_{|K|}(\Sigma) = 0$, then by Corollary 33, we have $\lambda^{\mathrm{simp}}_{|K|}(\Sigma) = \lambda_{|K|}(\Sigma) = 0$. Thus for arbitrarily large s, there exist simplicial operations on $|K|$ that cause the equations Σ to hold within $1/s$. Therefore the output stream of $\mathcal{B}$ will contain $\langle K, M, N, 1, s, \Sigma\rangle$ for some M and N. Therefore the output stream of $\mathcal{F}$, when started with K, will contain the pair (Σ, s). ∎

Corollary 40 *For a fixed finite simplicial complex K, the set of Σ with $\lambda_{|K|}(\Sigma) = 0$ is a Π_2-set.* ∎

8 Filters.

For any metric space Z, we may define the class of theories

$$\mathcal{L}_Z \;=\; \left\{\Sigma^* \,:\, \lambda_Z(\Sigma^*) > 0\right\}.$$

Σ^* is in this class iff there exists real $\varepsilon > 0$ so that any continuous operations on Z violate Σ^* by more than ε at some point of Z^ω. In this context, it is best to work exclusively with deductively closed sets of equations; hence our use of the notation Σ^*. By Lemma 2 of §2.4, if Z is compact, then the class

$\mathcal{L}_Z$ is a topological invariant. Hence one may also regard $\mathcal{L}_Z$ as well-defined for any compact metrizable topological space Z.

From Theorem 1 of §2.3, it is not hard to see that $\mathcal{L}_Z$ is an upward-closed subclass of the class of equational theories, ordered according to interpretability. That is, if Γ^* is interpretable in Σ^*, and if $\Gamma^* \in \mathcal{L}_Z$, then $\Sigma^* \in \mathcal{L}_Z$.

For some special spaces Z, such as $Z = [0, 1]$ we can also show that $\mathcal{L}_Z$ is a filter (i.e. that it is also closed under the meet operation for this ordering). As is well known, the meet of theories Γ and Δ in this context is the product theory $\Gamma \times \Delta$ that was described at the start of §5. For some Z, $\mathcal{L}_Z$ is not a filter, for example for $Z = S^1 \times Y$, as was seen in Theorem 22 of §5.2.

In keeping with the convention of §8, the following theorem will be stated only for deductively closed theories $\Gamma^\star$ and $\Delta^\star$. It is easily proved (e.g. from Condition (ii) at the start of §5) that $(\Gamma \times \Delta)^\star = (\Gamma^\star \times \Delta^\star)^\star$

Theorem 41 *If* $\Gamma^\star, \Delta^\star \in \mathcal{L}_{[0,1]}$, *then* $(\Gamma \times \Delta)^\star \in \mathcal{L}_{[0,1]}$. *Therefore* $\mathcal{L}_{[0,1]}$ *is a filter.*

Proof. For a proof by contradiction, let us suppose that $(\Gamma \times \Delta)^\star \notin \mathcal{L}_{[0,1]}$. Then $\lambda_{[0,1]}((\Gamma^\star \times \Delta^\star)^\star) = 0$, and hence $\lambda_{[0,1]}(\Gamma^\star \times \Delta^\star) = 0$. By Theorem 19 of §5.1, for any real ε with $0 < \varepsilon < 1/16$, we have $\lambda_{[0,1]}(\Gamma^\star) < 4\varepsilon$ or $\lambda_{[0,1]}(\Delta^\star) < 4\varepsilon$. This clearly implies that $\lambda_{[0,1]}(\Gamma^\star) = 0$ or $\lambda_{[0,1]}(\Delta^\star) = 0$. This conclusion contradicts our assumption that both $\lambda_{[0,1]}(\Gamma^\star)$ and $\lambda_{[0,1]}(\Delta^\star)$ are > 0. ▮

For Z product-indecomposable, there is a better-known filter, that of all theories Σ^* that are incompatible with Z. Our filter $\mathcal{L}_Z$ is of course a subset of that one—generally a proper subset—but no further relationship is known at this time. Notice also that the argument for Theorem 41 is quite general: for any space Z, if we have a result like Theorem 19 for Z, then we can prove a result like Theorem 41 for Z. For example Theorem 20 yields that $\mathcal{L}_Y$ is a filter, for Y being the figure-Y space.

Further up-sets may be proposed, for example using δ^- in place of λ.

9 Approximate satisfaction by differentiable operations.

Obviously one could make a whole new list of definitions here; we refrain from this action until it may be warranted by further developments. Instead, we content ourselves with sketching one example.

Recall that we proved in §8.1.1 of [31] that semilattice theory is not C^1-compatible with $\mathbb{R}$. Here we exhibit some C^1 approximants to the theory. For arbitrary real $p > 1$, and for arbitrary reals $a, b > 1$, define

$$\overline{\wedge_p}(a,b) \;=\; \left(a^p \,+\, b^p\right)^{\frac{1}{p}}.$$

It is not hard to verify that each $\overline{\wedge_p}$ is C^1, and

$$\lim_{p\to\infty} \overline{\wedge_p}(a,b) \;=\; a \wedge b,$$

uniformly in a and b. It is also not hard to show that for every bounded interval [c,d], and for every real $\varepsilon > 0$ there exists p such that

$$\left|\, \overline{\wedge_p}(a,b) \,-\, a \wedge b \,\right| < \varepsilon$$

for all a, b in $[c, d]$.

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