R. Hartshorne, **Geometry: Euclid and Beyond** (Undergraduate Texts in Mathematics), xii+526 pages, Springer Verlag, New York – Berlin – Heidelberg, 2000.

This fascinating book grew out of the junior-senior-level course the author taught in recent years on the classical geometries. A delightful combination of modern views with true appreciation of the achievements of the geometers of antiquity gives a unique flavor to the book.

The starting point in Chapter 1 is a detailed discussion of some portions of Euclid's *Elements*. A critical examination of Euclid's proofs and constructions leads up to Hilbert's axioms of incidence, betweenness, and congruence, the parallel postulate, and a few other axioms, which supply a rigorous foundation for Euclid's geometry (Chapter 2). It has to be pointed out that the axiom system for a Euclidean plane does not include any axioms on numbers. This approach has two advantages: it is faithful to the thinking of the ancient Greek and, more importantly, it allows the author to demonstrate how a plane can be coordinatized by a field that is determined by the arithmetic of line segments; it also allows to explore the effect of additional axioms on the geometry to the properties of the associated field (Chapters 3–4). Let me cite a few sentences from the introduction which explain this approach in more detail:

> "[...] the modern reader quickly becomes aware that Euclid does not use numbers in his geometry. [...]
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> The absence of numbers may seem curious to a student educated in an era in which the real numbers are all-powerful, [...]. In fact some modern educators have gone so far as to build the real numbers into the axioms for geometry with the "ruler postulate," which says that to each interval is assigned a real number, its **length**, and that two intervals are congruent if they have the same length. However, this use of the real numbers at the foundational level of geometry is far from the spirit of Euclid.
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> So we may ask, what role do numbers play in the development of geometry? As one approach to this question we can take the modern algebraic structure of a field (which could be the real numbers, for example), and show that the **Cartesian plane** formed of ordered pairs of elements of the field forms a geometry satisfying our axioms. But a deeper investigation shows that the notion of number appears intrinsically in our geometry, since we can define purely geometrically an **arithmetic of line segments**. We will show that (up to congruence) one can add two line segments to get another segment, and one can multiply two segments (once a unit segment has been chosen) to get another segment. These operations satisfy the usual associative, commutative, distributive laws, so that we obtain an **ordered field**, whose positive elements are the congruence equivalence classes of line segments."

This is one of the instances where the book emphasizes the use of modern algebra in understanding classical geometry and solving problems that arise naturally in classical geometry. Another instance occurs in Chapter 5 which discusses Euclid's notion of area and volume. According to a theorem of Bolyai and Gerwien any two rectilineal figures of the same area are equivalent by dissection in an Archimedean Euclidean plane; however, as Dehn showed using an algebraic invariant, the analogous statement is false for volumes. Both theorems are presented in the book with complete proofs.

Chapter 6 on constructibility uses Galois theory (the main results of which are summarized without proofs) to deduce necessary and sufficient conditions for constructibility with ruler and compass, and also with marked ruler and compass.

The topic of Chapter 7 is non-Euclidean geometry. One of the highlights of this chapter is the coordinatization of hyperbolic planes. In analogy with the theorem that every Euclidean plane is isomorphic to the Cartesian plane over its associated field (determined by the arithmetic of line segments), it is proved that every hyperbolic plane is isomorphic to the Poincaré model over its associated field (the field of ends).

The last chapter discusses regular and semiregular polyhedra and their symmetry groups.

The short summary above does not do full justice to the richness of the material covered in this book. The style is captivating, the presentation is elegant, and the proofs — even the proofs of the deepest theorems — are elementary enough to be accessible to undergraduates. There are over 500 illustrations to help the reader follow the text, and over 600 exercises to challenge the reader or stimulate his/her curiosity to explore further. I think that this is a book of rare mathematical and educational quality. I do hope, just as the author does, that "this material will become familiar to every student of mathematics, and in particular to those who will be future teachers".

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