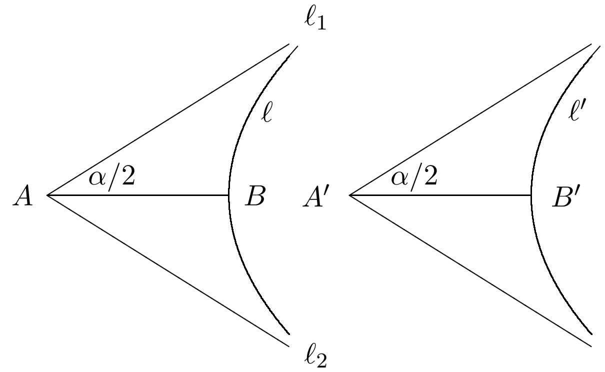
Mathematics 3210
Spring Semester, 2005
Homework notes, part 9
April 22, 2005

40.2 ALL.



We are given congruent angles at A and A'. (We let α denote their common angle.) We take ℓ and ℓ' to be the enclosing lines, respectively, of $\angle A$ and $\angle A'$, and we take B and B' to be the feet, respectively, of A on ℓ and of A' on ℓ'. We wish to show that $AB \cong A'B'$.

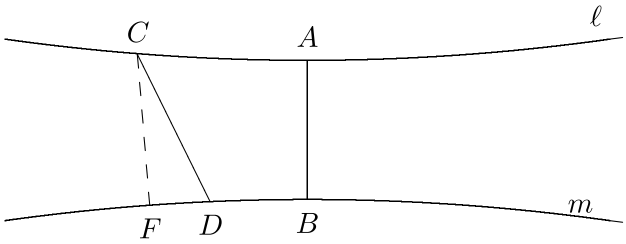
Consider the two limit triangles $\triangle AB\ell_1$ and $\triangle AB\ell_2$, where ℓ_1 and ℓ_2 are the two ends of ℓ. Their respective angles at B are congruent, and they share the side AB; hence by ASL (Exercise 34.10, page 317), their angles $\angle BA\ell_1$ and $\angle BA\ell_2$ are congruent. In other words, $\overrightarrow{AB}$ bisects the angle at A. Similar reasoning shows that $\overrightarrow{A'B'}$ bisects the angle at A'. Thus the four smaller angles in the diagram (two at A and two at B) are all congruent one to another. (Two of them are denoted $\alpha/2$ in the diagram.)

From the diagram we see immediately that

$$\alpha(AB) \;=\; \frac{\alpha}{2} \;=\; \alpha(A'B'),$$

where $\alpha(\cdot)$ denotes the angle of parallelism. It is now immediate from Proposition 40.1(b) that $AB \cong A'B'$.

40.3 Shortest segment between two lines.



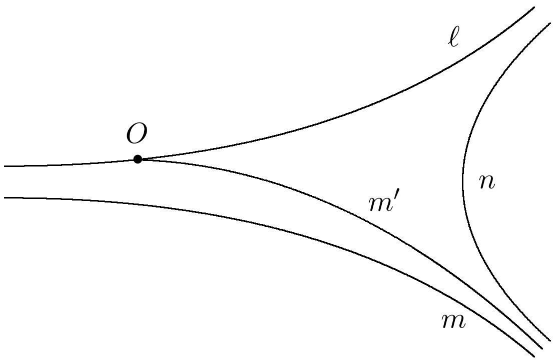
Segment AB is the unique segment perpendicular to both lines ℓ and m, with $A \in \ell$ and $B \in m$. For C any point in ℓ and D any point in m, we need to prove that $AB < CD$. Let F be the foot of the perpendicular from C to m. Now $CFAB$ is a Lambert quadrilateral, and hence $\angle FCA$ is acute, by Exercise[1] 34.2 on page 316.

Finally

$$CD \ > \ CF \ > \ AB,$$

with the first inequality from Euclid[2] I.16 and I.19, and the second inequality from Proposition 34.2, page 307, applied to the quadrilateral $CAFB$.

40.6 Completion of trilimit triangle.



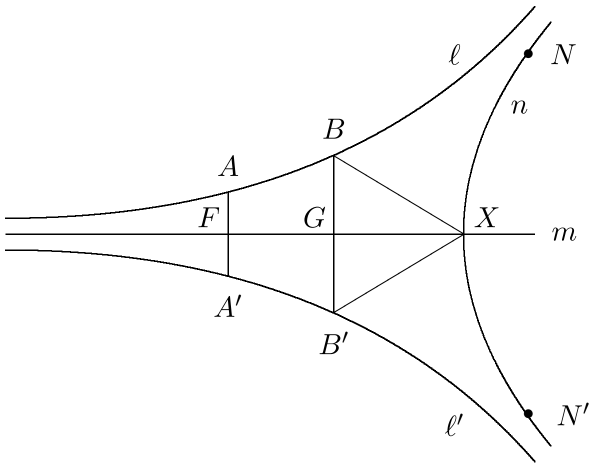
We are given two lines ℓ and m, limit-parallel in a given direction (to the left in the diagram). We need to find n that is limit-parallel both to the other end of ℓ and to the other end of m. Let O be an arbitrary point on ℓ, and let m' be a ray originating at O that is limit-parallel to m in a direction opposite to the given direction on ℓ.

[1]We skipped 34.2, but it is very easy; if interested, one could insert its proof here. If you wish to avoid Lambert quadrilaterals, you could extend $CFAB$ to a Saccheri quadrilateral with midline AB—by adding appropriate points C', F' with $C * A * C'$ and $F * B * F'$.

[2]These propositions of Euclid are confirmed for our context on page 101 of the text. Euclid I.16 is the Exterior Angle Theorem, and Euclid I.19 says that the larger side is opposite the larger angle.

So consider $\angle \ell O m'$. By Corollary 40.6, this angle has an enclosing line n. That is, there exists a line n such that n is limit-parallel to ℓ in one direction and limit-parallel to m' in the opposite direction. By transitivity of limit-parallelism (Proposition 34.11 on page 314), n is limit-parallel to m.

40.5 Parallel–orthogonal.



We are given limit-parallels ℓ and m (with limit parallelism to the left in the diagram), and need to find n that is limit-parallel to (the opposite end of) ℓ and perpendicular to m.

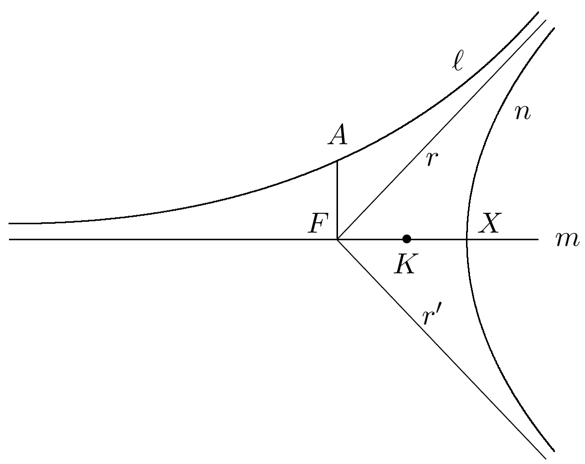
Let A and B be two points on ℓ. Let F and G be the feet of A and B. respectively, on m. There are unique points A' and B' on AF and BG, respectively, such that $A * F * A'$, $B * G * B'$, $AF \cong FA'$ and $BG \cong BG'$. We omit the easy proof (using, e.g., the Exterior Angle Theorem) that $A' \neq B'$. Thus there is a unique line ℓ' joining A' and B'. Now by Exercise 40.6 (established just above) there is a line n that is limit-parallel at one end to ℓ and at its other end to ℓ'. This n has the desired property of being limit-parallel to ℓ. It remains only to prove that n meets m at right angles.

We first note that points on ℓ get arbitrarily far from m, while points on n get closer and closer to m, so some points of n are on the ℓ-side of m. Likewise, some points of n are on the ℓ'-side of m. Therefore n and m intersect at a point P. Moreover n contains some points N that are on the ℓ-side of m and some points N' that are on the ℓ'-side of m; choose one of each for future reference.

If the quadrilaterals $ABFG$ and $A'B'FG$ are divided into triangles (with the segments AG and $A'G$), then a familiar argument—which we omit—will establish that $\angle XBA \cong \angle XB'A'$. Now consider the limit triangle $\triangle BX\ell n$ comprising the segment BX, and the rays of ℓ and n that are limit-parallel, and consider the corresponding limit triangle $\triangle B'X\ell' n$ on the opposite side of m. By the above-mentioned congruences, $\triangle BX\ell n$ and $\triangle B'X\ell' n$ have congruent angles at B and B', and their two segments are congruent: $BX \cong B'X$. Therefore, by ASL (Exercise 34.10), $\angle BXN \cong \angle B'XN'$.

We skip the straightforward proof that B is in the interior of $\angle GXN$ and B' is in the interior of $\angle GXN'$. From this, we may add congruent angles, to obtain $\angle GXN \cong \angle GXN'$. Since these are supplementary angles, each is a right angle.

Alternate proof for Exercise 40.5.



Let A be an arbitrary point on ℓ and F its foot on m. Choose K on m so that $\overrightarrow{FK}$ is opposite to the ray that is limit-parallel to ℓ. Let r be the ray emanating from F that is limit-parallel to ℓ and is not contained in m. The existence of the right angle at F tells us that $\angle KFr$ is acute. There is a unique ray r' on the opposite side of m from r, emanating from F, such that $\angle KFr' \cong \angle KFr$. Clearly $\overrightarrow{FX}$ is interior to $\angle Frr'$, and bisects it.

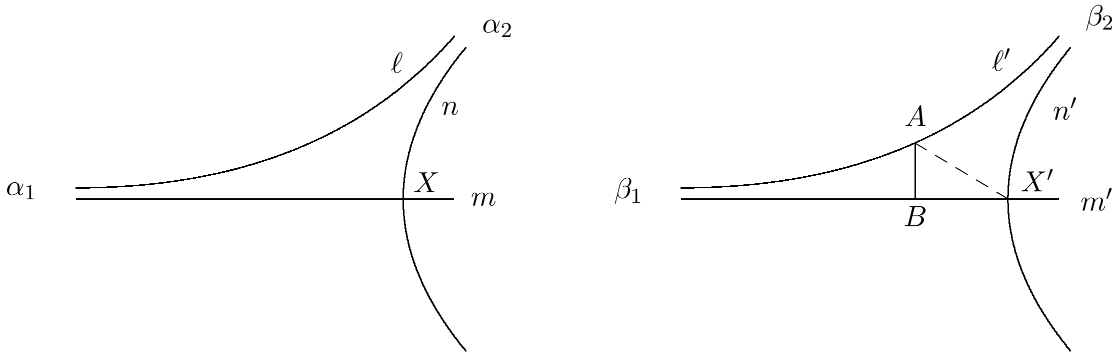
By Proposition 40.6, $\angle Frr'$ has an enclosing line, which we shall denote n. Clearly n is limit-parallel to ℓ; thus we need only show that n meets m at right angles. Clearly n contains points on either side of m, and so m and n intersect at a point X. In Exercise 40.2 we saw that the bisector of an angle meets the enclosing line at right angles. Therefore, m meets n at right angles.

Comments on 40.5.

1. In the alternate proof, you could also use Corollary 40.7 instead of Proposition 40.6.

2. The first proof here is the one that I presented in class on Wednesday 4/20. The alternate proof is one that I discovered on Thursday. Too bad I didn't have it soon enough to present on Wednesday; it is certainly easier. But, mathematics is like that, especially when it is taken as living rather than dead. Some students discovered this without my help!

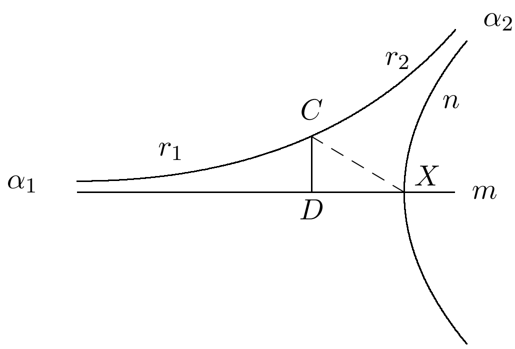
40.7 Distance from ℓ to a limit-parallel.



Given ℓ and m limit-parallel, there is, by Exercise 40.5, a line n orthogonal to m at a point X, and limit-parallel to the opposite end of ℓ. We let α_1, α_2 denote the ends of ℓ that are limit-parallel, respectively, to m and to n. As we know, ℓ is uniquely determined by joining α_1 and α_2; in other words it is uniquely determined as the enclosing line of $\angle\alpha_1 X \alpha_2$.

As for AB, let m' be a line orthogonal to AB at B, and let ℓ' be limit-parallel to m' through A (towards an end that we shall call β_1). A further application of Exercise 40.5 yields a line n' orthogonal to m' at a point X', and limit-parallel to the opposite end of ℓ' (which we shall call β_2).

We now work with a new diagram which shows the three lines ℓ, m and n, but which does not ostensibly show the line ℓ:

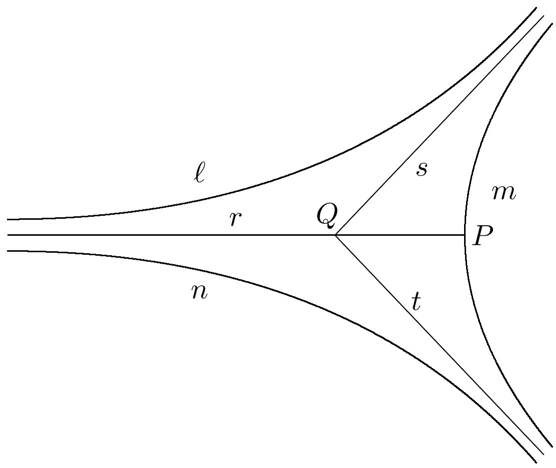


We choose D on line m so that $XD \cong X'B$ and such that $\overrightarrow{XD}$ has end α_1. We then choose C on the ℓ-side of m, so that $\overrightarrow{DC}$ is orthogonal to m at D, and so that $DC \cong BA$. We claim that C and D are the required points on ℓ and m. We certainly have $CD \cong AB$ and we certainly have the right angle at D. It only remains to show that C is on ℓ.

Let $r_1 = \overrightarrow{C\alpha_1}$ and $r_2 = \overrightarrow{C\alpha_2}$. We will show that r_1 and r_2 are collinear. Consider the limit triangles $\triangle\alpha_1 CD$ and $\triangle\beta_1 AB$. They have one pair of angles and one side congruent, and hence by ASL (Exercise 34.10) they also have $\angle\alpha_1 CD \cong \angle\beta_1 AB$. On the other hand, consider the limit triangles $\triangle\alpha_2 CX$ and $\triangle\beta_2 AX'$. From the two congruent triangles $\triangle ABX'$ and $\triangle CDX$, we have $CX \cong AX'$ and $\angle X'A\beta_2 \cong \angle XC\alpha_2$. And by the same congruent triangles, $\angle BAX' \cong \angle DCX$.

Thus the three angles at C are congruent, respectively, to the three angles at A. In the latter case, the three angles lie along the line ℓ', and hence span a straight angle. Hence the same is true of the three angles at C. Thus in fact rays r_1 and r_2 join to form a line joining ends α_1 and α_2. The only such line is ℓ, and hence $C \in \ell$.

40.8(a) Midlines of a trilimit triangle.



By Exercise 40.5, there is a unique line r that is limit-parallel to ℓ and n and that meets m orthogonally; we let P denote the point of intersection of r and m. Let σ_r denote reflection (page 152) in r. Now σ_r fixes P and r, and preserves the right angle at P, so it must map m into itself. Moreover σ_r preserves limit-parallelism, so it must map ℓ to a line that is limit-parallel to $\sigma_r[r] = r$ and to $\sigma_r[m] = m$ (but on the opposite side of r). Thus $\sigma_r[\ell] = n$. Further reasoning of this sort shows that in fact σ_r interchanges the two lines ℓ and n. But, in fact, we already know (from Exercise 34.11) that reflection in the midline interchanges ℓ and n. It is an easy consequence of rigid-motion theory that r *is* the midline of ℓ and n.

Let the other two midlines be s and t as indicated in the diagram. In particular let us consider the midline r and the two half-planes determined by the midline s. Let H_1 be the half-plane containing P, and let H_2 be the opposite half-plane. Clearly all of ℓ lies in H_2. Going out r in the direction of limit-parallelism with ℓ, by Exercise 40.7, one finds points on r arbitrarily close to points on ℓ, and moreover their distance to ℓ decreases as one move further out r, by Exercise 35.8. It is an easy consequence (we omit the details) that r contains points in H_2. Since r contains points in both half-planes H_1 and H_2, r must meet s. In fact this argument is valid for any two of the midlines, and so we may conclude that each pair of midlines has a point of intersection.

Now consider a point Q where two of the midlines meet, say $r \cap s = \{Q\}$. Let F, G and H (not shown in the diagram) be the feet of the perpendiculars from Q to ℓ, m and n, respectively. Since $Q \in r$, we have $QF \cong QH$; since $Q \in s$, we have $QF \cong QG$. By transitivity of congruence, we have $QG \cong QH$. Therefore Q lies on all three midlines, and is equidistant from ℓ, m and n.

Comments on the proof of 40.8(a).

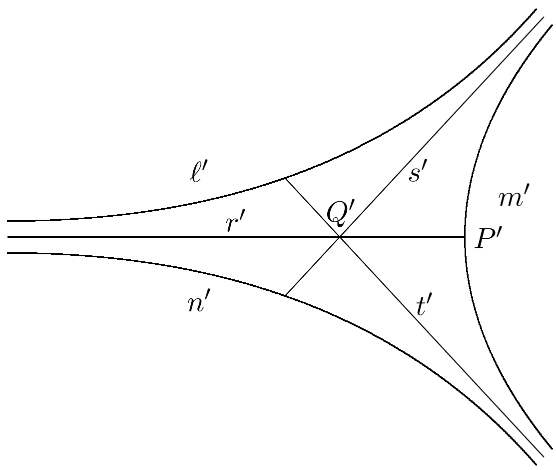
1. Here I freely used the rigid motion σ_r, and properties of σ_r. One can do the problem without rigid motions, but it is harder. So I decided to write up this (and 40.8(b) below) with rigid motions, as an illustration of the importance of this theory.

Rigid motions are defined (on page 149) as motions that preserve collinearity, betweenness, segment congruence and angle congruence. Their existence in a Hilbert plane is established in Proposition 17.4 on page 153. We have used them sporadically and informally, but we simply didn't have time to do the full theory. (Well, we could have traded those Euclidean constructions early in the semester for some work in §17, but those were important also.)

It is of course essential to our argument here that *rigid motions also preserve limit parallelism.* The easiest way for us to see this is with Exercise 34.9. In principle, one can remove rigid motions from any proof. One simply needs to view Proposition 17.4 and Exercise 34.9 as subroutines that could be spelled out as often as necessary in the proof. Thus a replacement proof would contain some triangle congruences and some use of Exercise 34.9.

2. In fact the author could have brought rigid motions into greater prominence. Several things in §§34–40 are easier to work out if one has rigid motions as a tool. As a good example of this, look for a simplification of Exercise 40.7 (just above) with rigid motions.

40.8(b) Congruence of all trilimit triangles.

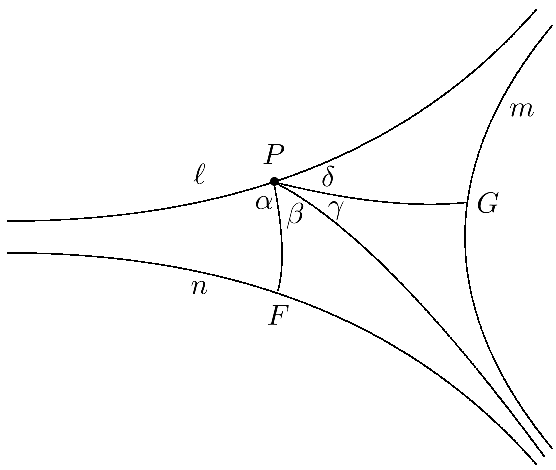


So let us compare the trilimit triangle of 40.8(a) with another trilimit triangle $\triangle \ell' m' n'$, for which we have shown the three midlines, labeled as before except with primes. Our objective is to show that $P'Q' \cong PQ$.

As we saw in Part (a), the reflection $\sigma_{r'}$ fixes r' and Q', interchanges ℓ' and n', and interchanges the two ends of m'. From this it is easy to see that $\sigma_{r'}$ interchanges s' and t'. From this it is immediate that the six angles shown surrounding Q' are congruent in pairs: the two on the left, the two in the center, and the two on the right. (Clearly $\sigma_{r'}$ interchanges each of these pairs.) Similar reasoning applies to $\sigma_{s'}$ and $\sigma_{t'}$, and now transitivity yields that all six angles around Q' are congruent. Thus each is an angle of $60°$.

Thus $\alpha(P'Q') = 60°$, where $\alpha(\cdot)$ denotes the angle of parallelism; clearly the same reasoning applies to the trilimit triangle in Part (a), and so $\alpha(PQ) = 60°$. Now $P'Q' \cong PQ$ by Proposition 40.1(b).

40.8(c) A right angle inside a trilimit triangle.



Let P be an arbitrary point of ℓ, with foot F on n and foot G on m. The figure contains segments PF and PG plus a ray originating at P that is limit-parallel to m and n. The four angles α, β, γ and δ are arrayed about P as indicated in the diagram. Clearly

$$\alpha \;=\; \alpha(PF) \;=\; \beta \qquad \text{and} \qquad \gamma \;=\; \alpha(PG) \;=\; \delta,$$

where $\alpha(\cdot)$ denotes the angle of parallelism. Thus

$$2(\beta + \gamma) \;=\; \alpha + \beta + \gamma + \delta \;=\; 180^\circ$$
$$\beta + \gamma \;=\; 90^\circ.$$

Thus $\angle FPG$ is a right angle.