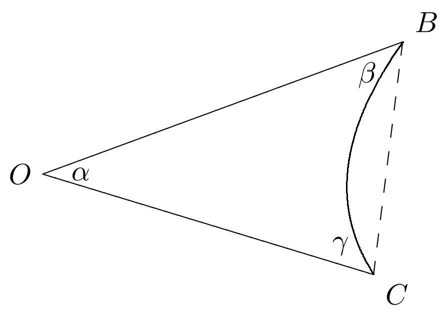
Mathematics 3210
Spring Semester, 2005
Homework notes, part 8
April 15, 2005

The underlying assumption for all problems is that all points, lines, etc., are taken within the Poincaré plane (or Poincaré model). (Therefore we will not repeat that assumption every time.) It is easier to prove things within a particular plane, such as Poincaré's, for an obvious reason. Such a proof will be less general than a proof from axioms (which covers *all* planes of a given type). Therefore, at this stage, it is an open problem for us whether these things can be carried into the axiomatic framework. (They can.)

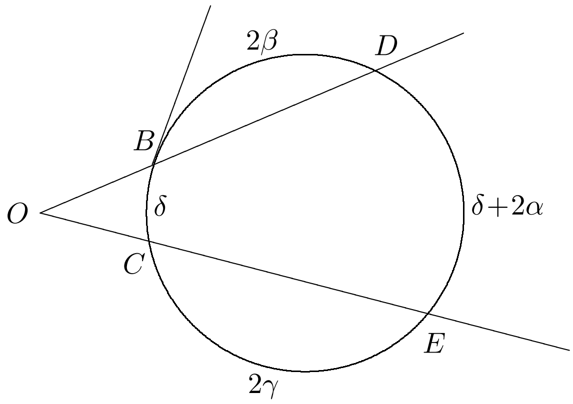
39.2. Angle sum $< 180°$**.** Let α, β, γ be the three angles of a triangle in the Poincaré plane P. There are enough rigid motions to move one of the vertices to the center O (without, of course, changing the three angles). Thus we have



Since the P-line BC, which is to say the arc BC, bends inward from the Euclidean line BC, it is clear that the P-triangle OPC has angle sum less than that of the Euclidean triangle OPC, which is to say, less than 180°. I'll accept this as an answer to Exercise 39.2.

There is however, a more thorough mathematical analysis of the situation, which will also serve as a motivating diagram for Exercise 39.6 below. Looking at the P-line BC as a full circle in the model, we see that β is the angle between a tangent line (depicted) and a chord (part of the line OB).

According to Euclid,



such an angle is half the arc enclosed between the tangent and the chord. (Where an arc is measured by the angle it subtends at the center of its circle.) Thus the measure of arc BD in the diagram is 2β, and we have so labeled it. Similarly, the measure of arc CE is 2γ, and we have so labeled it. We let δ be the measure of arc BC, and provisionally regard it as unknown. Finally, again by Euclid, the angle α at O must be half the difference of arcs DE and BC; easy algebra shows that arc DE must be $2\alpha + \delta$. (See the diagram above, where all arcs are labeled with their measures.)

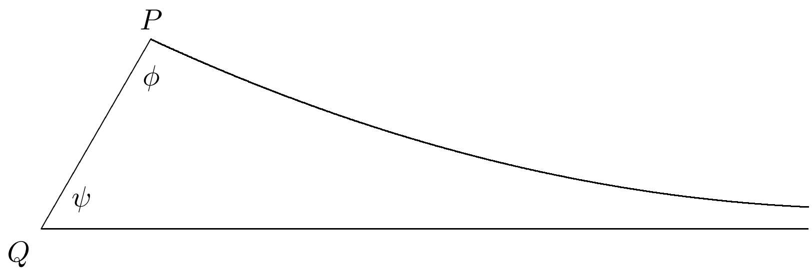
Adding angular measure around the circle, we have

$$\begin{aligned} 360^\circ &= \delta + 2\beta + (\delta + 2\alpha) + 2\gamma \\ &= 2(\alpha + \beta + \gamma + \delta); \\ 180^\circ &= (\alpha + \beta + \gamma) + \delta. \end{aligned}$$

Thus δ is revealed to be the defect of the original triangle. Since $B \neq C$, the arc BC is non-trivial, and hence $\delta > 0$. In other words, the angle sum of the original triangle is $< 180^\circ$.

39.5, The hyperbolic trigonometry of equilateral triangles. We begin with a figure and equation that are apparently not in the book, but are very easily derived from Proposition 39.13 (Bolyai's formula). A proof of this version of Bolyai's formula will be given in class. It is an easy consequence of Proposition 39.13.

Extended version of Bolyai's formula:

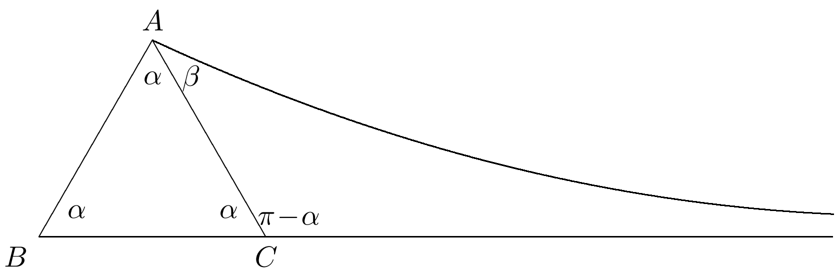


Given two limit-parallel rays making angles ϕ and ψ from line PQ, as indicated in the diagram. Then

$$\tan\frac{\phi}{2}\ \tan\frac{\psi}{2} \;=\; \mu(PQ)^{-1}. \tag{1}$$

We now turn to our equilateral triangle $\triangle ABC$, with side AB and with angle α at each vertex. Consider also the ray that is limit-parallel to $\overrightarrow{BC}$ at A; let it make an angle β with $\overrightarrow{AC}$.

(2)



We apply Equation (1) twice: once to segment AB and once to segment AC. On the left, we have[1]

$$\tan\frac{\alpha+\beta}{2}\ \tan\frac{\alpha}{2} \;=\; \mu(AB)^{-1} \;=\; \frac{1}{a}, \tag{3}$$

where a is introduced (following Hartshorne) simply as an abbreviation for $\mu(AB)$. And on the right, since $AC \cong AB$, we have

$$\tan\frac{\beta}{2}\ \tan\frac{(\pi-\alpha)}{2} \;=\; \frac{1}{a}, \qquad \text{or}$$

$$\tan\frac{\beta}{2} \;=\; \frac{1}{a}\ \tan\frac{\alpha}{2}. \tag{4}$$

[1] Discussing this in class on Wednesday 4/13, I mistakenly had $a = \mu(AB)^{-1}$ in place of the book's value $a = \mu(AB)$. In fact, the calculation can be completed that way, since the answer is symmetric in a and $1/a$. I didn't mark this on student papers. Just be aware that what is here is more correct.

Now let us introduce the further abbreviations $t = \tan(\alpha/2)$ and $b = \tan(\beta/2)$. In addition, we use the addition formula for tangents, which says

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{b+t}{1-bt}.$$

So now, Equations (3) and (4) say

$$t \cdot \frac{b+t}{1-bt} = \frac{1}{a}; \qquad b = \frac{1}{a}\,t\,.$$

It is now an easy matter to eliminate b (which is not wanted in the final answer):

$$t \cdot \frac{\frac{1}{a}t + t}{1 - \frac{1}{a}t^2} = \frac{1}{a};$$

$$t \cdot \frac{t + a\,t}{a - t^2} = \frac{1}{a};$$

$$at^2 + a^2\,t^2 = a - t^2;$$

$$t^2(1 + a + a^2) = a;$$

$$t^2 = \frac{a}{1 + a + a^2}; \tag{5}$$

$$\frac{t^2}{1 - t^2} = \frac{a}{1 + a^2}. \tag{6}$$

Comments on Exercise 39.5. 1. There is some cause to wonder what derivation Hartshorne intended for this exercise. Equation (1) does not seem to appear in the text, and so presumably there is another way to do it. (Perhaps involving Euclidean co-ordinates taken directly from the Poincaré model?) It's worth noting that Hartshorne revisits Equation (6) in Example 42.3.2 on page 407. (Here using sophisticated trigonometric methods, which nevertheless are very similar to calculations in the Poincaré model.)

2. Equation (6) is the answer solicited by the textbook. In some ways Equation (5) might be thought of as preferable, since it yields a more direct functional relationship between a and t. In any case, the two equations are easily interchangeable.

3. In Exercise 34.4 (March 18) we proved AAA for semi-hyperbolic geometry. Thus we already knew that two equilateral triangles with the same angle α have the same side-length. In other words taking $t = \tan(\alpha/2)$ and a the multiplicative side-length, as above, we already knew that t determines

a. So the progress made by this Exercise is the discovery of a *quantified relationship* between a and t. Consider equation (5): First note that we are talking only about $a > 1$. One easily checks, by taking derivatives, that for $a > 1$ the right-hand side of (5) is monotone down in a. Therefore it is one-to-one for a in this range. Therefore (5) uniquely determines a from t^2, and hence of course, from t.

4. What if we imagine the Poincaré plane as a model of the world we live in? What sorts of numbers are involved? We may represent[2] $a = \mu(AB)$ as $e^{d/L}$, where d is ordinary length, and L is a constant length. How large is L?

Suppose $\triangle ABC$ is an equilateral P-triangle with side L. Then $a = \mu(AB)^{-1} = e^{L/L} = e = 2.718\ldots$. Equation (5) then yields

$$t^2 \;=\; \frac{e}{1\,+\,e\,+\,e^2} \;=\; 0.244728\ldots\;,$$

so

$$\tan(\alpha/2) \;=\; t \;=\; 0.494700\ldots\;,$$

from which we obtain

$$\frac{\alpha}{2} \;=\; 0.459398\ldots\text{ radians} \;=\; 26.3216\ldots^\circ; \qquad \alpha \;=\; 52.64\ldots^\circ\;.$$

In short, L is the side of an equilateral triangle with angles less than 53°, and thus a defect of at least 21°. With presently available instrumentation, triangles as large as Earth's orbit have defect that is smaller than experimental error. Hence L must be astronomically large to accomodate a triangle with 21° of defect!

5. It's worth thinking about how Equation (5) relates to large and small values of the multiplicative distance a.

In our world, all lengths are small by hyperbolic standards. As we noted in Comment 4,

$$a \;=\; \mu(AB) \;=\; e^{d/L},$$

where d is ordinary distance between A and B, and L is a constant length, of astronomical size. So as far as formula (5) is concerned, $a \approx e^0 = 1$. In this case, Equation (5) yields

$$t^2 \;\approx\; \frac{1}{1+1+1} \;=\; \frac{1}{3},$$

or $\tan(\alpha/2) = t \approx 1/\sqrt{3}$. Therefore we know, from elementary trigonometry, that $\alpha/2 \approx 30^\circ$, or $\alpha \approx 60^\circ$. In other words, small equilateral triangles in the Poincaré model behave like equilateral triangles in the Euclidean world. (We already knew that small triangles have small defect, so we really already knew that small equilateral triangles have $\alpha \approx 60^\circ$; but it is worthwhile to see this result confirmed from another direction.)

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[2]This representation may be found on pages 402–403 of Hartshorne. The exponential is found in Exercise 41.14, and the extra constant (analogous to our L) is introduced in Exercise 41.15. Hartshorne delays introducing this representation, because fields (the number systems most relevant to geometry) are not necessarily closed under taking logarithms.

On the other hand, if we consider equilateral P-triangles with multiplicative side-length a *large*, then Equation (5) yields

$$t^2 \;=\; \frac{a}{a^2} \;\approx\; \frac{1}{a}.$$

In other words, for large a, and for α measured in radians, we have

$$\alpha/2 \;\approx\; \tan(\alpha/2) \;\approx\; 1/\sqrt{a},$$

so $\alpha \to 0$ as $a \to \infty$.

6. Suppose we replace the equilateral triangle with an arbitrary isosceles triangle, with base angle α appearing at B and C, and vertex angle γ at A. To study this isosceles triangle is to study the arbitrary right triangle $\triangle ABM$, where M is the midpoint of BC.

One may begin with a revision of the diagram in (2)—revised only to make the top angle γ. Now it is clear that versions of Equations (3–4) hold in this context. They are not quite as simple, since the new angle γ is now in the picture; nevertheless an analysis in the style of Exercise 39.5 is possible in this situation, leading to hyperbolic trigonometric formulas for an arbitrary right triangle.

7. We wish to point out that our treatment of Exercise 39.5 does not really rely on the Poincaré model. It's true, we did use Bolyai's formula, which is presented in Proposition 39.13 as a feature of the Poincaré model. The formula, however, is reproved, from the axioms, as Proposition 41.9 on page 396. Thus our treatment of Exercise 39.5 can be taken as coming straight from the axioms, and not relying in any special way on this model.

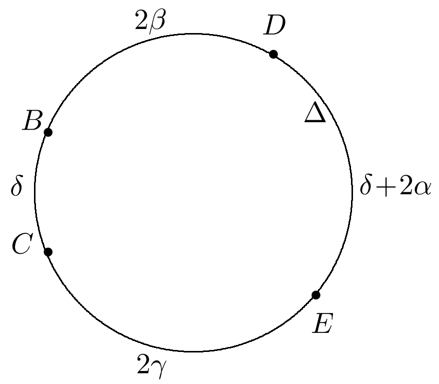
39.6. Constructing an arbitrary triangle of positive defect. We are given angles α, β, γ with $\alpha + \beta + \gamma < 180°$, and wish to construct a P-triangle with these three angles. First, we define

$$\delta \;=\; 180° - (\alpha + \beta + \gamma) \;>\; 0. \tag{7}$$

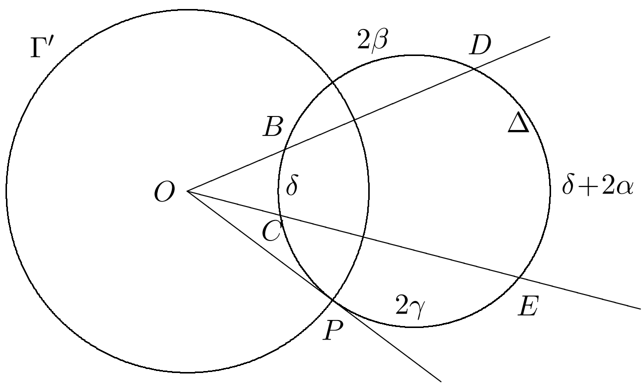
From (7) it follows that

$$\delta \;+\; 2\beta \;+\; (\delta \;+\; 2\alpha) \;+\; 2\gamma \;=\; 360°.$$

So we take a (Euclidean) circle Δ and points B, D, E, C that divide Δ into four arcs of lengths δ, 2β, $\delta + 2\alpha$, and 2γ, as indicated in this diagram:



Since arc DE is larger than arc BC, the lines BD and CE intersect (in Euclidean geometry!) at a point O outside Δ. Let P be a point on Δ such that OP is tangent to Δ, and let Γ' be the circle with center O and radius OP:

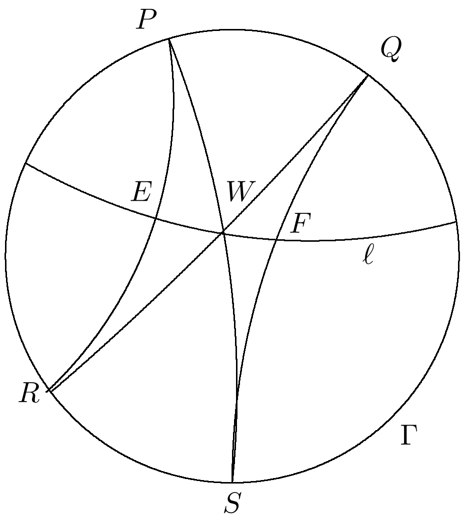


Finally, we do one more thing. The radius of Γ' (which was taken to be OP) may not be the same as that of our standard circle Γ that serves as the boundary of the Poincaré disk. Therefore, we simply rescale our figure to the radius of Γ. (For obvious reasons, we do not bother to depict this rescaling.) Now all our constructs are actually in the Poincaré disk. In particular the Euclidean lines OB and OC are P-lines, and the circle Δ (more properly, the part of Δ that is interior to Γ) is a P-line.

By the same reasoning that we used for Exercise 39.2 above (involving the angle between a chord and a tangent), the angle between OD and the Δ-tangent at B is β. In other words, the **P-angle** $\angle OBC$ is β. By similar reasoning, the other two P-angles of the **P-triangle** $\triangle OBC$ are γ and α.

Comment on Exercise 39.6 The most prevalent error on this homework set was the failure to recognize that you have to have the circle Δ and its four points B, D, E, C, with the correct angular spacing, *before* you can have the point O and the circles Γ', Γ centered at O. This is because O is obtained as the intersection of (Euclidean) lines BD and EC. To my knowledge, this is the only simple way to make it work; if you have an essentially different method that is correct, please let me know.

39.8. Unique common orthogonal line.



Let us be given P-lines PR and US, where P, Q, R, S all belong to Γ. From the assumption that PR and QS are not limit-parallel, we known that the four points are distinct. From the assumption that PR and QS do not meet, we know that one of the two Γ-arcs joining P and R contains neither Q nor S. So a path starting at R and tracing out Γ in the direction toward P encounters the four points either in the order $RPQS$ or in the order $RPSQ$. Without loss of generality, we may assume that the order is $RPQS$, as we have depicted here.

Thus P and S lie on opposite sides of RQ, which means that the P-lines PS and RQ meet at a point W interior to Γ. Let ℓ be the P-line that bisects the P-angle $\angle RWP$. (Angle bisectors exist in all Hilbert planes; therefore they exist in the Poincaré plane.) We claim that ℓ is the desired P-line meeting both PR and QS at right angles.

Since $\overrightarrow{EP}$ is limit-parallel to $\overrightarrow{WP}$, and since $\overrightarrow{ER}$ is limit-parallel to $\overrightarrow{WR}$, one ray of ℓ at W intersects RP, at a point E. Similarly, the opposite ray of ℓ meets QS at a point F.

Now consider the limit triangles $\triangle WEP$ and $\triangle WER$. They have congruent angles at E (since we took ℓ to be the angle bisector), and they have the side WE in common. Hence by ASL (Exercise 34.10, page 317; homework for 3/18/05), $\triangle WEP \cong \triangle WER$. Hence the two angles at E

are supplementary congruent angles, and hence right angles. Similar reasoning shows that there are right angles at F. Thus ℓ is the desired common perpendicular.

By Exercise 39.2 above, the Poincaré plane is semi-hyperbolic. Hence it has no rectangles. On the other hand, if there were *two* lines perpendicular to PR and QS, then we would have a rectangle. This contradiction shows that the common perpendicular is unique.

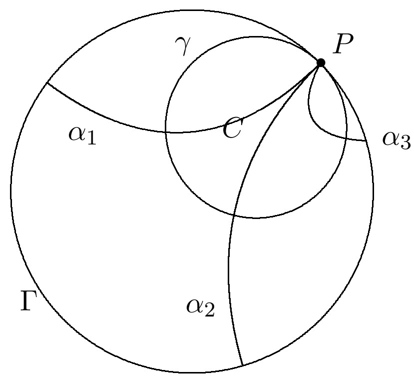
Comments on Exercise 39.8. 1. The existence of a common perpendicular will be proved again as Theorem 40.5 on page 377. Obviously such a proof, based only on the axioms of hyperbolic geometry, is more general, since it covers any hyperbolic plane, not just the Poincaré plane. Therefore, in a sense, the proof in Exercise 39.8 is redundant.

On the other hand, this proof illustrates how models can be useful in suggesting theorems to be proved in the axiomatic system. It also illustrates how the Poincaré model can serve as a reservoir of intuition for the subject of non-Euclidean geometry: the proof here is very intuitive and perspicuous, compared with that of 40.5 below.

2. Alternate finish to the proof. Construct W and ℓ as written above, and pick up the proof from there. Let us apply a rigid motion that brings W to the center of Γ. (Since rigid motions preserve angles, it will be enough to show that the angles at E and F are right angles in the transformed picture.) *. . . perhaps supply a diagram after the rigid motion . . .* . Now ℓ, PS and QR are ordinary Euclidean lines. In fact QR and PS are tangents to the Euclidean circular arc PER through W, and ℓ is their angle bisector. Therefore $WE = \ell$ goes through the center of Euclidean circle PER. Therefore the angle at E is a right angle. Similar reasoning holds for the angle at F.

I chose to write it up with Exercise 34.10, in hopes of illustrating that 34.10 (ASL) has some realm of application, but you may find this alternate approach easier. (It's good to understand them both.)

39.14(a). The horocycle.



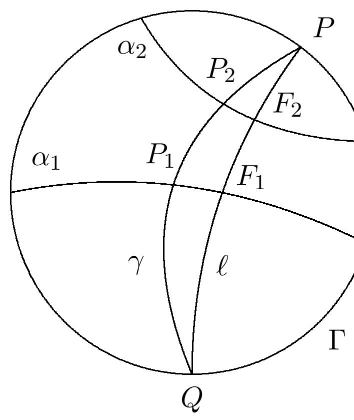
In the Poincaré plane defined by Euclidean circle Γ, we are given a Euclidean circle γ that is tangent to Γ at P. Let $\mathcal{F}$ be the family of all P-lines α that are limit-parallel to one another at the end P. We need to prove that in the Poincaré plane, γ is a curve that meets each $\alpha \in \mathcal{F}$ at right angles.

From the Euclidean point of view each $\alpha \in \mathcal{F}$ is simply a circle meeting Γ at right angles at P. (Three such circles, α_1, α_2 and α_3, are illustrated, although in fact obviously $\mathcal{F}$ is an infinite family of circles.) Still reasoning with Euclidean geometry, we consider any circle $\alpha \in \mathcal{F}$. It meets Γ at P, at right angles. But γ is tangent to Γ at P; hence α also meets γ at right angles at P. If two circles meet in one point at right angles, then their other meeting is also at right angles. Therefore α and γ intersect at right angles at a point of the P-model. Since the P-model is conformal, this means that the curve γ intersects the P-line α at right angles in the model. This is what we sought to prove.

Comment on Exercise 39.14(a). So from the purely hyperbolic point of view, γ is an interesting curve. It is not a P-line. No, it has a different description. There is a family $\mathcal{F}$ of P-lines that are limit-parallel all in the same direction. The curve γ is determined by the condition that it meets all these P-lines at right angles. (Thus it can't possibly be a P-line: in hyperbolic geometry, no two limit-parallels meet a single line at right angles.)

Such a curve is known as a *horocycle*. This is a kind of curve that does not exist (except trivially) in Euclidean geometry. It's not that horocycles are too exotic for us to have studied before, or have too complex an equation—it's that the concept is redundant in Euclidean geometry. In Euclidean geometry, $\mathcal{F}$ would simply be a family of parallels, and γ would simply be a line that is a common orthogonal transversal to $\mathcal{F}$.

39.14(b) The hypercycle.



In the Poincaré plane defined by Euclidean circle Γ, we are given a Euclidean circular arc γ that is neither tangent nor perpendicular to Γ, and that meets Γ at P and Q. We also consider the P-line ℓ that has P and Q as ends. Let $\mathcal{F}$ be the family of all P-lines α that meet the P-line ℓ at right angles. (Two such P-lines, labeled α_1 and α_2, are illustrated, although of course there are infinitely many $\alpha \in \mathcal{F}$.) We need to prove that in the Poincaré plane, γ is a curve that meets each $\alpha \in \mathcal{F}$ at right angles.

So, let α be a P-line in $\mathcal{F}$. (In the diagram, one might take $\alpha = \alpha_1$ or $\alpha = \alpha_2$—take your pick—although in fact α can be any P-line orthogonal to ℓ.) Let P be the point where α meets ℓ at right angles. Consider ρ_α, (Euclidean) inversion in the circle α. We know (by Proposition 37.5) that ρ_α is conformal, i.e. preserves angles, and we are given that ℓ meets α orthogonally. Thus $\rho_\alpha[\ell]$ meets $\rho_\alpha[\alpha]$ orthogonally. We also know that $\rho_\alpha[\alpha] = P$ and that $\rho_\alpha(P) = \alpha$; thus we can now say that $\rho_\alpha[\ell]$ meets α orthogonally at P. By Axiom C4, there is only one line meeting α orthogonally at P, namely ℓ. Therefore $\rho_\alpha[\ell] = \ell$. Moreover, since ρ_α interchanges the two sides of α, we have $\rho_\alpha(P) = Q$ and $\rho_\alpha(Q) = P$.

Now, what about $\rho_\alpha[\gamma]$? By Proposition 37.4, it is a circular arc, and this arc has P and Q as endpoints, by the calculations in the previous paragraph. By definition of circular inversion, γ and $\rho_\alpha[\gamma]$ lie on the same side of ℓ. Finally, since ρ_α preserves angles, the angle between $\rho_\alpha[\gamma]$ and $\rho_\alpha[\ell] = \ell$ is the same as the angle between γ and ℓ. There is only one arc starting at P, in the same halfplane as γ, and having the same angle with ℓ as γ; namely, the only such arc is γ itself. Therefore $\rho_\alpha[\gamma] = \gamma$.

Now by Proposition 37.3, the angles where γ meets α are congruent; since they are supplementary, they are right angles; in ohter words, γ is orthogonal to α. To recapitulate, we have proved that γ is orthogonal to every $\alpha \in \mathcal{F}$; that is to say, γ is orthogonal to every P-line α that is perpendicular to the given P-line ℓ.

Let us also prove that all points on γ are equidistant from ℓ. That is, given $P_1, P_2 \in \gamma$, we let F_1, F_2 be their respective feet on ℓ, and prove that $P_1F_1 \cong P_2F_2$. Let α (not shown in the diagram) be the perpendicular bisector of the P-segment F_1F_2. (I.e. α is a P-line meeting the P-line F_1F_2 at right angles at a point Q such that F_1Q is P-congruent to F_2Q. Since ρ_α preserves P-angles and P-congruence, it is clear that $\rho_\alpha(F_1) = F_2$ and $\rho_\alpha(P_1) = P_2$. Therefore $P_1F_1 \cong P_2F_2$.

Comments on 39.14(b).

1. So from the purely hyperbolic point of view, γ is an interesting curve. It is not a P-line. There is a family $\mathcal{F}$ of P-lines that are all perpendicular to a given line ℓ. The curve γ is determined by the condition that it meets all these P-lines at right angles. (Thus γ can't possibly be a P-line in hyperbolic geometry: if it were, we would obviously have a rectangle.)

γ may also be described hyperbolically as the locus of points that have a fixed distance from ℓ on a given side of ℓ.

Such a curve is known as a *hypercycle* or *equidistant curve*. As with the horocycle, there is no non-trivial instance of a hypercycle in Euclidean geometry, and hence it is not familiar to us.

2. There is one special case not yet discussed. The Euclidean segment PQ (in the diagram for 39.14(b)) is also a hypercycle for the Poincaré model Γ. (One easily proves, as above for the arc γ, that if α is a P-line orthogonal to the P-line PQ, then ρ_α maps the Euclidean segment PQ into itself (interchanging P and Q).

3. We first met equidistant curves in Exercise 33.7 (discussion of Clavius' Axiom). There it was seen that if an equidistant curve is a line, and if Archimedes' Axiom holds, then (P) holds, i.e. the geometry is Euclidean. Therefore, in our present context, no equidistant curve is a line.

4. Suppose that a man takes a walk in the hyperbolic world, always walking in a straight line. He is accompanied by his dog, who always walks a distance d to his left. The man is walking along a line. The dog is walking along an equidistant curve. The dog will experience his path as deviating from a line — always veering toward the right.

5. The figure for this exercise can help one realize how severely the Poincaré model distorts distances. Just realize that (more or less horizontal in this view) distance between the points on the two arcs joining P and Q is a constant in the picture, even though it appears to us to approach zero.