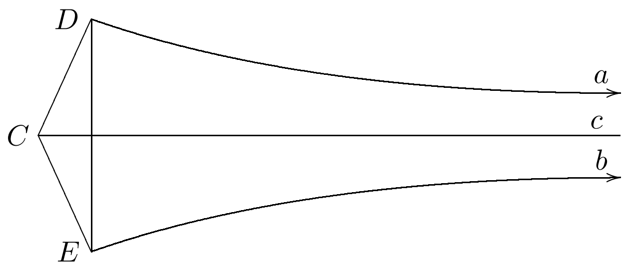
Mathematics 3210
Spring Semester, 2005
Homework notes, part 7
April 1, 2005

Note for today. Sometimes one's work can be expedited by taking time out to make a fresh diagram, leaving behind the constructs (points and lines, say) that have already served their purpose. We do this twice in our discussion of 34.11 just below.

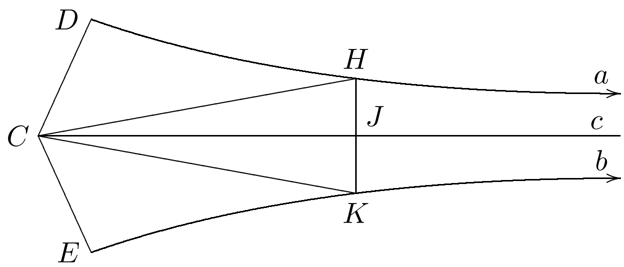
34.11 Bisector of a pair of limiting parallel rays.

Working with the book's diagram, one should begin by stating why the point C exists. ... *do this* ... . Then let G *(not shown here)* be the foot of the perpendicular from C to AB. Certain congruent triangles *(not shown here)* immediately imply that $CE \cong CG \cong CD$ ... *supply the details* ... . We no longer require points A, B, G, so they are not included in this diagram:



Here Cc is an altitude of the isosceles triangle $\triangle CDE$; hence the two angles at C are congruent to each other.

We now need only remember that $\angle DCc \cong \angle ECc$, that the angles at D and E are right angles, and that $CD \cong CE$. We no longer require the segment DE, so now we work with this diagram:

(1)


We begin by considering a point H on $\overrightarrow{Da}$, and a point K on $\overrightarrow{Eb}$ such that $DH \cong EK$. From these simple conditions, it is evident that

(2) $$\triangle CDH \cong \triangle CEK,$$

by SAS (the angles being the right angles at D and E).

Let us now prove, by contradiction, that $H \notin \overrightarrow{Cc}$ and $K \notin \overrightarrow{Cc}$. Without loss of generality, suppose that $H \in \overrightarrow{Cc}$. (This contradictory situation is not depicted.) Thus $\overrightarrow{CH} = \overrightarrow{Cc}$, and so, by (2),

$$\angle ECK \;\cong\; \angle DCH \;=\; \angle DCc \;\cong\; \angle ECc.$$

Thus $\overrightarrow{CK} = \overrightarrow{Cc}$, by the uniqueness part of Axiom C4. Moreover $CK \cong CH$ by (2), and hence $K = H$ by the uniqueness part of Axiom C1. Thus rays $\overrightarrow{Da}$ and $\overrightarrow{Eb}$ have a point in common, in contradiction to our assumption that lines Da and Eb are parallel. This contradiction establishes our claim that $H \notin \overrightarrow{Cc}$ and $K \notin \overrightarrow{Cc}$.

Therefore, *all points of* $\overrightarrow{Da}$ *lie on one side of line* Cc*, and all point of* $\overrightarrow{Eb}$ *lie on the opposite side.*

We now attend to the two specific parts of Exercise 34.11:

34.11(a) — $\overrightarrow{Da}$ and $\overrightarrow{Cc}$ are limit parallels. We just proved that the two rays do not intersect. For limit parallelism, we consider a ray $\overrightarrow{DX}$ that is interior to $\angle EDa$. We are given that $\overrightarrow{Da}$ and $\overrightarrow{Eb}$ are limit-parallel, and hence $\overrightarrow{DX}$ meets $\overrightarrow{Eb}$ at a point Y. By the above results, D and Y are on opposite sides of Cc. Therefore $\overrightarrow{DX}$ meets Cc. This is exactly what is required for limit parallelism

34.11(a) — alternate proof. For limit parallelism, one could instead use Remark 34.12.1, which says that if a line stays completely within $\angle A$ and within $\angle B$, then it too is limiting parallel. (See book for diagram and precise statement.) What one needs to apply 34.12.1 is our statement, above, that $\overrightarrow{Da}$ and $\overrightarrow{Eb}$ lie on the opposite sides of Cc.

34.11(b) — $\overrightarrow{Eb}$ is the reflection of $\overrightarrow{Da}$ in Cc. We return to the situation depicted in Diagram (1), and in particular to an arbitrary pair of points H and K with $DH \cong EK$.

We also have $DC \cong EC$ and $\angle D \cong \angle E$, from (2). Thus $\angle DCH \cong \angle ECK$ and $CH \cong CK$. By angle subtraction, $\angle HCJ \cong \angle KCJ$. Thus $\triangle HCJ \cong \triangle KCJ$, by SAS. Thus $HJ \cong KJ$, and the supplementary angles at J are congruent to each other—and thus are right angles.

In other words, the segment $\overline{HK}$ has line Cc as its perpendicular bisector. Thus, by definition, K is the reflection of H in Cc (and vice versa). In other words, we have shown that, starting with an arbitrary point $H \in \overrightarrow{Da}$, its Cc-reflection is a certain point $K \in \overrightarrow{Eb}$. And of course, we obviously have the same congruences if we begin with $K \in \overrightarrow{Eb}$ and then take $H \in \overrightarrow{Da}$ to satisfy $DH \cong EK$. Therefore, of the two rays $\overrightarrow{Da}$ and $\overrightarrow{Eb}$, each is the reflection of the other in Cc.

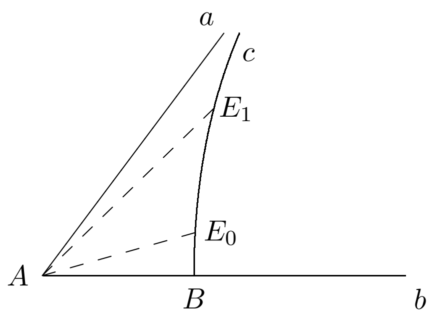
Comments on Exercise 34.11. 1. In our proof of 34.11(b), we hardly used the hypothesis that Da is limiting parallel to Eb. (We did however use it somewhat covertly; can you tell where?) This of course raises the question whether it's really necessary. Indeed, it is not! In fact for any two lines m and n, there exists a line ℓ such that m is the reflection of n in ℓ. (We have proved this for m limit-parallel to n, and it is very easy to prove if m intersects n. The remaining case—neither intersecting nor limit parallel—is nicely covered by Theorem 40.5 on page 377.)

2. Exercise 34.11 concerns the concurrence of two angle bisectors and a so-called midline. For the concurrence of three midlines, see Exercise 40.8 on page 385—which will be assigned for April 22. Obviously one could also discuss the concurrence of one angle bisector and two midlines, although I didn't happen to find this stated as an exercise.

3. The part about right angles at J (see Part (b) above) is essential. Neglecting to prove this was the most prevalent mistake on this week's homework.

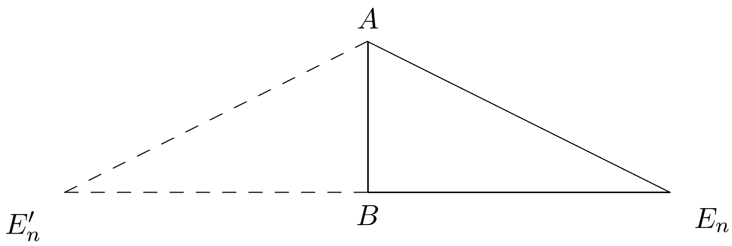
4. For Part (a) many forgot to prove the "limit" part—any ray at a sharper angle will meet the other line. Remember, on the last homework writeup I said, "Never forget . . . ."

35.6 Three small angles in a triangle.



We begin with the diagram from Exercise 35.3 (of the previous set). In that exercise it was proved that if we begin with any angle at A, we can find B on $\overrightarrow{Ab}$ so that Bc is parallel to Aa and the angle at B is a right angle. So, we apply that exercise to find such a figure with the angle at A given as $\frac{1}{2}\varepsilon$.

We now apply the recursive procedure that is outlined in the proof of Theorem 35.4 (diagram top of page 322). Briefly, E_0 is any point on $\overrightarrow{Bc}$, and E_1 is chosen so that $E_0E_1 \cong AE_0$. Then recursively, E_{n+1} is chosen so that $E_nE_{n+1} \cong AE_n$. As described in the proof of Theorem 35.4, the angles at the points E_n continue to decrease at least by a factor of 2. In the presence of Archimedes' Axiom (one of the assumptions for Exercise 35.6), we may conclude that $\angle AE_nB < \varepsilon$ for some n. Moreover, as confirmed in the proof of Theorem 35.4, we have $\angle BAE_n < \frac{1}{2}\varepsilon$.



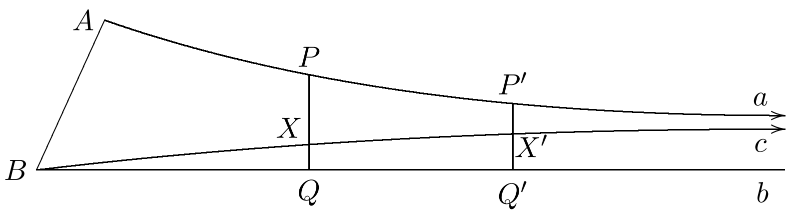
For the final phase, we use Axiom C1 for the existence of E'_n with $E_n * B * E'_n$ and with $BE'_n \cong BE_n$. By SAS, $\triangle ABE_n \cong \triangle ABE'_n$. It is now immediate that all three angles of $\triangle AE_nE'_n$ are less than ε.

Comments on Exercise 35.6.

1. To get small angles requires a huge triangle. To get the angle small at A requires proposition 35.5, whose proof, one may recall, involves essentially doubling $\triangle ABC$ until its defect exceeds 2RA. Then the construction of 35.4 requires the successive construction of points E_n, each one approximately twice as far from B as the previous.

2. On the next assignment (April 15), we will have Exercise 39.6, which asks you to prove that, in the Poincaré model, if α, β and γ are any three angles with sum $< 180°$, then there exists a triangle whose angles are α, β and γ. This is a stronger result in that it replaces inequalities with equalities. It is, however, a weaker result, in that it holds (so far) for *the Poincaré model only.* (Ultimately, it can be proved from the axioms of hyperbolic geometry.)

35.9 If distances decrease, the lines are limit-parallel.



We are given rays $\overrightarrow{Aa}$ and $\overrightarrow{Bb}$ that lie on the same side of AB, and do not meet. We assume that the perpendicular distance from $P \in \overrightarrow{Aa}$ decreases as P moves along the ray away from A. In other words,

$$\text{(3)} \qquad \text{If } P, P' \in \overrightarrow{Aa} \text{ with } A * P * P', \text{ then } P'Q' < PQ,$$

where Q and Q' are the feet, respectively, of P and P' on Bb. We claim that (3) implies that $\overrightarrow{Aa}$ is limiting parallel to $\overrightarrow{Bb}$.

For a proof, we assume that the two rays are not limit-parallel, and work toward a contradiction to (3). By Proposition 34.10 (on symmetry of the limiting-parallel relation), we may assume that $\overrightarrow{Bb}$ is not limit-parallel to $\overrightarrow{Aa}$ from B. Therefore the real limit-parallel from B, which we will denote $\overrightarrow{Bc}$, lies interior to $\angle ABQ$. Let P be a point on $\overrightarrow{Aa}$, and let Q be its foot on Bb. We omit the proof that P and Q are on opposite sides of Bc. Hence there is a point X on Bc with $P * X * Q$.

Now, according to Proposition 35.6 (this is the book's hint), the perpendicular distance from points on $\overrightarrow{Bc}$ to Bb can be made as large as we wish. In particular, there exists X' with $B * X * X'$ such that $X'Q' > PQ$ (where Q' is the foot of X' on Bb). (One does not perceive $X'Q' > PQ$ in the diagram, but after all this is a contradictory situation and hence hard to draw.) Now since $\overrightarrow{X'c}$ is limit parallel from X' to Aa, and since the ray opposite to $\overrightarrow{X'Q'}$ lies inside $\angle PX'c$, it must intersect $\overrightarrow{Pc}$. In other words, there exists P' with $A * P * P'$ and $P' * X' * Q'$. Thus $P'Q' > X'Q' > PQ$, and we have a violation of (3). This contradiction complete the proof of the Exercise.

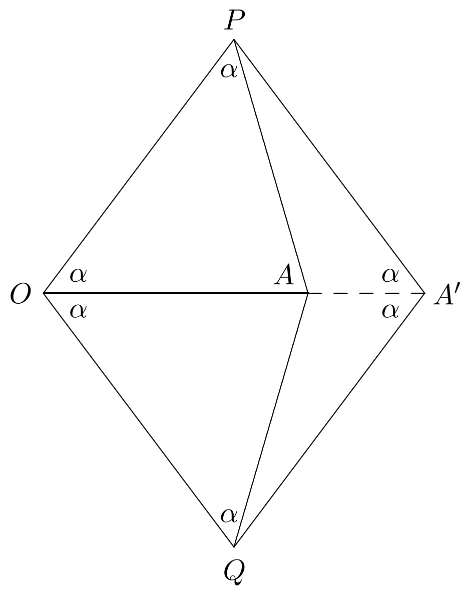
Comments on Exercise 35.9.

1.Theorem 40.5 on page 377 (mentioned already above) will provide another proof of 35.9. It tells us that if the two lines are not limit-parallels, then they have a common perpendicular. Then one may quickly invoke Proposition 34.4 (and its remark 34.4.1) to see that distances do not in fact always decrease.

2. A couple of students tried to prove this using Exercise 35.8 (its converse). It is almost always fruitless to prove a result from its converse. Here's why. Suppose we know (1) $A \Longrightarrow B$, and we wish to prove (2) $B \Longrightarrow A$. First of all, all that (1) can do for us is establish B; but B is what we're assuming in (2), so invoking (1) would be a waste of time. It gets worse. The only way we could invoke (1) is to somehow have A available; but once we have A available, our proof of (2) is complete, so there would be no need to invoke (1).

What *does* sometimes work, sometimes not always, is to take a known *proof* of (1) $A \implies B$, go through its steps, and turn each of those steps around, one by one, into its converse. 35.9 is probably not one of these cases.

37.2 Compass construction of the inverse point. The first step of the construction (a circle with center A) yields $AP \cong AO \cong AQ$. Since $P, Q \in \Gamma$, we have $OP \cong QO$. Finally, the last two circular arcs in the construction yield $OP \cong PA'$ and $A'Q \cong QO$. Thus we have the configuration



with

(4) $$OP \cong PA' \cong A'Q \cong QO \quad \text{and} \quad AP \cong AO \cong AQ.$$

Let us first prove that O, A and A' are collinear. Consider $\triangle POA$ and $\triangle QOA$. They are congruent by (4) and SSS, and thus $\overrightarrow{OA}$ bisects the large angle at O. By similar reasoning (with $\triangle POA'$ and $\triangle QOA'$), the ray $\overrightarrow{OA'}$ also bisects the large angle at O. Therefore $\overrightarrow{OA} = \overrightarrow{OA'}$. (We have not expressly proved that angle bisectors are unique, but we could easily do it now, in at least two different ways. We skip the details, for lack of time. If interested, write up a proof and show it to me.) Thus O, A and A' are collinear. This fact allows us to see that $\angle POA = \angle POA'$, and so on.

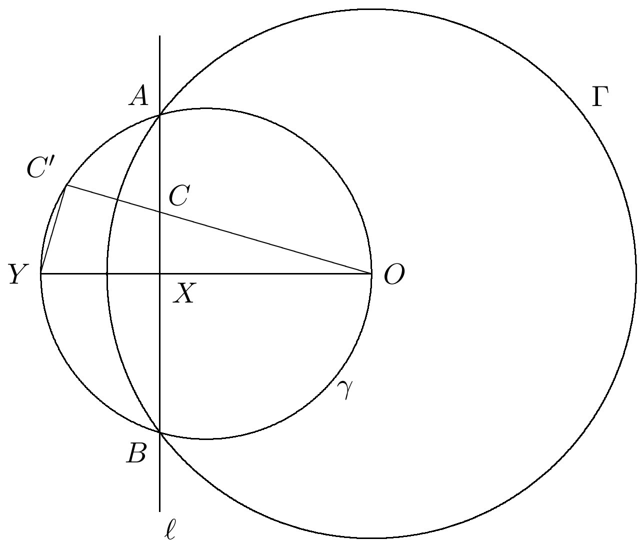
Now consider $\triangle POA$ and $\triangle POA'$. Each is an isosceles triangle, by (4), and they in fact have $\angle POA = \angle POA'$ as a base angle in common. Therefore all the base angles are congruent, as noted in the figure with α. (In fact, α also occurs three times in the lower half of the diagram, in accordance with the two SSS congruences that we mentioned above. We noted these three $\alpha's$ in the lower part of the diagram, but we don't need them.)

Now two isosceles triangles with the same base angle are similar in Euclidean geometry, and that certainly applies to $\triangle POA$ and $\triangle POA'$. Being similar, they have the same leg:base-ratio, and so we have:

$$\frac{OA}{OP} = \frac{OP}{OA'} \ .$$

This is the same as the first equation on page 335; as there, it immediately yields $OA \cdot OA' = r^2$, which is the defining condition for A' to be the inverse of A in the circle Γ.

37.3 Circular inverse of a line, simple construction.



Let X be the foot of O on $\ell = AB$, and let Y be the point, other than O, where OX meets γ. More generally, take C to be any point on $\overline{AB}$, and then define C' to be the point, other than O, where OC meets γ. We shall prove that C' is the inverse of C with respect to the circle Γ.

We skip the proof that OY is a diameter of the circle γ. (It is straightforward Euclidean geometry.) We can then see that $\triangle YC'O$ is inscribed in a semicircle of γ, and hence that $\angle YC'O$ is a right angle. (Euclid III.31.) Now consider $\triangle YC'O$ and $\triangle CXO$. They share the angle at O, and each has a right angle; hence they are similar. Comparing ratios of corresponding sides yields

$$\frac{OC}{OX} = \frac{OY}{OC'} \ ,$$

which in turn yields

$$OC \cdot OC' \; = \; OX \cdot OY. \tag{5}$$

Now the important thing about Equation (5) is that it hold for *every* C along the segment $\overline{AB}$. In particular, it holds when $C = A$, in which case C' is also equal to A. In this case, (5) yields

$$r^2 \; = \; OA \cdot OA \; = \; OX \cdot OY, \tag{6}$$

where r is the radius of Γ. Now (5) and (6) yield

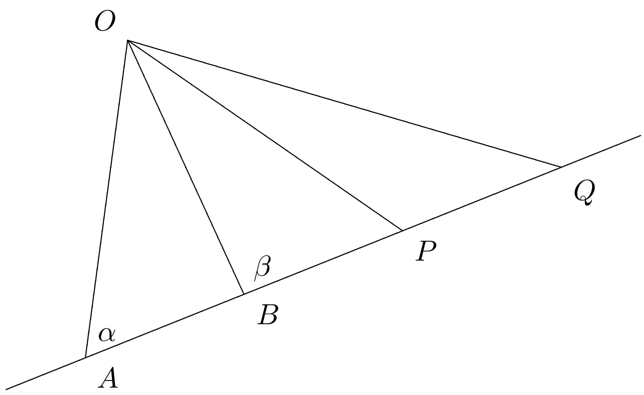
$$OC \cdot OC' \; = \; r^2,$$

which is the defining equation for C' to be the inverse of C in the circle Γ.

Afterthought. I only wrote up the case $X * C * A$. There is a slightly different picture, and different argument, for $X * A * C$.

37.14 Cross-ratio preserved under projectivity.

(a). The trigonometric formula.



The hint is to use the (strictly Euclidean[1]) law of sines:

$$\frac{\sin\gamma}{c} \; = \; \frac{\sin\delta}{d}, \tag{7}$$

where c and d denote two sides of a triangle, and γ and δ are the angles opposite those sides, respectively. Since we seek information about the cross-ratio, which involves segments $\overline{AP}$, $\overline{AQ}$, $\overline{BP}$ and $\overline{BQ}$, we shall apply (7) to the four triangles $\triangle OAP$, $\triangle OAQ$, $\triangle OBP$ and $\triangle OBQ$—in each case taking the segment of interest together with O as a third vertex.

[1]There is a corresponding law that applies in hyperbolic geometry—we don't need it here, and we probably won't get to it during the semester, but you can find it as Exercise 42.6 on page 411.

Of course, the question immediately arises[2]: to which angles and sides in these triangles shall we apply (7)? Looking at the form of the desired conclusion, we see that no angles besides α_P, α_Q, β_P and β_Q appear in the final answer. To make any other angles drop out, some cancelation must occur; so angles must be used twice. Thus it's a good bet to try using α twice and β twice (each occurs in two of our triangles). Once one has a clear idea of what to do, the calculation is easy:

Four applications of (7) yield

$$\frac{\sin\alpha_P}{AP} = \frac{\sin\alpha}{OP}; \quad \frac{\sin\alpha_Q}{AQ} = \frac{\sin\alpha}{OQ}; \quad \frac{\sin\beta_P}{BP} = \frac{\sin\beta}{OP}; \quad \frac{\sin\beta_Q}{BQ} = \frac{\sin\beta}{OQ}.$$

Looking at the form of the desired answer, we algebraically revise each of these four equations, obtaining

$$\sin\alpha_P = \frac{AP\cdot\sin\alpha}{OP}; \qquad \frac{1}{\sin\alpha_Q} = \frac{OQ}{AQ\cdot\sin\alpha};$$

$$\frac{1}{\sin\beta_P} = \frac{OP}{BQ\cdot\sin\beta}; \qquad \sin\beta_Q = \frac{BQ\cdot\sin\beta}{OQ}.$$

If we now multiply these four equations, we have immediate cancellation of $\sin\alpha$, $\sin\beta$, OP and OQ, yielding

$$\text{(8)} \qquad \frac{\sin\alpha_P\,\sin\beta_Q}{\sin\alpha_Q\,\sin\beta_P} = \frac{AP\cdot BQ}{AQ\cdot BP} = (AB, PQ),$$

the desired cross-ratio.

(b). Preservation of the cross-ratio. Equation (8) applies both to (AB, PQ) and to $(A'B', P'Q')$. However the angles α_P, α_Q, β_P and β_Q are the *same* as the corresponding angles for A', B', C' and D'. Hence $(AB, PQ) = (A'B', P'Q')$, by (8).

Comment on Exercise 37.14. To my mind, this is an example of a lovely and elegant piece of mathematics. It gives a surprising result, and upon first seeing that result, one could imagine that the calculations might be very complex. The emphasis was on finding just the right thing to calculate, and on calculating it as economically as possible. Contrary to the popular imagination—of a mathematician writing ever longer and messier equations—the true mathematician earnestly pursues an elegant way of getting to the answer as smoothly as possible.

[2]This entire paragraph is merely commentary on how we decided which triangles to use. From a purist point of view, such comments are irrelevant to a formal proof: once you have a proof, it's a proof, and its correctness doesn't depend on how you arrived at it. If, however, you are tying to explain things and share ideas, such commentary can be very worthwhile.