Mathematics 3210
Spring Semester, 2005
Homework notes, part 6
March 18, 2005

Please note that, as always, these notes are really not complete without diagrams. I have included a few, but the others are up to you.

34.4 AAA in semi-hyperbolic geometry. For this exercise we take semi-hyperbolicity in this simple form: *every triangle has positive defect.* We are given $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. If any pair of corresponding sides are congruent, then the proof may be finished by ASA. So let us assume that congruence holds for no pair of corresponding sides, and derive a contradiction.

Without loss of generality, we shall assume that $A'B' < AB$. Let us use Axiom C1 to locate a point B'' on $\overrightarrow{AB}$ and a point C'' on $\overrightarrow{AC}$ such that $AB'' \cong A'B'$ and $AC'' \cong A'C'$. By SAS, we have $\triangle AB''C'' \cong \triangle A'B'C'$. Since $AB'' \cong A'B' < AB$, we have $A * B'' * B$.

We claim we also have $A * C'' * C$. If not then we would have B'' and C'' on opposite sides of BC, so line BC would meet line $B''C''$ at a point which we shall call G. Since $A * C * C''$, $\angle ACB$ is and exterior angle to $\triangle GCC''$. Thus $\angle ACB > \angle AC''G = \angle AC''B''$. On the other hand, we already proved that $\triangle AB''C'' \cong \triangle A'B'C'$, and hence $\angle ACB \cong \angle A'C'B' \cong \angle AC''B''$. This contradiction establishes our claim that $A * C'' * C$.

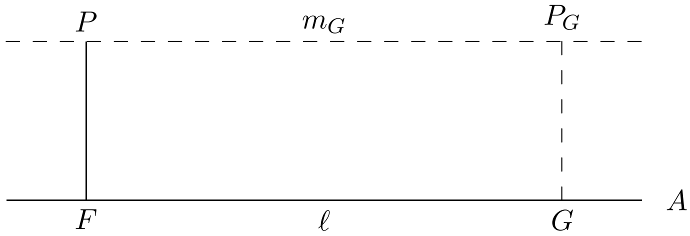
Thus lines $B''C$ and $B''C$ divide $\triangle ABC$ into three triangles: $\triangle AB''C''$, $\triangle B''BC''$, and $\triangle BC''C$. The first of these three has the same angles as the whole triangle $\triangle ABC$, and hence the same defect; by additivity of defect, the other two triangles have zero defect. This contradicts the fact (mentioned above) that in this context all triangles have positive defect. This contradiction completes the proof of AAA.

34.5 Infinitely many parallels in semi-hyperbolic geometry. Let us recall that in Exercise 7.4 we proved that every line has infinitely many points. I neglected to remark on it at the time, but in fact a close scrutiny of the proof reveals that we really proved this: *every* ***ray*** *has infinitely many points.*

So now, let us be given a line ℓ and a point[1] $P \notin \ell$. (Assuming a semi-hyperbolic plane; in other words, assuming the hypothesis of the acute angle) we shall construct infinitely many parallels to ℓ through P. Let F be the foot

[1] After writing this up I noticed that I have called the point P, where Hartshorne calls it A. I trust this is not too hard to sort out.

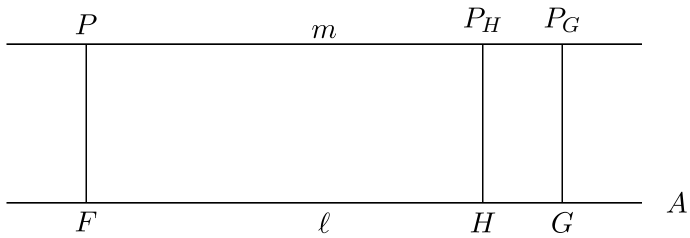
of the perpendicular from P to ℓ. Let $\overrightarrow{FA}$ be one of the two rays contained in ℓ and originating at F. For each $G \in \overrightarrow{FA} \smallsetminus \{F\}$, we define P_G to be the point that is uniquely defined by these three conditions: (i) P_G is on the P-side of ℓ; (ii) the line GP_G meets ℓ at right angles; (iii) $GP_G \cong FP$. (The unique existence of P_G follows from Axioms C1 and C4.) Finally, we define m_G to be the line joining P and P_G:



(Thus $FGPP_G$ is a Saccheri quadrilateral with right angles on line ℓ and acute angles at P and P_G.)

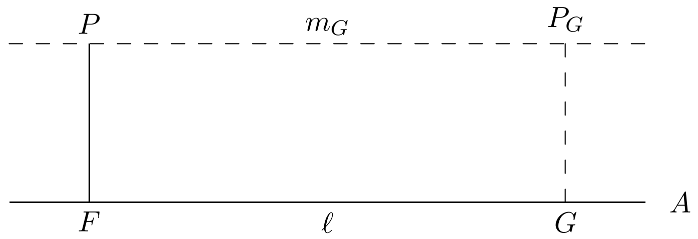
We have already seen that the top and bottom lines of a Saccheri quadrilateral are parallel (since each is perpendicular to the midline (not depicted)). Thus for each G, m_G is a parallel to ℓ through P. As we remarked above, there are infinitely many points on $\overrightarrow{FA}$, and so to complete the construction of infinitely many parallels to ℓ through P, it will suffice to show that $G \neq H$ implies $m_G \neq m_H$—in other words, that the correspondence $G \longmapsto m_G$ is one-to-one.

Suppose, for a contradiction, that $G \neq H$ and m_G and m_H form a single line m:



Thus we have three collinear points, P, P_H and P_G, with $FP \cong GP_G \cong HP_H$ (where F, G and H are the feet of the respective perpendiculars to ℓ). As we saw in Exercise 33.7(a) (or in Proposition 34.4(b)), this configuration leads to the hypothesis of the right angle (i.e. the semi-Euclidean hypothesis). We have, however, assumed either the semi-elliptic or the semi-hyperbolic hypothesis (acute angles or obtuse angles in Saccheri quadrilaterals). Hence we have a contradiction, and $m_G \neq m_H$ is proved.

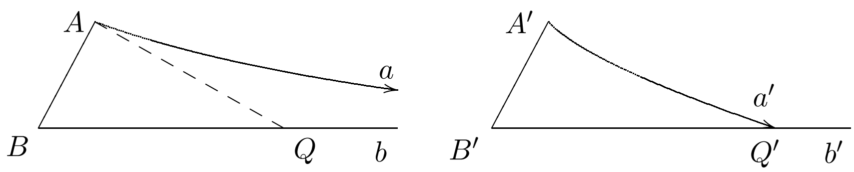
34.5, other methods. There are actually lots of methods for this, several of which involve Saccheri quadrilaterals in some way. But here's a non-Saccheri proof that seems easier than the one above (no need to appeal to 33.7(a) or 34.4(b)):



In place of the three conditions mentioned above, we now use (i) P_G is on the P-side of ℓ; (ii) the line GP_G meets ℓ at right angles; (iii) GP_G meets $m_G = PP_G$ at right angles. (The third condition changed.) The right angles at P_G and G tell us that m_G is parallel to ℓ. No two G give the same line m_G, for if they did, we would immediately have a rectangle ... *you should include a diagram of this* ... and of course there are no rectangles in semi-hyperbolic geometry. Thus, as before ... *write out the details* ... the infinitely many points G on $\overrightarrow{FA}$ give us infinitely many lines m_G.

34.9 ASAL. We are given $AB \cong A'B'$, $\angle ABb \cong \angle A'B'b'$, $\angle BAa \cong \angle B'A'a'$ and $\overrightarrow{Aa}$ limiting parallel to $\overrightarrow{Bb}$. We wish to prove that $\overrightarrow{A'a'}$ is limiting parallel to $\overrightarrow{B'b'}$.

We first prove, by contradiction, that $\overrightarrow{A'a'} \parallel \overrightarrow{B'b'}$. If not, then $\overrightarrow{A'a'}$ meets $\overrightarrow{B'b'}$ at a point Q':



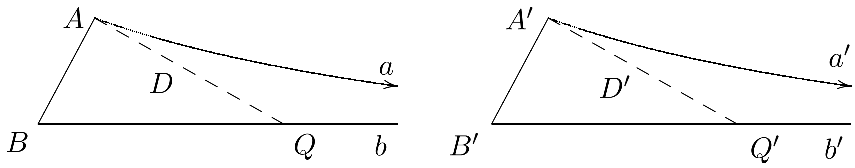
By Axiom C1, there is a point Q on $\overrightarrow{Bb}$ with $BQ \cong B'Q'$. Moreover, we are given that $AB \cong A'B'$ and $\angle ABQ \cong \angle A'B'Q'$; hence $\triangle ABQ \cong \triangle A'B'Q'$, by SAS. Thus

$$\angle BAQ \;\cong\; \angle B'A'Q' \;=\; \angle B'A'a' \;\cong\; \angle BAa$$

(where the last congruence was one of our assumptions). By the uniqueness part of Axiom C4, $\overrightarrow{AQ}$ and $\overrightarrow{Aa}$ are the same ray; in other words Q lies on

$\overrightarrow{Aa}$. This contradiction (to our assumption that $\overrightarrow{Aa} \parallel \overrightarrow{Bb}$) completes the proof of $\overrightarrow{A'a'} \parallel \overrightarrow{B'b'}$.

We now prove that $\overrightarrow{A'a'}$ is a limiting parallel to $\overrightarrow{B'b'}$. To this end, let $\overrightarrow{A'D'}$ be a ray interior to $\angle B'A'a'$. We need to prove that $\overrightarrow{A'D'}$ meets $\overrightarrow{B'b'}$.

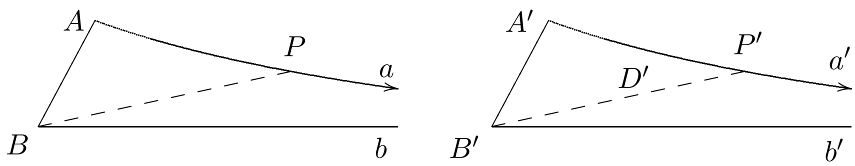


By Axiom C4, there exists D interior to $\angle BAa$ such that $\angle BAD \cong \angle B'A'D'$. Since $\overrightarrow{Aa}$ is a limiting parallel to $\overrightarrow{Bb}$, there exists Q that lies on the two rays $\overrightarrow{AD}$ and $\overrightarrow{Bb}$. By Axiom C1, there exists a point Q' on $\overrightarrow{B'b'}$ such that $B'Q' \cong BQ$. We are given that $AB \cong A'B'$ and that $\angle ABQ \cong \angle A'B'Q'$. Thus $\triangle ABQ \cong \triangle A'B'Q'$, by SAS. From this we deduce that

$$\angle B'A'Q' \;\cong\; \angle BAQ \;=\; \angle BAD \;\cong\; \angle B'A'D'.$$

By the uniqueness part of C4, $\overrightarrow{A'D'}$ and $\overrightarrow{A'Q'}$ are the same ray; in other words, Q' lies on $\overrightarrow{A'D'}$. This is therefore the desired point of intersection of $\overrightarrow{AD}$ and $\overrightarrow{Bb}$.

34.10 ASL. We are given $\angle BAa \cong \angle B'A'a'$, $AB \cong A'B'$, $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$ and $\overrightarrow{A'a'} |||\overrightarrow{B'b'}$. We need to prove that $\angle ABb \cong \angle A'B'b'$.



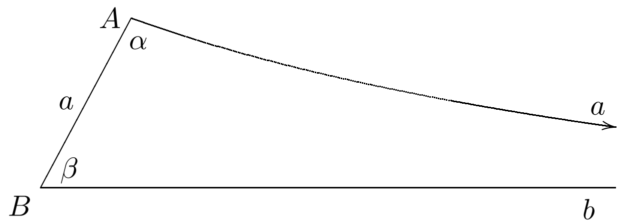
We shall assume, by way of contradiction, that $\angle ABb < \angle A'B'b'$. By definition of "<," there is a point D' interior to $\angle A'B'b'$ such that $\angle ABb \cong \angle A'B'D'$. Since $\overrightarrow{A'a'}$ is *limiting* parallel to $\overrightarrow{B'b'}$, we know that $\overrightarrow{B'D'}$ meets $\overrightarrow{A'a'}$ at a point P'. By Axiom C1, there exists P on $\overrightarrow{Aa}$ with $AP \cong A'P'$. We are given $BA \cong B'A'$ and $\angle BAP \cong \angle B'A'P'$. Hence $\triangle BAP \cong \triangle B'A'P'$ by SAS. Thus

$$\angle ABP \;\cong\; \angle A'B'P' \;=\; \angle A'B'D' \;\cong\; \angle ABb.$$

By the uniqueness part of Axiom C4, $\overrightarrow{BP}$ and $\overrightarrow{Bb}$ are the same ray. From this we see that $\overrightarrow{Bb}$ meets $\overrightarrow{Aa}$ at P, in contradiction to the fact that $\overrightarrow{Bb}$ and $\overrightarrow{Aa}$ are parallel rays. This contradiction completes the proof that $\angle ABb \cong \angle A'B'b'$.

Slight twist on the proof of 34.10. It works just as well to locate the point P on $\overrightarrow{Bb}$, with $BP \cong B'P'$. As before, one gets $\triangle BAP \cong \triangle B'A'P'$ by SAS. (But they're not the same two sides as before, and it isn't the same angle!). Then C4 is applied again (but not to the same rays as before).

Comment on the importance of Exercise 34.10. If we speak in the context of measured segments and measured angles, then 34.10 may be paraphrased as follows: *for configurations comprising two limiting parallel rays and a transversal segment—of length a, meeting the rays at angles α and β as indicated below—the quantities a and α determine β.*



In other words, *some relationship must hold between a, α and β.* One might of course seek to represent this relationship mathematically. One might say that non-Euclidean geometry reached its maturity when Bolyai and Lobachevskiĭ proved (independently) that

$$\tan\frac{\alpha}{2}\,\tan\frac{\beta}{2} \;=\; e^{-a/L}. \tag{1}$$

Here L is the length of the segment that corresponds to angles of $\alpha = 90°$ and $\beta = 40.359\ldots°$. (In other words, in our observed universe, L is a hitherto unknown, and inconceivably large, astronomical distance.) I can't find (1) in Hartshorne, but it is closely related to the formula[2] that appears in Proposition 39.13 on page 364 and again in Proposition 41.9 on page 396. (In the first instance (§39), it is proved in a single model of hyperbolic geometry; in the second instance (§41), it is proved from the axioms of hyperbolic geometry. Obviously the latter context yields a stronger result.)

Notice also this: once one has (1) available (which for us won't be for a long time), then Exercise 34.10 is immediately obvious.

[2] The μ in Hartshorne's formula is a sort of naïvely constructed exponential function.

A further comment about Exercise 34.10 is that it represents an extension of SAA-congruence to the limiting case where one of the two angles is, so to speak, a zero angle.

Comment on the proofs of 34.9 and 34.10. Never lose sight of the fact that limit parallelism has two components: (1) simple parallelism in the form of non-intersection, and (2) limit parallelism, which is to say that any ray at a smaller angle is not parallel. Both (1) and (2) must appear in any proof that involves limit parallelism in any essential way, such as 34.9 and 34.10. There is, however, a subtle difference in these two examples. In 34.9 we are proving that a certain pair of rays are limit parallels. Thus in 34.9, (1) and (2) were two things that we needed to prove, and they were in fact proved *separately*, one at a time. On the other hand, in 34.10 limit parallelism appears only in the hypothesis; the conclusion of 34.10 involves something else (a certain congruence of angles). In this case there is only one thing to prove, but along the way we make use of one instance of (1) (as given) and one instance of (2) (as given).

35.1 Triangle of small defect. Let $\triangle ABC$ be any triangle. By Exercise 7.6, there exists a point D with $A * D * C$. Standard arguments (we have been through them before) yield the non-collinearity of A, B, D and the non-collinearity of B, C, D. Additivity of defects (Lemma 34.8) says that

$$\delta(ABC) \;=\; \delta(ABD) \;+\; \delta(BCD).$$

Therefore one of the two new triangles $\triangle ABD$ and $\triangle BCD$ has defect $\leq 1/2\delta(ABC)$. In other words, *for every triangle, there is another triangle with no more than half the defect of the original triangle.* Repeating this construction n times yields: *for every triangle $\triangle ABC$, there is a triangle $\triangle DEF$ such that*

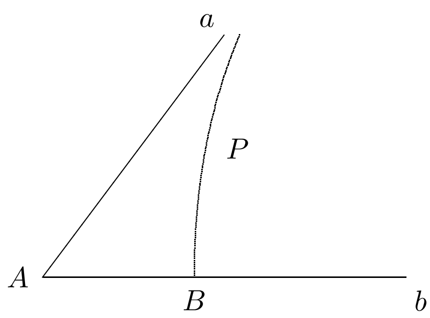
$$\delta(DEF) \;\leq\; \frac{1}{2^n}\,\delta(ABC) \tag{2}$$

Now, given $\varepsilon > 0$, we invoke Archimedes Axiom for the existence of n such that

$$\frac{1}{2^n}\,\delta(ABC) \;\leq\; \varepsilon, \tag{3}$$

and let $\triangle DEF$ be a triangle that satisfies (2). It is now evident from (2) and (3) that $\triangle DEF$ has defect $< \varepsilon$.

35.3 Parallel ray with arbitrary vertex angle.



We are given an angle comprising rays $\overrightarrow{Aa}$ and $\overrightarrow{Ab}$. We need to find $B \in \overrightarrow{Ab}$ so that the perpendicular to Ab at B is parallel to Aa. If $\angle aAb$ is obtuse, we replace $\overrightarrow{Aa}$ by its opposite ray. This has no affect on the conclusion (since lines Aa and Ab have not changed), but it allows us to work with an angle that is acute or a right angle.

The contrapositive of Proposition 35.5 says this: *In a Hilbert plane with (A), if (P) fails then Legendre's axiom fails for every α.* The blanket assumption for Exercises 35 includes (A) and the negation of (P); hence we may conclude that Legendre's axiom fails for the angle at A. In other words, there exists a point P in the interior of $\angle aAb$ such that *no* line through P meets both sides of the angle. Let B be the foot of the perpendicular from P to Ab. Since $\angle aAb$ is acute, the Exterior Angle Theorem tells us that $B \in \overrightarrow{Ab}$.

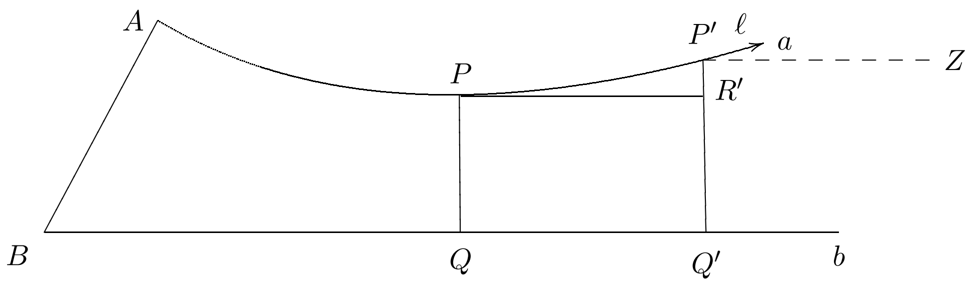
Now consider the line PB. It meets one side of our angle (namely $\overrightarrow{Ab}$); hence, by the way we chose P, it does not meet $\overrightarrow{Aa}$. On the other hand, it also does not meet the ray r opposite to $\overrightarrow{Aa}$, for the following reason. If there is a point G lying on both r and PB, consider $\triangle ABG$. It has a right angle at B and an obtuse or right angle at A. This violates Euclid I.17 (*Any two angles of a triangle are less than two right angles*).[3] Thus PB is the required line meeting $\overrightarrow{Ab}$ at right angles and not meeting Aa.

Comment about student answers to Exercise 35.3. I read the papers before I had worked out my own answer for 35.3, and I had not yet noticed the part about making sure one begins with an acute angle. Thus I neglected to mention anything about this on people's papers. It's a minor point, but something does have to be said. For example if you begin with an obtuse

[3]For it to fit precisely in his system, Hartshorne requires a slight rewording of I.17; see pages 101–102. Nevertheless, the sense is the same.

angle, then B, taken as the foot of an almost random point P, may easily fail to lie in $\overrightarrow{Ab}$.

35.8 The distance decreases along a limiting parallel ray. The blanket assumption for the Exercises of §35 is that (A) holds and (P) does not. It then follows easily from Propositions 35.2 and 35.4 that we have "the hypothesis of the acute angle," in other words, the upper angles of a Saccheri quadrilateral are always acute.

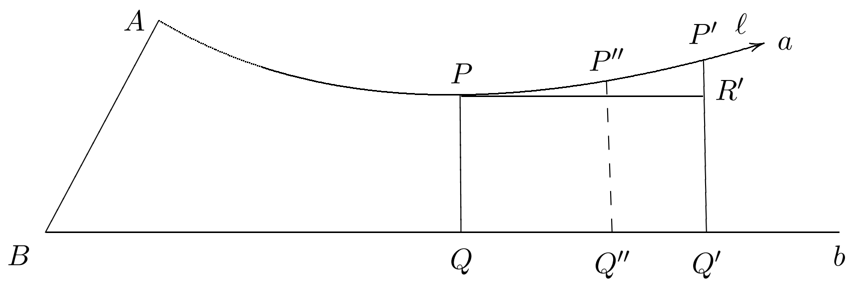


Suppose that $P, P' \in Aa$, with $A * P * P'$, and that Q and Q' are the feet of P and P', respectively, on line b. We must prove $P'Q' < PQ$. Supposing the desired conclusion to be false, namely assuming that $P'Q' \geq PQ$, we shall derive a contradiction to the assumption that $Aa|||Bb$.

Since $P'Q' \geq PQ$, there exists R' with $Q'R' \cong QP$ and with $Q' * R' * P'$ or $R' = P'$. Thus $QQ'PR'$ is a Saccheri quadrilateral, and hence $\angle PR'Q'$ is acute. It then follows that $\angle PP'Q'$ is acute, for one of two reasons: if $Q' * R' * P'$, then by the Exterior Angle Theorem; and if $R' = P'$, then $\angle PP'Q' = \angle PR'Q'$. Thus $\angle aP'Q' >$ RA.

Thus there exists a ray $\overrightarrow{P'Z}$ interior to $\angle aP'Q'$, such that $\angle Q'P'a =$ RA. By the right angles at P' and Q', line $P'Z$ is parallel to Bb. Hence $P'a$ is not the limiting parallel to Bb at P'. By Proposition 34.9, Aa is not the limiting parallel to Bb. This contradiction to our hypothesis finishes the proof.

35.8 Alternate proof.



Suppose that $P, P' \in Aa$, with $A * P * P'$, and that Q and Q' are the feet of P and P', respectively, on line b. We must prove $P'Q' < PQ$. Supposing the desired conclusion to be false, namely assuming that $P'Q' \geq PQ$, we shall derive a contradiction to the assumption that $Aa|||Bb$.

We first deal with the case that $P'Q' \cong PQ$. Supposing this to be the case, we relabel P' and Q' as P'' and Q'', and then notice that $PP''QQ''$ is a Saccheri quadrilateral, with acute angles at P and P''. Now choose P' with $P * P'' * P'$, and let Q' be its foot on Bb. By Proposition 34.4, $P'Q' > P''Q'' \cong PQ$. So, in either case $P'Q' > PQ$: either this was so in the first place, or it became so after the revision described in this paragraph.

Since $P'Q' > PQ$, there exists R' with $Q'R' \cong QP$ and with $Q' * R' * P'$. Thus $QQ'PR'$ is a Saccheri quadrilateral, and hence PR' is parallel to $QQ' = Bb$. Since $\overrightarrow{PR'}$ is interior to $\angle QPP'$, PP' is not the limit parallel at P. By Proposition 34.9, Aa is not the limiting parallel to Bb at A. This contradiction to our hypothesis finishes the proof.

Comment on the proofs of 35.8. In both proofs, it would have been handy to have a ready-made lemma that says, *in the semi-hyperbolic case, if two lines have a common perpendicular, then they are not limit parallel in any direction.* I think I have said it in class; it is fairly natural; people alluded to it in their work; and yet I cannot find it expressly stated in Hartshorne. (If you find it, let me know.)