Mathematics 3210
Spring Semester, 2005
Homework notes, part 5
March 4, 2005

Please note that, as always, these notes are really not complete without diagrams. I have included a few, but the others are up to you. (Although I would say that, if you write it the way I did, 33.1 really requires no diagram, since it is really accomplished by pure logic—the manipulation of contrapositives.)

I hasten to point out that Clavius' Axiom was mentioned (as were others in that section) as a historical footnote to the subject. In a certain historical period, many people were trying all sorts of dodges to get [what they hoped to be] a proof of Euclid's fifth postulate. All such would-be proofs contained, explicitly or implicitly, an extra assumption. Exercise 33.7, for example, is an inquiry into how such extra assumptions—or axioms—relate to one another (Clavius + (A) $\Longrightarrow$ (P), in this case). But our purpose (and that of the text from §34 onward) is to bravely go into the world where there is no guarantee about the fifth postulate (which is to say, no guarantee about (P)). Therefore we leave behind all things like Clavius Axiom.

10.9 RASS. If $BC \cong B'C'$, we have $\triangle ABC \cong \triangle A'B'C'$ by SAS. To complete the proof, we will suppose that $BC \ncong B'C'$ and deduce a contradiction. Without loss of generality, we may suppose that $BC < B'C'$. By Axiom C1, there exists C'' on $\overrightarrow{BC}$ with $BC'' \cong BC'$. By Axiom $C2$ we have $C \neq C''$. By Axiom 3 there exists D on line BC with $C * B * D$. We now have

$$(1) \qquad \angle ACC'' > \angle ABC'' \cong \angle ABD > \angle AC''D = \angle AC''C.$$

(The middle congruence is from the definition of a right angle, and the two inequalities are from the Exterior Angle Theorem.)

On the other hand, $\triangle ABC'' \cong \triangle A'B'C'$, by SAS, and so $AC \cong A'C' \cong AC''$; in other words, $\triangle ACC''$ is isosceles. Therefore, its base angles are congruent:

$$(2) \qquad \angle ACC'' \cong \angle AC''C.$$

The contradiction between (1) and (2) completes the proof of RASS.

10.9 RASS, second proof (sketch). Choose point C'' on the ray opposite to $\overrightarrow{BC}$, with $BC'' \cong B'C'$. From SAS we have $\triangle ABC'' \cong \triangle A'B'C'$. Thus we have $AC'' \cong A'C' \cong AC$ and $\angle C'' \cong \angle C'$. Thus $\triangle ABC''$ is an isosceles triangle whose base angles are congruent to $\angle C$ and $\angle C'$. Thus $\angle C \cong \angle C'$. Therefore the two original triangles are congruent by AAS.

10.9 RASS, third proof (sketch). Choose point A'' on the ray opposite to $\overrightarrow{BA}$, with $BA'' \cong BA$. Choose point A''' on the ray opposite to $\overrightarrow{B'A'}$, with $BA''' \cong B'A'$. By an easy use of SAS we see that $\triangle AA''C$ is an isosceles triangle whose base angles are congruent to the original $\angle A$. Likewise $\triangle A'A'''C'$ is an isosceles triangle whose base angles are congruent to the original $\angle A'$. These two isosceles triangles are clearly congruent by SSS, and hence $\angle A \cong \angle A'$. Therefore the two original triangles are congruent by SAS

11.5 Construction from triangle inequality, using (E). Let us suppose that we are given segments AB, CD and EF with

$$AB < CD + EF \tag{3}$$
$$CD < EF + AB \tag{4}$$
$$EF < AB + CD. \tag{5}$$

Moreover, we will choose our notation so that

$$AB \leq CD \leq EF. \tag{6}$$

(Once the notation is chosen in this way, Equations (3) and (4) become consequences of (5) and (6), and hence redundant.)

Define Γ and Δ to be the circles

$$\Gamma = \{P : EP \cong AB\} \qquad \text{(center } E\text{, radius } AB\text{)}, \tag{7}$$
$$\Delta = \{Q : FQ \cong CD\} \qquad \text{(center } F\text{, radius } CD\text{)}. \tag{8}$$

We are now going to invoke Axiom (E) on the top of page 108. So, we need to see that Δ contains a point inside Γ and also contains a point outside Γ. By Axiom C1, on $\overrightarrow{FE}$ there is a point P with $PF \cong CD$. Clearly P lies on Δ. Since $CD < EF$, we have $E * P * F$. Then by (5) we have

$$EP + PF = EF < AB + CD = AB + PF;$$

hence $EP < AB$, and so P lies inside Γ. On the other hand, there exists a point Q on the ray opposite to $\overrightarrow{FE}$, with $FQ \cong CD$. Again, Q lies on Δ, but this time $E * F * Q$. Thus we have

$$EQ = EF + CD > AB,$$

and so Q lies outside Γ.

Thus the hypotheses of (E) are satisfied, and hence also its conclusion. Thus there exists a point G that lies on the intersection of Δ and Γ. Let us now consider the triangle $\triangle EFG$. Since G lies on Γ, we have $EG \cong AB$. Since G lies on Δ we have $FG \cong CD$. And finally, of course $EF \cong EF$. Hence the three sides of this triangle are congruent to the three given segments, AB, CD and EF, which is what we set out to prove.

33.1 Proclus equivalent to Playfair. Let us write the Lemma of Proclus as follows:

If $m||n$, then (if m meets ℓ, then n meets ℓ).

Recall that an implication $A \Longrightarrow B$ is logically equivalent to its *contrapositive*: not $A \Longrightarrow$ not A. Thus the Lemma of Proclus is now seen as logically equivalent to this:

If $m||n$, then (if $n||\ell$, then $m||\ell$).

Which in turn is logically equivalent to:

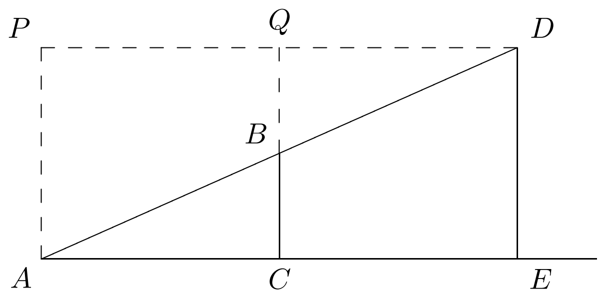
If $m||n$ and if $n||\ell$, then $m||\ell$.

This last statement, however, is none other than the assertion that parallelism is transitive, which we have already seen equivalent to Playfair's Axiom (P).

33.7(abc) The axioms of Clavius and Archimedes.

(a) It's best to approach this after having done Proposition 34.1 in the next section. The diagram for 33.7(a) contains three Saccheri quadrilaterals, $ABA'B'$, $ACA'C'$ and $BCB'C'$, and Proposition 34.1 can be applied to each of them separately. Doing this, we see that the four interior top angles are congruent to one another. Since two of these four are supplementary to each other, it follows that all four are right angles.

(b)



Given $AB \cong BD$, and right angles at C and E, with B and D on the same side of $\ell = CE$. By C4 and C1, there exists P on the B-side of ℓ with $PA \cong DE$ and with a right angle at A. Let Q be the midpoint of $\overline{PD}$. By Part (a), there are right angles at P and Q.

Now $\triangle PAD \cong \triangle EDA$, by RASS; hence $\angle PDA \cong \angle DAE$ and $PD \cong AE$.[1] Consider now the small triangles $\triangle ABC$ and $\triangle DBQ$. We just proved that the angle at D is congruent to the angle at A; moreover we have right angles at C and at Q, and we were given $AB \cong BD$. Therefore $\triangle ABC \cong \triangle DBQ$ by AAS.

[1] We don't need $PD \cong AE$ for (b), but we are saving it for (b$'$).

From this congruence we have $\angle DBQ \cong \angle ABC$. Let Q' satisfy $C*B*Q'$. Then $\angle DBQ' \cong \angle ABC$ by vertical angles, and hence $\angle DBQ' \cong \angle DBQ$ by transitivity. By the uniqueness part of C4, $\overrightarrow{BQ}$ is the same as $\overrightarrow{BQ'}$, which is to say, the opposite ray to $\overrightarrow{BC}$. Therefore C, B and Q are collinear, and $C*B*Q$. Again from $\triangle ABC \cong \triangle DBQ$, we have $CB \cong BQ$, and hence $CQ \cong 2\,BC$.

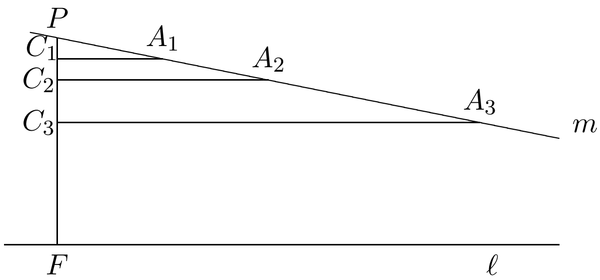
Finally, we have $DE \cong QC$ by Clavius' Axiom itself. Therefore, by transitivity, $DE \cong 2\,BC$

(b′) — Addendum to (b). Recall from the proof of (b) that $\triangle ABC \cong \triangle DBQ$, that Q is the midpoint of $\overline{PD}$, and that $PD \cong AE$. Thus

$$2\,AC \;\cong\; 2\,DQ \;\cong\; PD \;\cong\; AE.$$

(c) Assuming Archimedes' Axiom (A) and Clavius' axiom, we shall prove (P), expressed as follows: *Suppose that $P \notin \ell$ and that m and n are two lines through P. Then either m meets ℓ or n meets ℓ.*

Proof: Let F be the foot of the perpendicular from P to ℓ. By C4, we cannot have m and n both perpendicular to the line PF. We will assume the notation is taken so that m is not perpendicular to PF. This means that if we take points A and B on m with $A*P*B$, then one of the two supplementary angles $\angle APF$ and $\angle BPF$ is acute, and the other is obtuse. Let us take the notation so that $\angle APF$ is acute. Our proof will be complete when we have shown that $\overrightarrow{PA}$ meets ℓ.



We let $A_1 = A$, and we recursively define $A_2, A_3, \cdots$ (using C1) as points on the ray $\overrightarrow{PA_1}$ that satisfy these congruences:

$$PA_2 \;\cong\; 2PA_1; \quad PA_3 \;\cong\; 2PA_2; \quad \cdots \quad PA_{n+1} \;\cong\; 2PA_n \quad \cdots .$$

For each n, C_n is defined to be the foot of the perpendicular from A_n to line PF. The acuteness of $\angle FPA_n$ tells us that $C_n \in \overrightarrow{PF}$ (otherwise we would have a violation of the Exterior Angle Theorem).

By (b′) above—and hence indirectly by Clavius' Axiom—we have $PC_{n+1} \cong 2\,PC_n$ for each n. By the **Axiom of Archimedes** (finally, we get to use it!), $PC_n > PF$ for some n. In other words, $P*F*C_n$, and so C_n and P

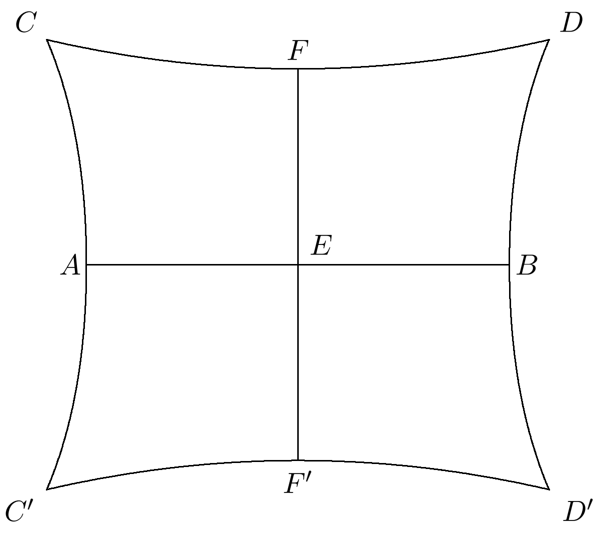
are on opposite sides of ℓ. On the other hand, because of the right angles at C_3 and at F, line C_nA_n is parallel to ℓ (by Euclid I.27). Therefore A_n and C_n are on the same side of ℓ.

From our two results about sides of ℓ, we see that A_n and P are on opposite sides of ℓ. Therefore the segment $\overline{PA_n}$ meets ℓ, so of course the ray $\overrightarrow{PA} = \overrightarrow{PA_n}$ meets ℓ; as desired.

Note on this proof of 33.7(c). One frequent mistake was to construct the points $C_1, C_2, \cdots$ along $\overrightarrow{PF}$, and then hope for the corresponding points $A_1, A_2 \cdots$ along $\overrightarrow{PA}$. This doesn't work—we have no axiom that will insure their existence. What does work is to first find the points $A_1, A_2 \cdots$ along $\overrightarrow{PA}$—all we need for this is C1—and then find $C_1, C_2, \cdots$ as the feet of perpendiculars onto PF. It is (b′) that guarantees the correct spacing of $C_1, C_2, \cdots$.

Remark on 33.7(c). From the author's setup of this exercise, one might think that he is hinting that one should be able to get (c) directly from (b), without a detour through (b′) as I have done. I don't see it, but maybe someone else does.

34.1 An inequality for Saccheri quadrilaterals.

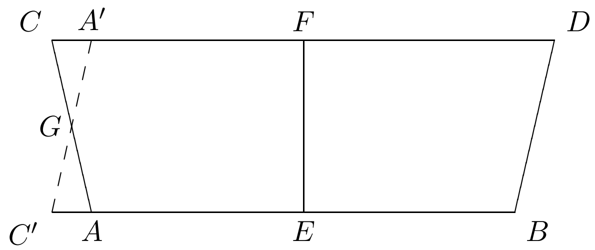


Sketch of proof: Suppose that $CD > AB$ in the Saccheri quadrilateral $ABCD$. We will prove that $\angle C$ is acute. Let EF be the midline of $ABCD$ (as previously described). Enlarge the figure on the opposite side of line AB as follows. C' is on the ray opposite to $\overrightarrow{AC}$ with $AC' \cong AC$; D' is on the ray opposite to $\overrightarrow{BD}$ with $BD' \cong BD$; and F' is on the ray opposite to $\overrightarrow{EF}$

with $EF' \cong EF$; A divide-and-conquer approach with triangles and SAS (such as we have seen several times) will easily yield a right angle at F' and $C'F' \cong CF$.

Thus $F'FC'C$ is itself a Saccheri quadrilateral, whose vertex angles are $\angle C$ and $\angle C'$, and whose midline is AE. We can easily prove (and I think it has appeared before) that $AE < CF$. It is immediate from Proposition 34.3(a) that $\angle C$ is acute.

34.1 Alternate method. (Sketch.)



Let the given Saccheri quadrilateral be $ABCD$, with right angles at A and B, with midline EF, and with $AB < CD$. (Note in particular that this diagram has distorted the angles. I really do intend right angles at A and B, and also at A' when we get that far.) We need to prove that $\angle C$ is acute.

By Axiom C1, there exists A' on $\overrightarrow{FC}$ such that $FA' \cong EA$, and there exists C' on $\overrightarrow{EA}$ such that $EC' \cong FC$. It is not hard to find some congruent triangles that will allow one to prove that $\angle A' \cong \angle A$ (we skip the details). Therefore we have a right angle at A'.

Since $EC' \cong FC > EA$, we have $C' * A * E$; thus C' and E are on opposite sides of line AC. Similar reasoning shows that $C * A' * F$, and thus that A' and F are on the same side of AC. Finally the line EF is parallel to AC (by Euclid I.27), and hence E and F are on the same side of AC. From these three half-plane relations, we may conclude that C' and A' lie on opposite sides of AC. Therefore, lines AC and $A'C'$ intersect at a point G.

Let us now consider the angles of $\triangle A'CG$. Obviously $\angle C$ is an interior angle of this triangle, whereas $\angle FA'G$ is an exterior angle (since $C * A' * F$). Hence, by the Exterior Angle Theorem, $\angle C < \angle FA'G = \text{RA}$.

34.3 Angle inscribed in semicircle. Hint. If you draw a segment from the center to each vertex, you will have two isosceles triangles. Thus one easily calculates that the angle sum of the triangle is twice the angle opposite the diameter of the circle. (You should supply a diagram and all the (easy) calculations.) The desired conclusion now follows immediately from Proposition 34.7 and the Definition on page 311.