Mathematics 3210
Spring Semester, 2005
Homework notes, part 4
February 25, 2005

Please note that, as always, these notes are really not complete without diagrams. I have included a few, but the others are up to you.

My first writeup illustrates a style of proof that we haven't seen before, but which is relatively common in the mathematics profession. It has the form of a commentary on a published proof. The general idea is that the published proof has almost what we want, so we can get away with describing only the modifications or additions that are needed. (In this case, an attention to the distinction between real numbers and rational numbers.)

It's implicit that one could, if pressed to do so, actually write out a self-contained proof, by taking the original proof and splicing in the modifications. This is not usually worth the trouble, and in any case, one would still need to cite one's source for the original.

For obvious reasons, this sort of thing doesn't lend itself well to an exam question, so you can be sure that I won't ask you for 8.6 on an exam.

8.6 (Skip I1–I3.) **Betweenness in the $\mathbb{Q}$-model.** For the betweenness axioms, we adapt Example 7.3.1 to the plane with co-ordinates from $\mathbb{Q}$ (the rational number system) . For instance, for distinct $a, b, c \in \mathbb{Q}$, we define $a * b * c$ to mean that either $a < b < c$ or $c < b < a$, and as before, we define $A * B * C$ to mean that either $a_1 * b_1 * c_1$ or $a_2 * b_2 * c_2$ (or both). The second paragraph of Example 7.3.1—establishing B1, B2 and B3—carries over word for word, and thus need not be repeated here. Hartshorne's proof of B4 (top of page 79) is valid here, except that we must be sure that the point of intersection found on $\overline{AB}$ or on $\overline{AC}$ *has rational co-ordinates.*

As indicated on page 79, the point of intersection is found by simultaneously solving two equations of the form $ax + by + c = 0$ (known as *linear equations.* In order to get the above-mentioned point of intersection in the rational plane we need two facts: (i) *if a line is determined by two rational points, then its coefficients a, b and c may be taken to be rational*; (ii) *if the point (x, y) is determined as the intersection point of two lines with rational coefficients, then $x, y \in \mathbb{Q}$.*

For (i), we shall assume without loss of generality that $a \neq 0$. The equation $ax + by + c = 0$ does not change its meaning if a, b and c are all multiplied by a non-zero constant. Therefore we shall do this in such a way as to guarantee that $a \in \mathbb{Q}$. Let (x_1, y_1) and (x_2, y_2) be two rational points on the line; in other words, suppose that

$$ax_1 + by_1 + c \;=\; 0; \qquad\qquad ax_2 + by_2 + c \;=\; 0.$$

A little algebra now yields

$$b \;=\; \frac{a(x_2 - x_1)}{y_2 - y_1}; \qquad\qquad c \;=\; -\,ax_1 - by_1.$$

Since a, x_1, y_1, x_2 and y_2 were all taken to be rational, it follows directly that b and c are rational as well.

For (ii), we merely remark that there is a well-known algebraic procedure for simultaneously solving two linear equations in two unknowns. [The details were on the board in class.] This procedure involves only the fundamental operations of addition, subtraction, multiplication and division, and hence it cannot introduce irrational numbers. This complete out proof of (ii), and hence the proof of B4 for the rational co-ordinate plane.

Congruence for line segments—C2 and C3. Here we adapt Example 8.4.1. The assertion about C2 (one sentence at the bottom of page 87) remains true here. The proof of C3 (first half of page 88), also remains true, word for word and symbol for symbol). (The main point is that no new points are introduced, and so one does not have to check the rationality of any points.)

The failure of C1 in the $\mathbb{Q}$-model. We consider one case of the axiom C1, just as it is stated on page 82. (The axiom is stated as a universal sentence, i.e. true for any A, B and C; hence to negate it I only have to find one ray and one set of points for which it is false.) Let $A = (0, 0)$ and $B = (1, 0)$. Let C also be $(0, 0)$, and take the ray r to be $\overrightarrow{CE}$, where $E = (1, 1)$.

Now Axiom C1 says that there exists a point $D \in r$ such that $AB \cong CD$. To see that C1 fails in the rational plane, we will simply show that no such D can have rational co-ordinates. Like every point on r, our point D has the form (t, t) for some $t \geq 0$. To say that $AB \cong CD$ is to say that $d(A, B) = d(C, D)$; in other words

$$1 \;=\; \sqrt{1^2 + 0^2} \;=\; \sqrt{t^2 + t^2} \;=\; \sqrt{2}\,t.$$

Thus $t = 1/\sqrt{2}$, which is not rational; hence there is no such D in the rational plane.

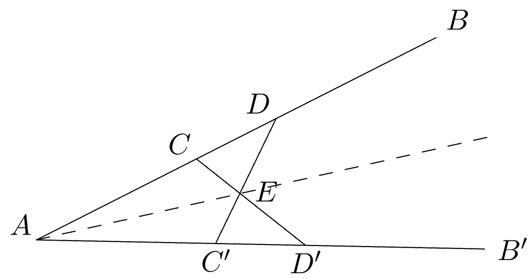
The text does not emphasize this here, but we can say that the import of this exercise is this simple fact: C1 cannot be proved from I1–I3, B1–B4 and C2–C3.

9.2 Since $\overrightarrow{AD}$ is interior to $\angle BAC$, we may apply the Crossbar Theorem to see the existence of a point D' on $\overrightarrow{AD}$ with $B * D' * C$. It is of course true that $\overrightarrow{AD'} = \overrightarrow{AD}$ Since $\overrightarrow{AE}$ is interior to $\angle D'AC = \angle DAC$, we may apply the Crossbar Theorem to see the existence of a point E' on $\overrightarrow{AE}$ with $D' * E' * C$. It now follows from Exercise 7.1(b) (second conclusion) that $B * E' * C$. (*Apologies—on the assignment page I suggested using 7.1(a). If you look at it closely, however, you will see that 7.1(b) is what is needed.*)

Now obviously B and E' are on the same side of line AC, and C and E' are on the same side of line AB. Therefore $\overrightarrow{AE} = \overrightarrow{AE'}$ is in the interior of $\angle BAC$.

Alternate proofs for 9.2. There are some alternate versions of this, but all really seem to come down to the same considerations. For instance, you can replace every statement of the form $B * C * D$ with an equivalent statement of the form "$\overrightarrow{AC}$ is interior to $\angle BAD$." You can decline to cite Exercise 7.1 in favor of recapitulating its proof (i.e. invoking line separation.) I won't say any more about these here, except that (if correctly presented) they are valid alternatives.

10.1 Angle bisector.



Given $\angle BAB'$. Choose a point $C \neq A$ on $\overrightarrow{AB}$. Use C1 to obtain a point C' on $\overrightarrow{AB'}$ such that $AC' \cong AC$. Choose a point $D \neq C$ on the ray opposite to $\overrightarrow{CA}$. Use C1 to obtain a point D' on the ray opposite to $\overrightarrow{C'A}$ such that $C'D' \cong CD$.

Use a straight-edge to draw the two lines $C'D$ and CD'.

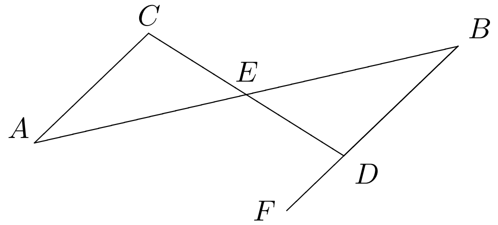
Consider now the line CD' and the two half-planes that it determines. The betweenness relations $A * C * D$ and $A * C' * D'$ tell us that A and D are on opposite sides of CD' and that A and C' are on the same side of CD'. Therefore D and C' lie on opposite sides of CD'. In other words segment $\overline{C'D}$ intersects line CD' at a point which we shall denote E.

The angle bisector is obtained by using a straight-edge to draw the line AE.

Commentary (optional) on the construction: Similarly, segment $\overline{CD'}$ intersects line $C'D$ at a point which we shall denote F. Clearly the four points C, D, C' and D' are not collinear, and hence the two lines $C'D$ and CD' have at most one point of intersection (by the incidence axioms). Clearly both E and F line on both of these lines; hence $E = F$. Now we see that $E = F$ is an interior point of both segments $\overline{C'D}$ and $\overline{CD'}$.

Proof that AE bisects the angle. Omitted.

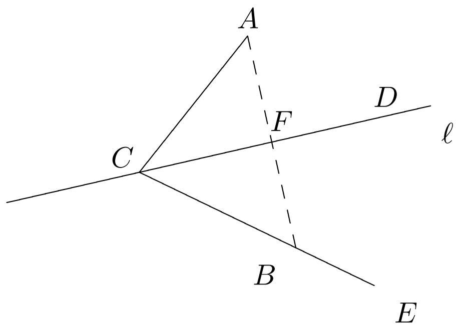
10.2 Midpoint of a segment.



Given $\overline{AB}$. Let C be any point not on line AB. Let H be the half-plane of AB that does not contain C. By C4, there is a unique ray $\overrightarrow{BF}$ in H with $\angle ABF \cong \angle BAC$. By C1, there is a unique point D on $\overrightarrow{BE}$ with $BD \cong AC$. Since D and C lie on opposite sides of line AB, the segment $\overline{CD}$ meets the line AB at a point E. This E is the desired midpoint.

Proof that E is the midpoint. Omitted.

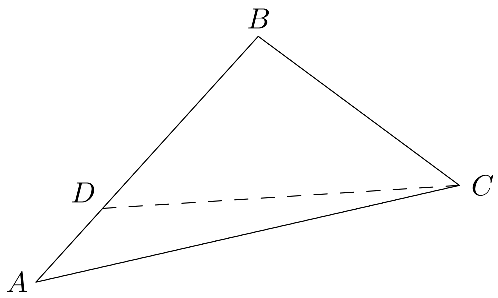
10.4 Foot of perpendicular from A to ℓ.



Given line ℓ and point $A \notin \ell$. Let C, D be two points on ℓ. Let H_1 be the ℓ-halfplane determined by A, and let H_2 be the opposite halfplane. By C4, there is a ray $\overrightarrow{CE}$ such that $E \in H_2$ and $\angle ECD \cong \angle ACD$. By C1, there is a point B on $\overrightarrow{CE}$ such that $CB \cong CA$. Since A and B are in opposite halfplanes, the line AB intersects ℓ, and by non-collinearity, the intersection is unique, and may be called F. The angle $\angle AFC$ is a right angle, and the construction is complete.

Proof that $\angle AFC$ is a right angle. Omitted.

10.6 Larger side opposite larger angle.



Without loss of generality, we may assume that $BA > BC$. Thus we wish to prove that $\angle A < \angle C$. By Axiom C1, there is a point D on $\overrightarrow{BA}$ such that $BD \cong BC$. Clearly A, D and C are non-collinear and thus form a triangle, and D, B and C are also non-collinear... [*supply reasons for the non-collinearity assertions*]. Thus $\triangle DBC$ is an isosceles triangle, and so $\angle CDB \cong \angle DCB$, by Euclid I.5 (which is proved for our context on pages 97–98).

Since $BA > BC \cong BD$, we have $B{*}D{*}A$. Therefore $\angle CDB$ is an exterior angle of $\triangle CAD$, and so $\angle CAB < \angle CDB$. Moreover $\overrightarrow{CD}$ is interior to $\angle ACB$ because ... [*supply a proof of the interior assertion*], and so $\angle DCB < \angle ACB$. Thus finally we have

$$\angle A \;=\; \angle CAB \;<\; \angle CDB \;\cong\; \angle DCB \;<\; \angle ACB \;=\; \angle C.$$

Warning on 10.6 Don't try copying the angle at A into the angle at C. That way lies a lot of trouble and false conclusions. (I marked some of these, but not all of them.) Do it this way: copy the segment BC.

10.11 Any finite set lies in a halfplane. We begin with these two lemmas:

Lemma 1: *If $\ell || m$, then ℓ lies entirely in one of the two half-planes determined by m.* Proof: Suppose not, then there are points $A, B \in \ell$ with A in one m-halfplane and B in the other m-halfplane. By definition of plane separation, the segment $\overline{AB}$ meets m. Thus ℓ meets m. Thus ℓ and m are not parallel. This contradiction completes the proof of Lemma 1.

Lemma 2: *Let H_1 and H_2 be the two half-planes associated to a line ℓ. If m is a line contained in H_2, then H_1 and ℓ are contained in one of the two* *half-planes of* m. Proof: Suppose we had points $A, B \in H_1 \cup \ell$, such that A and B are in opposite m-halfplanes. Thus there exists $C \in \overline{AB} \cap m$. On the other hand, $A, B \in H_1 \cup \ell$ implies $\overline{AB} \subseteq H_1 \cup \ell$, and we are given $m \subseteq H_2$. Thus

$$C \in (\overline{AB} \cup \ell) \cap m \ \subseteq\ (H_1 \cup \ell) \cap H_2.$$

The last conclusion is contradictory, since we know that H_1, H_2 and ℓ are disjoint sets. This contradiction completes the proof of Lemma 2.

Returning to Exercise 10.11, we proceed by induction. Suppose we are given points P_1, P_2, P_3, $\ldots$, and we wish, for each n, to find a line ℓ_n such that the first n of these points all lie in one of the two ℓ_n-halfplanes.

We proceed by induction. First of all, the truth of the assertion is obvious for $n = 1$. Now, the so-called "inductive step." Let us suppose that we have a line ℓ_{n-1} for which the desired conclusion holds for the first $n - 1$ points. We need to establish the conclusion for the first n points (possibly finding a new line for ℓ_n). Let the two halfplanes of ℓ_{n-1} be denoted H_1 and H_2. We can obviously choose the notation so that points $P_1, \cdots, P_{n-1}$ all lie in H_1. Now if $P_n \in H_1$, we have the desired conclusion for $\ell_n \ell_{n-1}$, and there is nothing more to prove.

So, let us examine the case where $P_n \notin H_1$. By Euclid I.31, there exists a line $m || \ell_{n-1}$ through P_n. (If $P_n \in \ell_{n-1}$, then simply take $m = \ell_{n-1}$.) Let the two sides of m be denoted K_1 and K_2, with K_1 denoting the side that contains P_{n-1}. By Lemma 2, $H_1 \cup \ell_{n-1} \subseteq K_1$. (In case $m = \ell_{n-1}$, we have $H_1 = K_1$.) Let $Q \in K_2$; by Euclid I.31, there exists a line $r || m$ through Q. By Lemma 1, r lies entirely in the m-halfplane K_2, and by Lemma 2, $K_1 \cup m$ lies in a single r-halfplane, which we shall call J.

We now have

$$\{P_1, \cdots, P_{n-1}\} \ \subseteq\ H_1 \ \subseteq\ K_1 \ \subseteq\ J,$$

and moreover, by construction, $P_n \in m \subseteq J$, and so, in fact,

$$\{P_1, \cdots, P_n\} \ \subseteq\ J.$$

Thus we may take r for the desired line ℓ_n, and the inductive argument is complete.

Please do not refer to Axiom (P) (or "the parallel postulate") in doing this problem. First of all, it has not been assumed, and so it is not available. Second, (P) refers to the ***uniqueness*** *of parallels, which we don't need here. What we need is their* ***existence****, which is given to us by Euclid I.31 (mentioned on page 102). Generally speaking, (P) will not be assumed or available to us for the rest of the semester. In fact, the whole point is that (P) is what we are leaving behind.*

Alternate proof (???). In class on Wednesday, there were some attempts at speaking about a proof that doesn't use Euclid I.31. I don't believe I found a complete version of such a proof in looking over the papers. If you think you have one, discuss it with me, and we can post it here. (I do have a slightly different proof, but it still uses I.31.)

Some interesting related facts. 1. In Euclidean geometry, for every finite set there is a triangle whose interior contains that set. In non-Euclidean geometry, there is a set of *four* points that is not contained inside any triangle. **2.** Let X denote the union of two intersecting lines. In Euclidean geometry, there is no halfplane that contains all of X. In non-Euclidean geometry, there is always such a halfplane (although its edge may be light-years from the center of the X).