Mathematics 3210
Spring Semester, 2005
Homework notes, part 3

Please note that, as always, these notes are really not complete without diagrams. I have left that work to you.

Just a little reminder that Euclid's axioms have been left behind. From now on, all steps in proofs must be justified by the incidence, betweenness and congruence axioms, and whatever axioms may come after them.

In looking over this assignment, I noted many places where people were just a tad sloppy about the location of a point through axiom C1. [Since it seemed minor, I left most of them unmentioned, but it would be good to tighten up on this.] Suppose, for instance, that in my writeup of 8.1(a) immediately following, I had said this instead of the third sentence: "On this same ray [namely $\overrightarrow{AB}$], there is a unique point Y with $XY \cong EF$." [This is analogous to what I read on several papers.] First off, it sounds like it's right because it uniquely specifies a point with a ray and a distance. (With a diagram, you can see that there may be two Y's on this ray with $XY \cong EF$.) For uniqueness Axiom C1 requires *the distance from the vertex of the ray*. So in the case at hand, we need to limit consideration to the ray opposite to $\overrightarrow{XA}$; I have implicitly done this by requiring $A * X * Y$.

8.1(a) On ray $\overrightarrow{AB}$ there is a unique point X with $A*B*X$ and $BX \cong CD$ (Axiom C1). The segment AX is by definition equal to $AB + CD$. On this same ray there is a unique point Y with $A * X * Y$ and $XY \cong EF$. The segment AY is by definition equal to $(AB + CD) + EF$.

On ray $\overrightarrow{CD}$ there is a unique point Q with $C * D * Q$ and $DQ \cong EF$. The segment CQ is by definition equal to $CD + EF$. Now let us compare $CQ = CD+EF$ with the segment BY constructed above. We have $B*X*Y$, $C * D * Q$, $BX \cong CD$ and $XY \cong EF \cong DQ$. Therefore, by Axiom C3, $BY \cong CQ = CD + EF$.

So now let us consider the three collinear points A, B and Y. We have $A * B * Y$ and $BY \cong (CD + EF)$. Therefore, by definition of "+," segment AY is equal to $AB + (CD + EF)$. Thus we have

$$(AB + CD) + EF \ = \ AY \ = \ AB + (CD + EF).$$

Comment. Notice that the equation contains four +-signs; therefore you might think that you need to perform the +-construction—invocation of Axiom C1 to get a new point in the right place—four times. Actually, because *of the equality, you only need to perform it three times. The three times were for X, for Y and for Q.*

8.2 Our proof of the first part is by contradiction: assume that $AE \not\cong CF$. According to Proposition 8.4(ii), either $AE < CF$ or $AE > CF$. Without loss of generality, we shall assume that $AE < CF$.

By Axiom C1, there exists F' on the ray $\overrightarrow{CF}$ such that $CF' \cong AE$. Since $AE < CF$, we have $C * F' * F$. We now apply Proposition 8.3, using $A * E * B$, $AB \cong CD$ and $AE \cong CF'$ to obtain $C * F' * F$ and $EB \cong F'D$.

On the other hand, From $EB \cong AE < CF \cong FD$, we deduce that $EB < FD$. Again by Axiom C1, there exists F'' on the ray $\overrightarrow{DF}$ such that $DF'' \cong EB$. Since $BE < FD$, we have $D * F'' * F$, or $F * F'' * D$.

Now consider sides of the line CD relative to the point F. By $C * F * D$, C and D are on opposite sides of F. By $C * F' * F$, C and F' are on the same side of F. By $F * F'' * D$, D and F'' are on the same side of F. Therefore F' and F'' are on opposite sides of F. Therefore $F' \neq F''$.

On the other hand, F' and F'' are points on $\overrightarrow{DF}$ such that $DF' \cong EB \cong DF''$. By the uniqueness part of Axiom C1, we have $F' = F''$. This contradiction completes our proof of the first part.

Then it is required to show that a midpoint, if one exists, is unique. So suppose we have two midpoints: $A * M_1 * B$, $AM_1 \cong M_1A$ and $A * M_2 * B$, $AM_2 \cong M_2A$. We apply the previous part, with $C = A$ and $D = B$, obtaining $AM_1 \cong AM_2$. By Axiom C1 (uniqueness part), $M_1 = M_2$.

Many people forgot to include a proof that midpoints are unique; in some cases I neglected to mention this in grading them. Please be aware that such a proof was asked for in this problem.

8.2, Second proof. Again by contradiction: assume that $AE \not\cong CF$. According to Proposition 8.4(ii), either $AE < CF$ or $AE > CF$. Without loss of generality, we shall assume that $AE < CF$.

Thus there is a point F' with $C * F' * F$ and $CF' \cong AE$. By subtraction of segments (Proposition 8.3), $F'D \cong EB$. Thus

$$CF \cong FD < F'D \cong EB \cong AE,$$

where the displayed inequality comes easily from $F' * F * D$. Since congruences respect inequality (Proposition 8.4) we have $CF < AE$, in contradiction to our assumption that $AE < CF$. This contradiction completes the proof of the first part. The second part (uniqueness of midpoints) is handled as above.

8.2, Third proof, assuming we have already done 8.3. Again by contradiction: assume that $AE \ncong CF$. According to Proposition 8.4(ii), either $AE < CF$ or $AE > CF$. Without loss of generality, we shall assume that $AE < CF$.

It is immediate from the definitions that $AB = AE + AE$ and that $CD = CF + CF$. Therefore, by Exercise 8.3,

$$\begin{aligned} AB \ &= \ AE + AE \ < \ CF + AE \\ &\cong \ AE + CF \ < \ CF + CF \ = \ CD. \end{aligned}$$

This contradiction (to $AB \cong CD$) completes the proof of the first part. The second part (uniqueness of midpoints) is handled as above.

8.3 Given $AB < CD$, we first apply Prop 8.4(a) to get $AB < DC$. By definition of "$<$," there is a point U with $D * U * C$ and $DU \cong AB$. For the addition of segments, there are unique points G and H, with $A * B * G$ and $C*D*H$ and $BG \cong DH \cong EF$. Then $AB+EF = AG$ and $CD+EF = CH$. Now we also have $AB \cong UD$, $BG \cong DH$ and $A * B * G$. By Exercise 8.1(b) we $U * D * H$. Therefore, by the addition axiom C3, we have $AB + EF = AG < HC \cong CH = CD + EF$.

8.4 Let X on ray r correspond to X' (also known as $\phi(X)$) on ray s that is defined by the condition $BX' \cong AX$. By Axiom C1, this X' is well-defined: there is one such X' and only one such X'. The correspondence has an inverse: given X' on ray s, we may let X' correspond to the unique X on s that satisfies $AX \cong BX'$; this two-way correspondence obviously establishes a one-one correspondence between X and X'.

First proof that $X \longleftrightarrow X'$ respects betweenness and congruence: We begin with two points X and Y that satisfy $A * X * Y$, i.e. for which $AX < AY$. Since we have $BX' \cong AX$ and $BY' \cong AY$, an appeal to Proposition 8.3 immediately yields $B*X'*Y'$ (i.e. $BX' < BY'$). and $X'Y' \cong XY$. Thus we have proved that the desired result about congruence, and a special case of betweenness.

For the general case of betweenness, let us suppose that X, Y, Z on r satisfy $X * Y * Z$. Thus X and Z are on opposite sides of Y. Without loss of generality we will assume that A is on the X-side of Y; in other words $A*X*Y$. By Exercise 8.1(a), we have $A*Y*Z$. By the previous paragraph, we have $B * X' * Y'$ and $B * Y' * Z'$. By 8.1(b), we have $X' * Y' * Z'$.

Alternate proof of preserving. Let us be given $A * X * Y * Z$. Now define X'', Y'', Z'' as follows. X'' is the unique point (by C1) on s so that $BX'' \cong AX$. (Thus $X'' = X'$, by uniqueness.) Next, Y'' is defined to be the unique point on the ray opposite $X''B$ such that $X''Y'' \cong XY$. By the addition axiom, $AY \cong BY''$. By uniqueness, $Y'' = Y'$. Thus $X'Y' \cong XY$, and preservation of congruence is established.

Continuing, define Z'' to be the unique point on the ray opposite $Y''X''$ such that $Y''Z'' \cong YZ$. In particular, we have $X'' * Y'' * Z''$. By the addition axiom, $AZ \cong BZ''$. By uniqueness, $Z'' = Z'$. Now, by substituting equals for equals, $X'' * Y'' * Z''$ immediately yields $X' * Y' * Z'$. Thus betweenness is preserved.

8.5(a) By Proposition 7.2 (line separation), a line ℓ through O comprises two rays; call them $\overrightarrow{OX_1}$ and $\overrightarrow{OX_2}$. According to Axiom C1, $\overrightarrow{OX_1}$ contains exactly one point C_1 with $OC_1 \cong OA$, which is to say that $\overrightarrow{OX_1}$ meets the circle in the single point C_1. Similarly, $\overrightarrow{OX_2}$ meets the circle in a single point C_2. Therefore ℓ meets the circle in the two-point set $\{C_1, C_2\}$.

8.5(b) Suppose that ℓ and m are distinct lines through O. By part (a) ℓ meets the circle at a point Q, and m meets the circle at a point R. We claim that $Q \neq R$ are non-collinear. If not, then ℓ and m would be two lines containing the two points O and Q, in contradiction to Axiom I1. In other words, we see that distinct lines through O contain distinct points of the circle. Therefore, to complete the Exercise, it will be enough to demonstrate that there are infinitely many lines through O.

Let n be a line that does not contain O (incidence axioms). As in the previous paragraph, it follows from Axiom I1 that distinct points on n determine distinct lines through O. (If Q, R are distinct points on n, then $OQ \neq OR$.) Since n has infinitely many points (by Exercise 7.4), there must be infinitely many lines through O.