Mathematics 3210
Spring Semester, 2005
Homework notes, part 2

Here is a little hint about proofs in the Hilbert axiomatization. There is no need to include steps that command one to draw the line or segment between two points, or to draw an angle, etc. In fact the word "draw" does not occur in the vocabulary or the formal apparatus of the Hilbert-style geometry—which of course is the style of modern mathematics in general. (The line connecting distinct points A and B exists by Axiom I1. Its existence is primordial, existing as soon as we have A and B.)

Of course such steps may have a psychological value: if not overdone, they serve to bring your attention to the right place. If you wish to include them, they can be seen as doing no harm, but you should remain aware that they don't accomplish anything. In a discursive proof of the modern style, the trend would be more to say something like, "now let us consider the line that joins A and B."

Please be careful about existential axioms like B2: "there exists a point C such that $A * B * C$." *All* you know about C is that $A * B * C$. It is not necessarily equal to any point that you already have. It is a *new point.* Thus, if you already had a point called C in your discussion, you must call the new point by a name other than C.

6.6(a) We choose Example 6.1.3, on page 68. It was proved there that this is an incidence geometry. From the diagram there, it is clear that $EA \| DC$ and $DC \| EB$ (in each case, there is no point of intersection). If parallelism were an equivalence relation in this example, we could apply transitivity to obtain $EA \| EB$. But this is manifestly false in the example, for the lines EA and EB have the point E in common. (Note: one small misunderstanding was common here. For an *example* in geometry, you need to be speaking of a definite plane. In §6, the kind of plane we are talking about is an incidence geometry.)

(b) *If we assume the parallel postulate (P), then parallelism is an equivalence relation.* This could be phrased as a proof by contradiction, but it works a little more smoothly to prove instead the contrapositive: if parallelism fails to be an equivalence relation, then (P) is false. Now it is evident from the definition (top of page 68) that reflexivity and symmetry hold regardless of (P). So, if parallelism is not an equivalence relation, it is because there is a failure of transitivity. In other words, there exist lines ℓ, m and n with $m \| \ell$, $\ell \| n$, but m not parallel to n. The latter assertion means simply that there is a point Q that lies on $m \cap n$. Now our failure of Axiom (P) is clear: lines m and n are two parallels to ℓ through the point Q.

(c) *Conversely, if parallelism is an equivalence relation in a given incidence geometry, then (P) must hold in that geometry.* Again the contrapositive: we will assume that (P) fails and then show that parallelism is not an equivalence relation. Well to say that (P) fails is simply to say that there are a line ℓ and a point Q with two parallels, m and n to ℓ through Q. In other words, we have $m\|\ell$, $\ell\|n$, but m not parallel to n. This failure of transitivity means that parallelism is not an equivalence relation. (Note this: when a proposition and its converse are both true—as in (b) and (c) here—it is sometimes (not always) possible for each of the two proofs to have the same steps, but in the reverse order. If you look at my proofs of (b) and (c), you will see that this is what happened here.)

7.1(a) (use Prop. 7.2). Let ℓ denote the line joining A and B. To prove $A * B * D$ from the assumptions, let us use Prop. 7.2 to divide ℓ at B. The assumptions (together with clauses (a) and (b) of the proposition) give you that A and C are on opposite sides of B, and that C and D are on the same side of B (on ℓ). Since ℓ has only two sides relative to B, you should be able to move to the desired conclusion from here. The other parts are similar (and you should do them).

7.2 (use Ex. 7.1(a)) If such C and D existed, we would then have (a) $C * A * D$ and $C \in \overline{AB}$ and $D \in \overline{AB}$. The latter condition means that either (i) $D = A$ or (ii) $D = B$ or (iii) $A * D * B$. Clearly (i) contradicts B1, and (ii) (together with $A * C * B$) contradicts B3. Finally (iii) and (a), using 7.1(a), yield $C * A * B$, another contradiction to B3.

7.2, alternate proof (not using 7.1(a)). We are given $A * C * B$; hence B and C lie on the same side of A in the line AB. We are also given that $A * D * B$; hence B and D lie on the same side of A. Since C and D both lie on the B-side of A, we may say that C and D lie on the same side of A.

On the other hand $C * A * D$ says that C and D lie on opposite sides of A, in contradiction to what we just proved. Thus $C * A * D$ is impossible for $C, D \in \overline{AB}$.

7.4 Infinitely many points on a line ℓ. By Axiom I2, we may select two points A_0 and A_1 on ℓ. By Axiom I3, there exists A_2 on ℓ so that $A_0 * A_1 * A_2$. It follows from Axiom A1 that the three points we have so far are distinct.

Proceeding recursively, let us suppose that we have already defined points $A_0 \cdots A_n$ which are all distinct and satisfy $A_k * A_{k+1} * A_{k+2}$ for each appropriate k. We now apply Axiom B2 to get a point A_{n+1} such that $A_{n-1} * A_n * A_{n+1}$. We need to see that A_{n+1} is distinct from each previously constructed A_k. Beginning with

$$A_k * A_{k+1} * A_{k+2}$$
$$A_{k+1} * A_{k+2} * A_{k+3},$$

we apply exercise 7.1(a) to get

$$A_k * A_{k+2} * A_{k+3}.$$

Combining this with

$$A_{k+2} * A_{k+3} * A_{k+4},$$

we get

$$A_k * A_{k+3} * A_{k+4}.$$

Carrying on in this manner[1],we finally get

$$A_k * A_n * A_{n+1},$$

which tells us (Axiom 1) that A_{n+1} is distinct from A_k. Since we can do this for every k, we see that A_{n+1} is a new point. Now we can do all of this for every n, thus creating infinitely many new points A_n on ℓ. Say goodbye to those finite geometries of the previous section.

7.4, alternate writeup. This version is a little less formal and rigorous, but it conveys the idea, and may be a little easier to understand. The axioms immediately give us two points on ℓ, which we will call A and B. Next, we apply Axiom B2 to get a point C with $A * B * C$. By Axiom B1, the three points A, B and C are distinct. By axiom B3, there exists D on ℓ with $B * C * D$. Axiom B1 tells us that B, C and D are distinct, but we still have to face the possibility that A and D might be the same point. For this, we invoke Exercise 7.1(a). It tells us that $A * C * D$; and hence, by Axiom B1, $A \neq D$. We now have *four*, and now we go for a fifth.

By Axiom B2, there is a point E on ℓ with $C * D * E$. Axiom B1 tells us that C, D and E are distinct, but there is still the possibility that $E = A$ or $E = B$. By construction (previous paragraph) we have $B * C * D$. This, together with $C * D * E$ and Exercise 7.1(a) yields $B * C * E$, and so $B \neq E$. And this last betweenness assertion, together with $A*B*C$, yields $A*B*E$. Hence finally $A \neq E$, and so we have five points.

Our next move would be to get a sixth point F with $D*E*F$. The proof that F is distinct from the other five is similar to what has come before, except that now we need separate arguments for $E \neq A$, $E \neq B$ and $E \neq C$. It should now be clear that these arguments extend indefinitely through points G, H and so on, ultimately yielding infinitely many points on ℓ.

7.4, third proof. Here we give a presentation that involves the extra work of proving a lemma, but then is much more straightforward and conceptual.

Lemma *If $A * B * C$, then $\overline{AB} \subset \overline{AC}$, i.e. the segment $\overline{AB}$ is a proper subset of the segment $\overline{AC}$.* Proof of lemma. First $\overline{AB} \subseteq \overline{AC}$. For this inclusion of sets, we will assume that $Q \in \overline{AB}$, and prove that $Q \in \overline{AC}$. We leave the case of $Q = A$ and $Q = B$ to the reader. Thus it remains to prove

[1]Officially, the method here would be a proof by mathematical induction, which you may have studied already. We have enough else to do this semester, so we won't spend any time talking about inductive proofs.

that if $A * Q * B$ and $A * B * C$, then $A * Q * C$. Consider the two sides of point B on line AB. Clearly A and Q are on the same side of B, while A and C are on opposite sides of B. Therefore C and Q are on opposite sides of B; whence $Q * B * C$. Now from $A * Q * B$ and $Q * B * C$, using Exercise 7.1(a), we have the desired relation $A * Q * C$. As for the fact that it is a proper subset, we clearly have $C \in \overline{AC}$ but not in $\overline{AB}$.

Now the proof of 7.4 (infinitely many points on a given line ℓ) from the Lemma. We know by I1 that there are two points A_1 and A_2 on ℓ. Then by repeated use of Axiom B2, we have points A_3, A_4, …, such that

$$\begin{gathered} A_1 * A_2 * A_3 \\ A_1 * A_3 * A_4 \\ A_1 * A_4 * A_5 \\ \vdots \end{gathered}$$

By the lemma, we have

$$\overline{A_1A_2} \subset \overline{A_1A_3} \subset \overline{A_1A_4} \subset \overline{A_1A_5} \subset \cdots,$$

which is an infinite increasing sequence of subset of ℓ. A finite set cannot have such a sequence of subsets; hence, ℓ is an infinite set.

7.6 Choose a point D so that A, B and D are non-collinear (Axiom I3.) We now make three successive applications of Axiom B2: Choose E so that $B * D * E$, then F so that $A * E * F$, and then G so that $F * D * G$. We want to have G in the interior of $\angle ADB$. This means we need to see (i) that A and G are on the same side of line BD, and (ii) that B and G are on the same side of line AD. I will do (i) for you (you should do both of them). By construction, E lies on BD. Therefore, since $A * E * F$, we have A and F on opposite sides of BD. Then since $F * D * G$, we have F and G on opposite sides of BD. Therefore A and G lie on the *same* side of BD.

Thus we have seen that G is in the interior of $\angle ADB$. Therefore, by the Crossbar Theorem, the ray $\overrightarrow{DG}$ intersects the line AB at a point C between A and B, as desired.

(Notes: 1. I have not supplied a diagram for 7.6, but since I have based my constructions on the axioms, I am confident that if you follow the instructions you will arrive at a correct diagram. 2. This construction may not be the simplest one available. In trying for a simpler construction make sure that you never assume what we are trying to prove.)

7.9 (use Ex. 7.6) Let us be given $\triangle ABC$. By Exercise 7.6, there exists a point D with $A * D * C$. A second application of 7.6 yields a point E with $B * E * D$. We claim that E is an interior point of $\triangle ABC$, and hence that this interior is not empty. For this we need: (i) E is on the B-side of AC, (ii) E is on the A-side of BC, and (iii) E is on the C-side of BA. I will do (i) and (ii); you should do all three.

(i): Line BE meets line AC at one point only, namely D. We have $B * E * D$, and hence D is not on $\overline{BE}$. Thus line AC does not meet $\overline{BE}$. Thus E is on the B-side of AC.

(ii): Similarly, line BC does not meet segment $\overline{ED}$, and line BC does not meet segment $\overline{AD}$. Thus E and D and A are all on the same side of BC. In particular, E is on the A-side of BC.

7.10 Ray $\overrightarrow{AD}$ is in the interior of $\angle BAC$; by the crossbar theorem it meets an interior point E of side $\overline{BC}$. Consider now $\triangle AEB$; line ℓ contains an interior point (namely D) of side $\overline{AE}$, and hence by Axiom B4, it contains a point of $\overline{AB}$ or a point of $\overline{BE}$. But of course $\overline{BE} \subset \overline{BC}$, so we really have a point of $\overline{AB}$ or a point of $\overline{BC}$, as required.