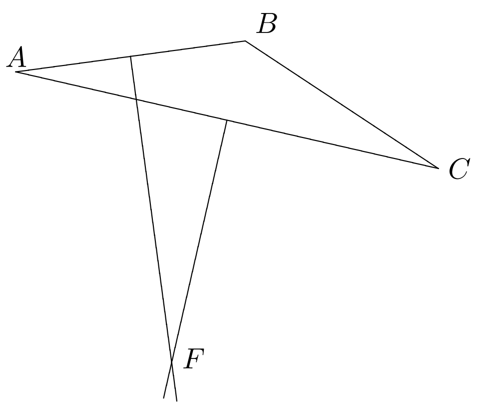
Mathematics 3210
Spring Semester, 2005
Homework notes, January 31, 2005

Be advised that these are remarks about the solutions to some of the assigned problems in 3210. In many cases they do not constitute full write-ups. (That is still up to you.)

Here are five points I made in class on January 24. They are worth remembering when proving anything in mathematics.

1. Write nothing false. This means write nothing that can't be backed up with a proof. (Being true, but you can't prove it, isn't good enough.) It's better to say "I'm stuck here" than to try to fake it.
2. Don't guess. (Closely related to the previous one.)
3. Don't assume the conclusion. There are subtle ways that this can happen, so always be on guard.
4. Avoid any extraneous pictorial conclusions.
5. If you get stuck, make sure that you have used all the information (hypotheses) that was given in the statement of the problem. Also, even if you have what you think is a proof, it is a good idea to review whether you have used everything that was given. (If not, there are two possibilities: (i) your proof is wrong, or (ii) what you were given was more than you need.)

1.9



It helps to begin with just *two* of the perpendicular bisectors, say the bisectors of AB and AC, as indicated. Then use an isosceles triangle to prove that $AF = FB$, and use a second isosceles triangle to prove that $AF = FC$. (Add some further lines to the picture if that helps you.) From this you can deduce that $FB = FC$ and hence that F is on the perpendicular bisector of BC.

1.13 It is useful to add the center to each circle and three radii in each circle. You then have lots of isosceles triangles to work with. There is a less elementary proof (discovered by some class members) that uses Euclid III.32 (which talks about the angle between a tangent and a chord.

1.14 On this problem it helps to begin with just *two* of the angle bisectors, and call their intersection F. You can then prove that F lies on some of the other angle bisectors, and proceed from there.

2.12 Construct the perpendicular to ℓ at B. Construct the perpendicular bisector of the segment AB. Intersect these two lines to find the center of the required circle.

2.15 I did this one in class.

3.1 The swindle in this false proof is subtle. All the alleged congruent triangles are actually congruent! Here's what really happens: The proof exhibits two cases, which are supposed to cover all the possibilities. Case 1 has F and G interior to their respective segments $\overline{AB}$ and $\overline{AC}$. (In this case, the lengths AB and AC are each formed by addition of two smaller lengths.) Case 2 has F and G each exterior to their respective segments $\overline{AB}$ and $\overline{AC}$. (In this case the lengths AB and AC are each formed by subtraction of appropriate lengths.)

The problem is that the author has (deliberately) neglected to mention Case 3: it may be that F is *interior* to $\overline{AB}$ while G is *exterior* to $\overline{AC}$ (or the other way around). In that case we would have to add lengths on the left and subtract lengths on the right. *(A sum of equals is equal to a difference of equals? Ridiculous.)* And, of course, Case 3 is really the only one that occurs for non-isosceles triangles.

A precise drawing for a non-isosceles triangle will of course reveal Case 3. Nevertheless, the deception has been made easier by the fact that the two bisectors—angle bisector and perpendicular bisector—are nearly parallel, and the intersection point of such lines is difficult to locate with any precision. Therefore it is hard for the eye to detect a phony picture.

At this point it should be clear why we need an axiomatic approach to betweenness.

(Incidentally, the final sentence is OK: "If E lands at the point D, or if the angle bisector at A is parallel to the perpendicular to AB at D, the proof becomes even easier ... ." This possibility concerns only the special case of a triangle that was isosceles in the first place, so there is no problem here.)

3.3 The core idea for this problem may be found in the proof of Proposition 24.4 on page 214.

4.8 (Paper-fold pentagon.) This one seems to require a lot of patience and fortitude (mathematics is sometimes like this). No really short submission was even close to correct, and I suspect there is no really short slick proof. (I could of course be proved wrong on this!) Among the longer proofs that were submitted, I tended to spot-check the findings, and did not pursue each piece of logic to its end. Thus there may be some undetected small errors, but if I made no marks on your page it is probably close to correct.

Beware of Euclid I.38 which says that triangles on the same base between parallel lines are equal. In that Theorem "equal" does not mean "congruent"; it means only that they have the same content or area. One is tempted to use this on the folded pentagon, since there are some conveniently located triangles of this type. Also, do not confuse Euclid I.27 with Euclid I.29. The former says that if alternate interior angles are equal, then the lines in question are parallel. Its *converse* is I.29: if the lines are parallel, then the angles are congruent. Now I.27 is a theorem of neutral geometry, that is to say, true both in Euclidean and in non-Euclidean geometry, whereas I.29 fails in non-Euclidean geometry.

In writing up a problem like 4.8, it is important to state clearly all the steps. It is not much use to have merely a diagram in which one has marked off various congruent angles and segments, because there remains no indication of the order in which these things were found.

I have one solution to this, which I will hand out in class.

6.1 There are two possibilities: (i) six two-point lines, and (ii) one three-point line and three two-point lines. The book does expressly request it, but as ever in mathematics it would be nice to have some reasons for why there are no other possibilities. (E.g. No four-element line is possible here (because ...). We cannot have two three-element lines (because ...). If there is exactly one three-element line, then ... (because ...). If there is no three-element line, then we must have ... (because ...).)

6.3(a) Start with three non-collinear points A, B, C. Let Q be a third point on line AB; ditto R for BC and S for CA. At this point you need to check that all six points are distinct. There are a number of potential equalities to check, each of which leads to a violation of the axioms. Finally, let X be a third point on line BS. You need to check that it is distinct from A, Q, C and R. We now have seven points.

To make these seven into a plane, let X also lie on the lines AR and CQ, and stipulate that there is one more line, namely $\{Q, R, S\}$. Thus the seven lines are $\{A, Q, B\}$, $\{B, R, C\}$, $\{C, S, A\}$, $\{A, X, R\}$, $\{B, X, S\}$, $\{C, X, Q\}$, and $\{Q, R, S\}$. For a drawing, try putting A, B and C at the corners of a triangle. Put Q halfway between A and B; similarly for R and S. Then put X in the center. All the lines will look linear in the plane, except for $\{Q, R, S\}$, which must be represented as a circle.

The seven-point projective plane is unique up to isomorphism, and is known as the Fano plane, after its discoverer. You can find thousands of references to it on Google.

6.5(a) Suppose that m and m' are two lines that intersect at A. We will show that m and m' have the same number of points. We do this by exhibiting a one-one correspondence between the points on m and the points on m'. Pick a point $B \neq A$ on m and a point $B' \neq A$ on m'; let ℓ be the unique line joining B and B'. Now for any C on m other than A and B, we will construct a corresponding C' on m'. Let r be the unique parallel to ℓ through C, and let C' be the point where r intersects m'. (To do this, we need to see that r and m' are not parallel. If they were, m' and ℓ would both be parallel to r through B', in contradiction to (P$'$).) To complete the correspondence, we let $A' = A$. Now for each point X on m there is a point A' on m'.

To get an inverse map, we reverse the procedure: given C' on m' , we let r be parallel to ℓ through C', and then define C to be the point of intersection of r and m.

That completes the proof for two lines that intersect. If lines m and n are parallel, there is clearly a line s that intersects each of them. Therefore, two applications of the first part give us the same number of points in m and in n.

6.5(b) Let m and ℓ be two lines that intersect in point A. For each point Q in the plane, we define two coordinates, $x(Q)$ and $y(Q)$. For $x(Q)$ we draw a line parallel to ℓ through Q and let $x(Q)$ be its point of intersection with m. For $y(Q)$ we draw a line parallel to m through Q and let $y(Q)$ be its point of intersection with ℓ. (You need to prove—using (P$'$)—that the desired points of intersection really exist.) Thus we have a correspondence between points of the plane and pairs (x, y) of points from m and ℓ. (You should add a proof that no two points have the same pair (x, y) and that every pair (x, y) corresponds to some Q—one further use of (P$'$).) Since there are $n \times n = n^2$ pairs of co-ordinates (n possibilities for x, and n possibilities for y), there must be exactly n^2 points in the plane.