Chapter Six
Hilbert's Seventh Problem and Transcendental Functions

So far we have not said much about an important portion of Hilbert's Seventh Problem, wherein he said

we expect transcendental functions to assume, in general, transcendental values for [...] algebraic arguments [...] we shall still consider it highly probable that the exponential function $e^{i\pi z}$ *[...] will [...] always take transcendental values for irrational algebraic values of the argument* z

In other words, Hilbert speculated that if $f(z)$ is a transcendental function and if α is an irrational algebraic number then $f(\alpha)$ is a transcendental number.

The function $e^{i\pi z}$, which Hilbert explicitly mentions, is covered by the Gelfond-Schneider Theorem because $e^{i\pi} = -1$ is an allowable value of α. A simple question is : For which numbers γ do we already know that the function $e^{\gamma z}$, in Hilbert's words, *always take transcendental values for irrational algebraic values of the argument* z. The partial answer we can already give to this question comes in two parts. The first part preceded Hilbert"s lecture. The Hermite-Lindemann Theorem established the transcendence of e^{α} for any non-zero algebraic number α. It follows, of course, that if in the original question we take γ to be any non-zero algebraic number then the function $e^{\gamma z}$ certainly takes on transcendental values for any non-zero algebraic values of the argument z. The second part of our answer to this question comes from the Gelfond-Schneider Theorem. If γ is the natural logarithm of any algebraic number $\alpha \neq 1$ then the function $e^{\gamma z} = \alpha^z$ also takes on transcendental values for any irrational algebraic values of the argument z. Thus we have the partial answer to the original question: For any $\gamma \in \{\alpha, \log \alpha : \alpha$ an algebraic number different from 0 or $1\}$ the function $e^{\gamma z}$ takes on transcendental values for any irrational algebraic values of the argument z. We note that although $\gamma = i\pi$ is in the above set of values it is, unfortunately, still a fairly small set of values; for example it is certainly countable.

The disclaimer *in general* in Hilbert's posing of his problem that transcendental functions should take transcendental values at irrational algebraic numbers saved him from possible embarrassment when counter-examples to the most general interpretation of this conjecture were given. We will not consider this topic here but notice that it is easy to exhibit a transcendental function that is algebraic at any number of prescribed algebraic numbers. For example, if $\alpha_1, \alpha_2, \ldots, \alpha_L$ are algebraic numbers, then

$$f(z) = e^{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_L)} \tag{1}$$

produces an algebraic value for each of $z = \alpha_1, \alpha_2, \ldots, \alpha_L$.

Despite the simplicity of the above counterexample to one interpretation of Hilbert's conjecture, a slight generalization of the question posed by Hilbert has proven to be a fruitful one for transcendental number theory. Instead of considering the values of a single function it is natural to consider the values of two functions simultaneously. The reason this is a natural generalization of Hilbert's question is because this point of view is already implicit in his question. Asking whether a transcendental function $f(z)$ takes on transcendental values for algebraic values of the argument is equivalent to asking: Can the functions $f(z)$ and z be simultaneously algebraic? (Or, more precisely, can the functions $f(z)$ and z be simultaneously algebraic, possibly with a finite number of exceptions?) The Hermite-Lindemann Theorem says that e^z and z are not simultaneously algebraic except for $z = 0$. The Gelfond-Schneider Theorem has two possible statements in terms of functions and their values. One says that for an algebraic number $\alpha \neq 0, 1$ the functions z and α^z are not simultaneously algebraic except when z is a rational number. Another version of the Gelfond-Schneider Theorem says that if β is an irrational algebraic number then the functions e^z and $e^{\beta z}$ cannot be simultaneously algebraic except when $z = 0$.

In this lecture we consider the question of when two algebraically independent functions can be simultaneously algebraic, and see that some important, but far from definitive, steps have been taken towards answering this question. We will then expand our dictionary of available functions beyond z and e^z to include elliptic functions. We first reframe the Gelfond-Schneider Theorem to involve two functions at two points.

Theorem. *Given an irrational $\xi \in \mathbf{C}$, the two functions $e^{\xi z}$ and z cannot be simultaneously algebraic at two $\mathbf{Q}$-linearly independent complex numbers x_1 and x_2.*

Note: In other words at least one of the four numbers

$$x_1, x_2, e^{\xi x_1}, e^{\xi x_2}$$

is transcendental.

Proof. If all four of the numbers $x_1, x_2, e^{\xi x_1}$, and $e^{\xi x_2}$ are algebraic then the numbers ξx_1 and ξx_2 are logarithms of algebraic numbers, namely of $e^{\xi x_1}$ and $e^{\xi x_2}$, respectively. We also observe that by our hypothesis, ξx_1 and ξx_2 are $\mathbf{Q}$-linearly independent. Thus, by the second version of Hilbert's seventh problem, the ratio $\frac{\xi x_1}{\xi x_2} = \frac{x_1}{x_2}$ is transcendental, which contradicts the assumption that x_1/x_2 is algebraic. Therefore at least one of the four numbers in the theorem must be transcendental.

The Six Exponentials Theorem

Taking z as one of the functions under consideration allows us to restate what might be called *classical* transcendence theorems. To move into the modern era we want to expand the type of functions under consideration, and before we move on to the so-called Weierstrass $\wp-$function we consider two, algebraically independent exponential functions.

With these results as inspiration, in this section we deduce a theorem that straddles the line between *classical* and *modern* transcendental number theory—the so-called *Six Exponentials Theorem*. This result is classical in that its proof is simply an elaboration of our proof of the Gelfond–Schneider Theorem and yet is modern in that it examines the transcendence of values of special functions rather than of particular numbers.

Theorem *Let $\{x_1, x_2\}$ and $\{y_1, y_2, y_3\}$ be two* **Q***-linearly independent sets of complex numbers. Then at least one of the six numbers*

$$e^{x_1y_1},\quad e^{x_1y_2},\quad e^{x_1y_3},\quad e^{x_2y_1},\quad e^{x_2y_2},\quad e^{x_2y_3}$$

is transcendental.

For example, if we consider the sets $\{1, e\}$ and $\left\{e, e^2, e^3\right\}$, then the Six Exponentials Theorem implies that at least one of the following numbers is transcendental:

$$e^{e}, e^{e^2}, e^{e^3}, e^{e^4}\ .$$

Restatement of the Six Exponentials Theorem. *Two algebraically independent exponential functions, e^{x_1z} and e^{x_2z} cannot be simultaneously algebraic at three* **Q***−linearly independent complex numbers $y_1, y_2,$ and y_3.*

Sketch of proof The proof of the Six Exponentials Theorem closely parallels Schneider's solution of Hilbert's seventh problem, so we will be brief.

Step 1. Assume that all of the values $e^{x_iy_j}$ are algebraic. Thus for any $P(x, y) \in \mathbf{Z}[x, y]$, we notice that the values of the function $F(z) = P(e^{x_1z}, e^{x_2z})$ will be algebraic when evaluated at y_1, y_2, y_3, or any $\mathbf{Z}$−linear combination of them. That is, for any integers k_1, k_2, and k_3, the quantity $F(k_1y_1 + k_2y_2 + k_3y_3)$ is an algebraic number.

Step 2. Apply Siegel's Lemma to find a nonzero integral polynomial

$$P(x, y) = \sum_{m=0}^{D_1-1}\sum_{n=0}^{D_2-1} a_{mn}x^m y^n\ ,$$

having "modestly sized" integral coefficients, such that if we let

$$F(z) = P(e^{x_1z}, e^{x_2z})\ ,$$

then $F(z) = 0$ for all $z \in \{k_1y_1 + k_2y_2 + k_3y_3 : 0 \le k_j < K\}$. Before proceeding to the next step, we note that since the two functions e^{x_1z} and e^{x_2z}, which we compose with $P(x, y)$ in order to produce $F(z)$, are so similiar, it is not natural to take $D_1 > D_2$ or $D_2 > D_1$. Thus we now declare that $D_1 = D_2$ and denote this common value by D.

Step 3. As we have seen there are several ways to obtain a nonzero value that will, if everything is set up correctly, lead to a contradictory nonzero integer.

In this proof we use a zeros estimate based on the observation that $F(z)$ is not identically zero. Specifically, the following lemma ensures that the function $F(z)$ has an advantageous nonzero value.

Lemma. *There exists a positive integer M such that*

$$F(k_1y_1 + k_2y_2 + k_3y_3) = 0 \ ,$$

for all $0 \le k_j < M$, while there exists some triple k_1^*, k_2^*, k_3^*, satisfying $0 \le k_j^* \le M$, such that

$$F(k_1^*y_1 + k_2^*y_2 + k_3^*y_3) \ne 0 \ .$$

(The proof of this lemma is an exercise at the end of the chapter.)

Step 4. It is possible to use the nonzero algebraic number $F(k_1^*y_1+k_2^*y_2+k_3^*y_3)$ to obtain a nonzero integer whose absolute value is less than 1.

The Schneider-Lang Theorem.

In this section we consider a conjecture which is an natural analogue of Hilbert's, albeit for two functions. This conjecture captures the essence of Hermite's result, the Hermite-Lindemann Theorem, and the Gelfond–Schneider Theorem.

First Conjecture. *Two algebraically independent functions should not be simultaneously algebraic at a point, unless there is some special reason. That is, if $f(z)$ and $g(z)$ are algebraically independent functions, then for just about any $z_0 \in \mathbf{C}$, at least one of the values $f(z_0)$ or $g(z_0)$ should be a transcendental number.*

Our earlier example of the function $f(z) = e^{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_L)}$, which is algebraically independent of the function $g(z) = z$, shows the need for the phrase "unless there is some special reason" in the above conjecture. That said, many of our previous results point to the truth of this admittedly vague conjecture; for example, Hermite's theorem implies that z and e^z are not simultaneously algebraic, *except* at $z = 0$.

Refined Conjecture. *Two algebraically independent functions cannot simultaneously be algebraic at very many different complex numbers.*

Both our reframing of the Gelfond-Schneider Theorem and the Six Exponentials Theorem point to the linear independence of the points under consideration as being a reasonable hypothesis. While this point of view remains an important one, there is another point of view that leads to an important result. This point of view is to refine the phrase "simultaneously algebraic" in the above refined conjecture.

Before we examine this portion of the Refined Conjecture we point out that the example we gave above indicates that the number of points at which the two functions are simultaneously algebraic must depend on some specific properties of the functions themselves. Specifically, if $P(z)$ is any nonzero polynomial with rational coefficients, of degree $d \geq 1$, then the algebraically independent functions

$$f(z) = z \quad \text{and} \quad g(z) = e^{P(z)}$$

are simultaneously algebraic at the d zeros of $P(z)$. So in this example "too many" must be connected with the degree of $P(x)$.

What cannot be taken away from the above example is how the degree of $P(z)$ plays a role in determining an upper bound on the number of simultaneous algebraic points for z and $e^{P(z)}$. There is a clue in our earlier zeros estimate based on the order of growth of the function. Recall that that result established that if a nonzero entire function $F(z)$ satisfies $|F|_R \leq e^{R^\kappa}$ for all sufficiently large R, then F cannot have more than $R^{\kappa+\epsilon}$ zeros in the disks of all complex numbers z satisfying $|z| \leq R$ a R approaches infinity. This result implies that for any particular complex number β, such a nonconstant entire function $F(z)$ satisfies

$$\operatorname{card}\Big\{z \in \mathbf{C} : F(z) = \beta \text{ with } |z| \leq R\Big\} < R^{\kappa+\epsilon} \ .$$

So the number of times such an entire function can attain any particular algebraic value is bounded by a function of R and κ. Extending this observation, it is reasonable to imagine that κ might influence how many times a particular entire function takes values in any fixed *set* of algebraic numbers, or more generally in any finite extension K of $\mathbf{Q}$.

We have never formally described the exponents κ that satisfy the above. To do so we first say that an entire function $F(z)$ has *finite order of growth* if there exists a positive constant κ such that for all sufficiently large $|z|$,

$$|f(z)| < e^{|z|^\kappa} \ .$$

In the above situation $f(z)$ is said to have a finite order of growth; we then define the *order of* $f(z)$ by

$$\rho = \inf\Big\{\kappa > 0 : |f(z)| < e^{|z|^\kappa} \text{ for all sufficiently large } |z|\Big\} \ .$$

When considering the simultaneous algebraic values of two algebraically independent functions, the orders of growth of the two functions could play a role. Surprisingly, however, we will see that if we wish to give an upper bound for the number of points in a disk at which two algebraically independent functions simultaneously take values from a prescribed collection of algebraic numbers, neither the radius of the disk nor the cardinality of the set of algebraic values appears in our bound. Instead, the upper bound depends only on the degree of the field extension of $\mathbf{Q}$ containing the given set of algebraic numbers and the orders of growth of the functions. Stated only slightly more precisely, if $f_1(z)$

and $f_2(z)$ are *sufficiently nice* functions with finite orders of growth ρ_1 and ρ_2, respectively, and K is an algebraic extension of $\mathbf{Q}$ of degree d, then

$$\mathrm{card}\Big\{z \in \mathbf{C} : f_1(z) \in K \ \text{ and } f_2(z) \in K\Big\} \le (\rho_1 + \rho_2)d \ .$$

We are now able to produce a special case of an important result due to Serge Lang (1964) known as the Schneider–Lang Theorem. In 1949 Schneider proved two general theorems concerning two algebraically independent function being simultaneously algebraic at numbers, but Lang's formulation is particularly succient. In this version the theorem deals with meromorphic functions with finite orders of growth that satisfy polynomial differential equations with algebraic coefficients. We first restrict our attention to entire functions; we will return to the more general formulation of the Schneider–Lang Theorem below.

Theorem (A First Schneider–Lang Theorem) *Suppose that $f_1(z)$ and $f_2(z)$ are two algebraically independent entire functions with finite orders of growth, each of which satisfies an algebraic polynomial differential equation. That is, there exists a number field F and a finite collection of functions*

$$f_3(z), f_4(z), \dots, f_J(z)$$

such that the differential operator $\frac{d}{dz}$ maps the ring $F[f_1(z), f_2(z), \dots, f_J(z)]$ into itself. Then for any number field E containing F,

$$\mathrm{card}\Big\{z \in \mathbf{C} : f_1(z) \in E,\ f_2(z) \in E, \dots, f_J(z) \in E\Big\}$$

is finite.

Moreover, if ρ_1 denotes the order of growth of $f_1(z)$ and ρ_2 denotes the order of growth of $f_2(z)$, then one can give a quantitive version of this result, namely,

$$\mathrm{card}\Big\{z \in \mathbf{C} : f_1(z) \in E,\ f_2(z) \in E, \dots, f_J(z) \in E\Big\} \le (\rho_1 + \rho_2)[E : \mathbf{Q}] \ .$$

Before outlining the proof of this result, let's see how even this special form of the Schneider–Lang Theorem can be applied to produce many of our earlier results.

The deduction of the Gelfond–Schneider Theorem from the Schneider–Lang Theorem. As before we begin by assuming that each of α, β and α^β is algebraic and let E be the number field given by $E = \mathbf{Q}\left(\alpha, \beta, \alpha^\beta\right)$. If we put $f_1(z) = e^z$ and $f_2(z) = e^{\beta z}$. Since β is irrational, we know that $f_1(z)$ and $f_2(z)$ are algebraically independent functions. Also, each of these functions satisfies an algebraic differential equation,

$$\frac{dy}{dz} = y \ \text{ and } \frac{dy}{dz} = \beta y \ ,$$

respectively. So if we take $K = \mathbf{Q}(\beta) \subseteq E$, then we see that $K[f_1(z), f_2(z)]$ is closed under differentiation. Thus we may apply the First Schneider–Lang Theorem and deduce that there are only *finitely* many points $z \in \mathbf{C}$ such that $f_1(z) \in E$ and $f_2(z) \in E$. However, we note that for all integers k,

$$f_1(k \log \alpha) = \alpha^k \in E \quad \text{and} \quad f_2(k \log \alpha) = \left(\alpha^\beta\right)^k \in E \ ,$$

which contradicts the previous sentence. Therefore we conclude that α^β is transcendental.

A sketch of the proof of the First Schneider-Lang Theorem We fix a number field E and let

$$\Omega = \{z \in \mathbf{C} : f_1(z) \in E,\ f_2(z) \in E, \dots, f_J(z) \in E\} \ .$$

Our aim is to show that Ω is a finite set. We establish this by assuming that $\{w_1, w_2, \dots, w_L\}$ is a set of distinct elements from Ω and showing that L cannot be too large.

An application of Siegel's Lemma allows us to solve a system of linear equations to find a nonzero polynomial $P(x, y)$ such that the function $F(z) = P(f_1(z), f_2(z))$ vanishes at each of the points $w_1, w_2, \dots, w_L$, each with multiplicity T. Since $f_1(z)$ and $f_2(z)$ are algebraically independent functions, $F(z)$ is not identically zero. Thus we let t_0 be the smallest positive integer such that there exists an index l_0, $1 \leq l_0 \leq L$, satisfying

$$\frac{d^{t_0}}{dz^{t_0}} F(w_{l_0}) \neq 0 \ .$$

If we let $\Gamma = \frac{d^{t_0}}{dz^{t_0}} F(w_{l_0})$, then by the hypothesis it follows that there exists a polynomial $P(x_1, x_2, \dots, x_J)$ with coefficients in E such that

$$\Gamma = P(f_1(w_{l_0}), f_2(w_{l_0}), \dots, f_J(w_{l_0})).$$

Thus we conclude that Γ is an algebraic number in E. Instead of dealing with having to multiply by a denominator of Γ to obtain an algebraic integer we henceforth assume Γ is an algebraic integer.

Then if we let $\Gamma_1 (= \Gamma), \Gamma_2, \dots, \Gamma_d$ denote the conjugates of Γ we know that

$$N = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_d,$$

is a rational integer.

It is relatively straightforward to estimate each of the values $|\Gamma_j|, j = 2, \dots, d$. As we have before we may employ the Maximum Modulus Principle to estimate $|\Gamma_1|$. Specifically, we consider the function

$$G(z) = \frac{F(z)}{\prod_{l=1}^{L} (z - w_l)^{t_0 - 1}}$$

on a disk of radius $t_0^{1/(\rho_1+\rho_2)}$ to get an upper bound for $|\Gamma|$ involving ρ_1, ρ_2, and $\deg(\Gamma)$. Applying our bounds it is possible to show that if $L > (\rho_1+\rho_2)[E:\mathbf{Q}]$, then the integer N satisfies $0 < |N| < 1$. This contradiction establishes the result.

As we have already noted the full Schneider-Lang Theorem applies not just entire functions but meromorphic. The order of growth of a meromorphic function $f(z)$ can be defined in one of two ways, and both depend on the observation that the poles of a meromorphic function are isolated. (If the poles of a particular meromorphic function were not isolated, then the reciprocal function, which is also a meromorphic function, would have a convergent set of zeros and thus must be identically zero.) This tells us that we can find a non-zero entire function $g(z)$ whose zeros are precisely the poles of $f(z)$. One then defines the order of growth of $f(z)$ to be the order of growth of the *entire* function $g(z)f(z)$.

Another way to define the order of growth of a meromorphic function that has a finite order of growth, which yields the same order of growth as the above definition, is to say that f has finite order of growth if there exists a number κ so that

$$\max\{|f(z)| : |z| = R\} \leq e^{R^\kappa} \;,$$

for all values of R that avoid the poles of $f(z)$. The infimum of all such κ is the order of growth of the function.

More formally, the order of growth of a meromorphic function $f(z)$ equals

$$\limsup_{R\to\infty} \frac{\log\log\max\{|f(z)| : |z| = R\}}{\log R} \;.$$

The Schneider–Lang Theorem *Let $f_1(z)$ and $f_2(z)$ denote two algebraically independent meromorphic functions with finite orders of growth ρ_1 and ρ_2, respectively. If $f_1(z)$ and $f_2(z)$ satisfy polynomial algebraic differential equations over a number field F; that is, there exists a finite collection of functions $f_3(z), f_4(z), \ldots, f_J(z)$ such that the differential operator $\frac{d}{dz}$ maps the ring $F[f_1(z), f_2(z), \ldots, f_J(z)]$ into itself. Then for any number field E containing F,*

$$\mathrm{card}\Big\{z \in \mathbf{C} : f_1(z) \in E, \ldots,\; f_J(z) \in E\Big\} \leq (\rho_1+\rho_2)[E:\mathbf{Q}] \;.$$

While we do not prove the Schneider-Lang Theorem, we do remark that its proof is essentially an elaboration of our sketch of the proof of the First Schneider-Lang theorem. The new idea, when making analytic estimates, is to multiply each of the meromorphic functions by the appropriate entire function that vanishes at its poles.

Elliptic Functions

The above section provided a statement of the Schnieder-Lang Theorem for meromorphic functions but did not consider any examples. We end this chapter with arguably the second most important function in number theory, behind the usual exponential function e^z, the meromorphic Weierstrass $\wp-$function. Just as there are several characterizations of e^z, there are several characterizations of $\wp(z)$. We will use these characterizations/properties in establishing transcendence results associated with $\wp(z)$ and, given what we have already seen, they are not surprisingly both analytic and algebraic in nature.

A series representation for $\wp(z)$. For any nonzero $w \in \mathbf{C}$, we know that there exists an entire function that is periodic modulo $\mathbf{Z}w$; namely the function $f(z) = e^{\frac{2\pi i}{w} z}$. A critical difference between the function $\wp(z)$ and e^z is that $\wp(z)$ has *two* $\mathbf{Q}$-linearly independent periods (such a function is said to be *doubly periodic*). Moreover, just as there is an exponential function periodic modulo $\mathbf{Z}w$ for any nonzero $w \in \mathbf{C}$, given any two $\mathbf{Q}$-linearly independent complex numbers w_1 and w_2 satisfying $w_2/w_1 \notin \mathbb{R}$, there exists a Weierstrass $\wp$-function that is periodic modulo the lattice $W = \mathbf{Z}w_1 + \mathbf{Z}w_2 \subseteq \mathbf{C}$.

Liouville demonstrated that an entire, bounded function must be a constant. It follows that the only doubly periodic functions that can be represented by an everywhere-convergent power series are the constant functions. Thus there cannot exist as attractive a power series for $\wp(z)$ as there is for e^z.

However, the complex numbers for which any non-constant, doubly periodic is not defined must form a *discrete* subset in the complex plane. The Weierstrass $\wp$-function is normalized so that the points at which it is not defined are precisely its periods. Moreover, the behavior of $\wp(z)$ at the periods $w \in W$ will be well understood—we will see that it has essentially the same behavior as the function $1/(z-w)^2$ for z near w.

We do not develop this theory here but it more-or-less follows from the above brief discussion that the Weierstrass $\wp$-function is represented by a series of the form:

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in W'} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

where W' denotes the nonzero elements of $W = \mathbf{Z}w_1 + \mathbf{Z}w_2$.

Although it is not immediately obvious from the above series representation, $\wp(z)$ is indeed a periodic function with respect to the lattice W. (This assertion can be established through a simple trick. Define two new functions by

$$f_1(z) = \wp(z+w_1) - \wp(z) \text{ and } f_2(z) = \wp(z+w_2) - \wp(z)\ ,$$

where $W = \mathbf{Z}w_1 + \mathbf{Z}w_2$. Then $f_1(z)$ and $f_2(z)$ are meromorphic functions whose derivatives are identically zero; thus, they are constant functions. Evaluating $f_1(-w_1/2)$ and $f_2(-w_2/2)$, and using the easily observed fact that $\wp(z)$ is an even function demonstrates that these functions are identically zero.)

The derivative of $\wp(z)$. The first step in uncovering a relationship between $\wp(z)$ and $\wp'(z)$ is to look at the so-called Laurent series of $\wp(z)$ centered at $z = 0$.

Since $\wp(z)$ is an even function we can deduce that the coefficients of the odd powers of z must all equal 0. Thus we may express the Laurent series for $\wp(z)$ about $z = 0$ as

$$\wp(z) = \frac{1}{z^2} + c_0 + c_2 z^2 + c_4 z^4 + \cdots \ .$$

But we also know that

$$\wp(z) - \frac{1}{z^2} = \sum_{w \in W'} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \ ,$$

and the right-hand side of the above identity vanishes at $z = 0$, so we may conclude that $c_0 = 0$. Thus

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \cdots \ ,$$

formal differentiation of which yields the convergent series

$$\wp'(z) = -\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + \cdots \ .$$

It is a fairly difficult exercise in most graduate complex variables courses to show that it follows from the above two expressions that for all complex numbers $z \notin W$,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \ ,$$

where g_2 and g_3 have the explicit representations

$$g_2 = 20c_2 = 60 \sum_{w \in W'} \frac{1}{w^4} \qquad g_3 = 28c_4 = 140 \sum_{w \in W'} \frac{1}{w^6}.$$

It is part of the theory of that the polynomial $4x^3 - g_2 x - g_3$ has distinct roots.

An application of the Schneider-Lang Theorem to $\wp(z)$. With this brief introduction we are already in a position to deduce transcendence results about numbers associated with a Weierstrass $\wp$−function from the Schneider-Lang Theorem. Many of the results it is possible to obtain in this way were established by Schnieder in 1934, before the formalization of the Schneider-Lang Theorem. We first consider one of Schneider's results form 1934, an elliptic analogue of Lindemann's Theorem.

Theorem *Suppose that the coefficients g_2 and g_3 of the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \ ,$$

are algebraic. Let W denote the lattice of periods for $\wp(z)$. Then every nonzero element of W is transcendental.

We note for historical accuracy that in 1932 Siegel proved that if the coefficients g_2 and g_3 of the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \ ,$$

are algebraic, and $W = \mathbf{Z}w_1 + \mathbf{Z}w_2$, then either w_1 or w_2 is transcendental.

Our deduction of the above theorem from the Schneider-Lang Theorem requires the following elementary lemma.

Lemma. *Suppose that we factor the polynomial differential equation of $\wp(z)$ over the complex numbers as*

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \qquad (2)$$

If we write the period lattice for $\wp(z)$ as $W = \mathbf{Z}w_1 + \mathbf{Z}w_2$, then reordering e_1, e_2, and e_3, if necessary, it follows that

$$\wp\left(\frac{w_1}{2}\right) = e_1 \ , \quad \wp\left(\frac{w_2}{2}\right) = e_2 \ , \quad \textit{and} \quad \wp\left(\frac{w_1 + w_2}{2}\right) = e_3 \ .$$

Proof. For simplicity let $w_3 = w_1 + w_2$. In view of the factorization (2), we need only show that for each $n = 1, 2, 3$, $\wp'(\frac{w_n}{2}) = 0$. Since $\wp(z)$ is even, $\wp'(z)$ is an odd function, and thus, using the fact that $w_n \in W$,

$$\wp'\left(\frac{w_n}{2}\right) = \wp'\left(\frac{w_n}{2} - w_n\right) = \wp'\left(-\frac{w_n}{2}\right) = -\wp'\left(\frac{w_n}{2}\right) \ .$$

This establishes the lemma.

The proof of the transcendence of the nonzero periods of a Weierstrass $\wp-$function with algebraic g_2 and g_3 then follows from applying the Schneider-Lang Theorem to the functions $f_1(z) = z$ and $f_2(z) = \wp(z)$ at the points $w/2, w/2 + w, w/2 + 2w \ldots$. The one complication is that $w/2$ might also be in W, and so be a pole of $\wp(z)$ (in other words it is very well possible that $w = mw_1 + nw_2$ where both m and n are even). If this is the case divide w by a sufficiently large power of 2, $\frac{w}{2^k} = m'w_1 + n'w_2$, with not both m' and n' even. Then let $w' = m'w_1 + n'w_2$ and consider z and $\wp(z)$ at the points $w'/2, w'/2 + w', w'/2 + 2w', \ldots$.. The transcendence of w follows from the transcendence of w'.

Corollary. *The real number*

$$\frac{\Gamma(1/4)^2}{\sqrt{\pi}}$$

is transcendental.

There are several equivalent definitions of the gamma function; perhaps the simplest is as an improper integral: For any complex number z with $\mathrm{Re}(z) > 0$, we define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \ .$$

While it is not obvious from this definition, it can be shown that $\Gamma(z)$ is defined and analytic for all z with positive real part.

In order to establish the above corollary we need two identities for the Γ function, which we state without proof.

First identity. For positive real numbers a and b,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \ , \tag{3}$$

and,
Second identity. For a complex number z for which neither z nor $1-z$ is a negative integer,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{4}$$

With these two identities in hand the proof of the above corollary depends on showing that $\frac{\Gamma(1/4)^2}{\sqrt{\pi}}$ is algebraically dependent on an explicity computed period of the specific elliptic curve

$$y^2 = 4x(x-1)(x+1) = 4x^3 - 4x \ .$$

In order to connect $\frac{\Gamma(1/4)^2}{\sqrt{\pi}}$ with a period of $\wp(z)$, we show that the improper integral

$$I = \int_1^\infty \frac{dx}{\sqrt{4x^3-4x}} \tag{5}$$

equals one-half of a nonzero period of the elliptic curve $y^2 = 4x^3 - 4x$. Following this we show that the numbers I and $\frac{\Gamma(1/4)^2}{\sqrt{\pi}}$ are algebraically dependent.

Lemma *The number $2I$ is a nonzero period of the Weierstrass $\wp$-function that satisfies the differential equation $y' = 4y^3 - 4y$.*

Proof. To establish this lemma, we first claim that the Weierstrass $\wp$-function associated with the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 4y^3 - 4y$$

inverts the integral of (5). This means that

$$z = \int_{\wp(z)}^\infty \frac{dx}{\sqrt{4x^3-4x}} \ ,$$

(where the path of integration is any simple curve that does not contain a zero of the denominator, that is, which does not contain the numbers $-1, 0$, and 1).

We offer a sketch of this claim that is perhaps not a rigorous as it could be. Let ζ be a variable. Then we can express z as a function of ζ through the integral

$$z = \int_{\zeta}^{\infty} \frac{dx}{\sqrt{4x^3 - 4x}} \ , \tag{6}$$

with the same restriction on the path of integration. Differentiation of this expression yields

$$\frac{d\zeta}{dz} = \sqrt{4\zeta^3 - 4\zeta}$$

If we square this we see that ζ, as a function of z, satisfies the same differential equation as the Weierstrass elliptic function. Thus

$$\zeta = \wp(z + a) \quad \text{for some constant } a.$$

But a can be determined by examining the limit of the integral as ζ approaches ∞. As $\zeta \to \infty$, we have that $z \to 0$. Thus a must be pole w for $\wp(z)$. Recalling that $\wp(z)$ is periodic modulo its lattice of poles W, we see that $\zeta = \wp(z + w) = \wp(z)$, which implies the validity of (6).

We now return to the proof of the lemma. The lower limit of integration, $x = 1$, in the integral I from (6) is a zero of the polynomial $4x^3 - 4x$, and thus we have that for some n, $\wp(w_n/2) = 1$. Therefore

$$I = \int_{\wp(w_n/2)}^{\infty} \frac{dx}{\sqrt{4x^3 - 4x}} = \frac{w_n}{2} \ ,$$

and so $2I = w_n \in W \setminus \{0\}$, which completes our proof.

Since $2I$ is a nonzero period of a Weierstrass $\wp-$function where g_2 and g_3 are algebraic, e know that I is transcendental. We next show how this leads to the transcendence of $\frac{\Gamma(1/4)^2}{\sqrt{\pi}}$.

Claim. The numbers I and $\frac{\Gamma(1/4)^2}{\sqrt{\pi}}$ are algebraically dependent.

We establish this claim by making the change of variables $x = \frac{1}{\sqrt{u}}$ in the integral representation for I. This leads to

$$I = -\frac{1}{2} \int_1^0 \frac{u^{-\frac{3}{2}}\, du}{\sqrt{u^{-\frac{3}{2}} - u^{-\frac{1}{2}}}} = \frac{1}{2} \int_0^1 u^{-\frac{3}{4}} (1-u)^{-\frac{1}{2}}\, du \ .$$

This last integral is precisely the one appearing in the first identity above. So we may conclude that

$$\frac{1}{2} \int_0^1 u^{-\frac{3}{4}} (1-u)^{-\frac{1}{2}}\, du = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})} \ .$$

We leave it as an exercise to show that the Claim then follows through an application of the second identity given, which should lead you to the equation

$$I = \frac{1}{2\sqrt{2}} \frac{\Gamma(\frac{1}{4})^2}{\sqrt{\pi}} \ . \tag{7}$$

Additional remarks about $\wp(z)$. Unfortunately we do not yet know enough about the $\wp-$function to deduce elliptic analogues of the Hermite-Lindemann or Gelfond-Schneider theorems from the Schneider-Lang Theorem. We are missing two things. The first is a $\wp-$version of the *addition formula* $e^{x+y} = e^x e^y$ that holds for the usual exponential function. As one might expect the analogous formula for $\wp(z)$ is a more complicated matter. Indeed, it is:

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\Big(\frac{\wp'(z_2) - \wp'(z_1)}{\wp(z_2) - \wp(z_1)}\Big)^2. \tag{8}$$

The verification of this formula is an application of Liouville's result that an bounded, entire function must be constant. (We outline this as an exercise.)

The second missing piece of mathematics is an elliptic analogue of the algebraic independence of e^z and $e^{\beta z}$ when β is an irrational number. We do not propose to develop this theory here but only report that if β is a complex number then $\wp(z)$ and $\wp(\beta z)$ are algebraically independent if and only if βW is not contained in W.

Theorem (Elliptic version of the Hermite-Lindemann Theorem) Suppose α is a nonzero algebraic number and that $\wp(z)$ has algebraic g_2 and g_3. Then $\wp(\alpha)$ is transcendental.

Proof. Apply the Schneider-Lang Theorem to the functions z and $\wp(z)$ at the points $\alpha, 2\alpha, 3\alpha, \ldots$.

Theorem (Elliptic version of the Gelfond-Schneider Theorem) Suppose β is an algebraic number so that βW is not contained in W. Suppose further that $\wp(z)$ has algebraic g_2 and g_3. If $\wp(u)$ is algebraic then $\wp(\beta u)$ is transcendental.

Proof A thought-provocating exercise.

Exercises.

1. Suppose $F(z)$ is a nonzero, meromorphic function. Suppose y_1, y_2, y_3 are **Q**-linearly independent complex numbers. Show that there exists a positive integer M such that

$$F(k_1 y_1 + k_2 y_2 + k_3 y_3) = 0 \ ,$$

for all $0 \leq k_j < M$, while there exists some triple k_1^*, k_2^*, k_3^*, satisfying $0 \leq k_j^* \leq M$, where

$$F(k_1^* y_1 + k_2^* y_2 + k_3^* y_3) \neq 0 \ .$$

2. In this exercise you will be asked to deduce the addition formula (8). Fix a $y \notin W$ and define the function $f(z)$ by

$$f(z) = \wp(z+y) + \wp(z) - \frac{1}{4}\Big(\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\Big)^2 .$$

First, show that the function

$$\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}$$

has a pole of order 1 at any element of the set

$$W_y = \{w, w+y, w-y : w \in W\} .$$

Then conclude that $f(z)$ does not have a pole at any point in W_y. Second, show that $f(z)$ is a bounded entire function, and letting $y \to 0$, conclude that $f(z) = -\wp(y)$.

3 . Suppose $\wp(z)$ is a elliptic function with g_2 and g_3 being algebraic. Let w be a nonzero period of $\wp(z)$. Show that both e^w and $\wp(\pi)$ are transcendental. (Warning: Establishing the transcendence of the second value is more subtle than the transcendence of the first.)

4. Prove the zeros estimate used in the sketch of the proof of the Schneider-Lang Theorem.