

ON THE CHARACTERISTIC MAP OF FINITE UNITARY GROUPS

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1. INTRODUCTION

In his seminal work [8], Green described a remarkable connection between the class functions of the finite general linear group $\mathrm{GL}(n, \mathbb{F}_q)$ and a generalization of the ring of symmetric functions of the symmetric group S_n . In particular, Green defines a map, called the characteristic map, that takes irreducible characters to Schur-like symmetric functions, and recovers the character table of $\mathrm{GL}(n, \mathbb{F}_q)$ as the transition matrix between these Schur functions and Hall-Littlewood polynomials [18, Chapter IV]. Thus, we can use the combinatorics of the symmetric group S_n to understand the representation theory of $\mathrm{GL}(n, \mathbb{F}_q)$. Some of the implications of this approach include an indexing of irreducible characters and conjugacy classes of $\mathrm{GL}(n, \mathbb{F}_q)$ by multi-partitions and a formula for the degrees of the irreducible characters in terms of these partitions.

This paper describes the parallel story for the finite unitary group $\mathrm{U}(n, \mathbb{F}_{q^2})$ by collecting known results for this group and examining some applications of the unitary characteristic map. Inspired by Green, Ennola [4, 5] used results of Wall [21] to construct the appropriate ring of symmetric functions and characteristic map. Ennola was able to prove that the analogous Schur-like functions correspond to an orthonormal basis for the class functions, and conjectured that they corresponded to the irreducible characters. He theorized that the representation theory of $\mathrm{U}(n, \mathbb{F}_{q^2})$ should be deduced from the representation theory of $\mathrm{GL}(n, \mathbb{F}_q)$ by substituting “ $-q$ ” for every occurrence of “ q ”. The general phenomenon of obtaining a polynomial invariant in q for $\mathrm{U}(n, \mathbb{F}_{q^2})$ by this substitution has come to be known as “Ennola duality”.

Roughly a decade after Ennola made his conjecture, Deligne and Lusztig [3] constructed a family of virtual characters, called Deligne-Lusztig characters, to study the representation theory of arbitrary finite reductive groups. Lusztig and Srinivasan [17] then computed an explicit decomposition of the irreducible characters of $\mathrm{U}(n, \mathbb{F}_{q^2})$ in terms of Deligne-Lusztig characters. Kawanaka [14] used this composition to demonstrate that Ennola duality applies to Green functions, thereby improving results of Hotta and Springer [11] and finally proving Ennola’s conjecture.

This paper begins by describing some of the combinatorics and group theory associated with the finite unitary groups. Section 2 defines the finite

unitary groups, outlines the combinatorics of multi-partitions, and gives a description of some of the key subgroups. Section 3 analyzes the conjugacy classes of $U(n, \mathbb{F}_{q^2})$ and the Jordan decomposition of these conjugacy classes.

Section 4 outlines the statement and development of the Ennola conjecture from two perspectives. Both points of view define a map from a ring of symmetric functions to the character ring C of $U(n, \mathbb{F}_{q^2})$. However, the first uses the multiplication for C as defined by Ennola, and the second uses Deligne-Lusztig induction as the multiplicative structure of C . The main results are

I. (Theorem 4.2) The Deligne-Lusztig characters correspond to power-sum symmetric functions via the characteristic map of Ennola.

II. (Corollary 4.2) The multiplicative structure that Ennola defined on C is Deligne-Lusztig induction.

Section 5 computes the degrees of the irreducible characters, and uses this result to evaluate various sums of character degrees (see [18, IV.6, Example 5] for the $GL(n, \mathbb{F}_q)$ analogue of this method). The main results are

III. (Theorem 5.1) An irreducible χ^λ character of $U(m, \mathbb{F}_{q^2})$ corresponds to

$$(-1)^{\lfloor m/2 \rfloor + n(\lambda)} s_\lambda \quad \text{and} \quad \chi^\lambda(1) = q^{n(\lambda')} \frac{\prod_{1 \leq i \leq m} (q^i - (-1)^i)}{\prod_{\square \in \lambda} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})},$$

where s_λ is a Schur-like function, and both $n(\lambda)$ and $\mathbf{h}(\square)$ are combinatorial statistics on the multi-partition λ .

IV. (Corollary 5.2) If \mathcal{P}_n^\ominus indexes the irreducible characters χ^λ of $U(n, \mathbb{F}_{q^2})$, then

$$\sum_{\lambda \in \mathcal{P}_n^\ominus} \chi^\lambda(1) = |\{g \in U(n, \mathbb{F}_{q^2}) \mid g \text{ symmetric}\}|.$$

V. (Theorem 5.3) We give a subset $\mathcal{X} \subseteq \mathcal{P}_{2n}^\ominus$ such that

$$\sum_{\lambda \in \mathcal{X}} \chi^\lambda(1) = (q+1)q^2(q^3+1) \cdots q^{2n-2}(q^{2n-1}+1) = \frac{|U(2n, \mathbb{F}_{q^2})|}{|\mathrm{Sp}(2n, \mathbb{F}_q)|}.$$

Section 6 uses results by Ohmori [20] and Henderson [10] to adapt a model for the general linear group, found by Klyachko [16] and Inglis and Saxl [12], to the finite unitary group. The main result is

VI. (Theorem 6.2) Let $U_m = U(m, \mathbb{F}_{q^2})$, where q is odd, and let Γ_m be the Gelfand-Graev character of U_m , $\mathbf{1}$ be the trivial character of the finite symplectic group $Sp_{2r} = \mathrm{Sp}(2r, \mathbb{F}_q)$, and R_L^G be the Deligne-Lusztig induction functor. Then

$$\sum_{0 \leq 2r \leq m} R_{U_{m-2r} \oplus U_{2r}}^{U_m} (\Gamma_{m-2r} \otimes \mathrm{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1})) = \sum_{\lambda \in \mathcal{P}_m^\ominus} \chi^\lambda.$$

That is, in the theorem of Klyachko, one may replace parabolic induction by Deligne-Lusztig induction to obtain a theorem for the unitary group.

These results give considerable combinatorial control over the representation theory of the finite unitary group, and there are certainly more applications to these results than what we present in this paper. Furthermore, this characteristic map gives some insight as to how a characteristic map might look in general type, using the invariant rings of other Weyl groups.

2. PRELIMINARIES

2.1. The unitary group and its underlying field k . Let $\mathbb{K} = \overline{\mathbb{F}}_q$ the algebraic closure of a finite field with q elements and let $\mathbb{K}_m = \mathbb{F}_{q^m}$ denote the finite subfield with q^m elements. Let $\mathrm{GL}(n, \mathbb{K})$ denote the general linear group over \mathbb{K} , and define Frobenius maps

$$(2.1) \quad \begin{array}{ccc} F : \mathrm{GL}(n, \mathbb{K}) & \longrightarrow & \mathrm{GL}(n, \mathbb{K}) \\ (a_{ij}) & \mapsto & (a_{ji}^q)^{-1}, \end{array} \quad \text{and} \quad \begin{array}{ccc} F' : \mathrm{GL}(n, \mathbb{K}) & \longrightarrow & \mathrm{GL}(n, \mathbb{K}) \\ (a_{ij}) & \mapsto & (a_{n-j, n-i}^q)^{-1}. \end{array}$$

Then the unitary group $U_n = \mathrm{U}(n, \mathbb{K}_2)$ is given by

$$(2.2) \quad U_n = \mathrm{GL}(n, \mathbb{K})^F = \{a \in \mathrm{GL}(n, \mathbb{K}) \mid F(a) = a\}$$

$$(2.3) \quad \cong \mathrm{GL}(n, \mathbb{K})^{F'} = \{a \in \mathrm{GL}(n, \mathbb{K}) \mid F'(a) = a\}.$$

In fact, it follows from the Lang-Steinberg theorem (see, for example, [2]) that there exists $y \in \mathrm{GL}(n, \mathbb{K}_2)$ such that $\mathrm{GL}(n, \mathbb{K})^F = y\mathrm{GL}(n, \mathbb{K})^{F'}y^{-1}$.

We define the multiplicative groups \mathbb{M}_m as

$$\mathbb{M}_m = \mathrm{GL}(1, \mathbb{K})^{F^m} = \{x \in \mathbb{K} \mid x^{q^m - (-1)^m} = 1\}.$$

Note that $\mathbb{M}_m \cong \mathbb{K}_m^\times$ only if m is even. We identify \mathbb{K}^\times with the inverse limit $\varprojlim \mathbb{M}_m$ with respect to the norm maps

$$N_{mr} : \mathbb{M}_m \longrightarrow \mathbb{M}_r \\ x \longmapsto xx^{-q} \dots x^{(-q)^{m/r-1}}, \quad \text{where } m, r \in \mathbb{Z}_{\geq 1} \text{ with } r \mid m.$$

If \mathbb{M}_m^* is the group of characters of \mathbb{M}_m , then the direct limit

$$\mathbb{K}^* = \varinjlim \mathbb{M}_m^*$$

gives the group of characters of \mathbb{K}^\times . Let

$$\Theta = \{F\text{-orbits of } \mathbb{K}^*\}.$$

A polynomial $f(t) \in \mathbb{K}_2[t]$ is *F-irreducible* if there exists an F -orbit $\{x, x^{-q}, \dots, x^{(-q)^d}\}$ of \mathbb{K}^\times such that

$$f(t) = (t - x)(t - x^{-q}) \dots (t - x^{(-q)^d}).$$

Let

$$(2.4) \quad \Phi = \{f \in \mathbb{K}_2[t] \mid f \text{ is } F\text{-irreducible}\} \xrightarrow{1-1} \{F\text{-orbits of } \mathbb{K}^\times\}.$$

The set Φ has an alternate description, as given in [4]. For $f = t^d + a_1 t^{d-1} + \cdots + a_d \in \mathbb{K}_2[t]$ with $a_d \neq 0$, let

$$\tilde{f} = t^d + a_d^{-q}(a_{d-1}^q t^{d-1} + \cdots + a_1^q t + 1),$$

which has the effect of applying F to the roots of f in \mathbb{K} . Then a polynomial $f \in \mathbb{K}_2[t]$ is F -irreducible if and only if either

- (a) f is irreducible in $\mathbb{K}_2[t]$ and $\tilde{f} = f$ (d must be odd in this case), or
- (b) $f(t) = h(t)\tilde{h}(t)$ where h is irreducible in $\mathbb{K}_2[t]$ and $\tilde{h}(t) \neq h(t)$.

2.2. Combinatorics of Φ -partitions and Θ -partitions. Fix an ordering of Φ and Θ , and let

$$\mathcal{P} = \{\text{partitions}\} \quad \text{and} \quad \mathcal{P}_n = \{\nu \in \mathcal{P} \mid |\nu| = n\}.$$

Let \mathcal{X} be either Φ or Θ . An \mathcal{X} -partition $\nu = (\nu(x_1), \nu(x_2), \dots)$ is a sequence of partitions indexed by \mathcal{X} . The *size* of an \mathcal{X} -partition ν is

$$(2.5) \quad \|\nu\| = \sum_{x \in \mathcal{X}} |x| |\nu(x)|, \quad \text{where} \quad |x| = \begin{cases} |x| & \text{if } \mathcal{X} = \Theta, \\ d(x) & \text{if } \mathcal{X} = \Phi, \end{cases}$$

$|x|$ is the size of the orbit $x \in \Theta$, and $d(x)$ is the degree of the polynomial $x \in \Phi$. Let

$$(2.6) \quad \mathcal{P}_n^{\mathcal{X}} = \{\mathcal{X}\text{-partitions } \nu \mid \|\nu\| = n\}, \quad \text{and} \quad \mathcal{P}^{\mathcal{X}} = \bigcup_{n=1}^{\infty} \mathcal{P}_n^{\mathcal{X}}.$$

For $\nu \in \mathcal{P}^{\mathcal{X}}$, let

$$(2.7) \quad n(\nu) = \sum_{x \in \mathcal{X}} |x| n(\nu(x)), \quad \text{where} \quad n(\nu) = \sum_{i=1}^{\ell(\nu)} (i-1) \nu_i.$$

The *conjugate* ν' of ν is the \mathcal{X} -partition $\nu' = (\nu(x_1)', \nu(x_2)', \dots)$, where ν' is the usual conjugate partition for $\nu \in \mathcal{P}$.

The *semisimple part* ν_s of $\nu = (\nu(x_1), \nu(x_2), \dots) \in \mathcal{P}_n^{\mathcal{X}}$ is

$$(2.8) \quad \nu_s = ((1^{|\nu(x_1)|}), (1^{|\nu(x_2)|}), \dots) \in \mathcal{P}_n^{\mathcal{X}},$$

and the *unipotent part* ν_u of $\nu \in \mathcal{P}_n^{\mathcal{X}}$ is given by

$$(2.9) \quad \nu_u(\mathbb{1}) \quad \text{has parts} \quad \{|x| \nu(x)_i \mid x \in \mathcal{X}, i = 1, \dots, \ell(\nu(x))\}$$

where

$$\mathbb{1} = \begin{cases} \{1\} & \text{if } \mathcal{X} = \Theta, \\ t-1 & \text{if } \mathcal{X} = \Phi, \end{cases}$$

$\mathbb{1}$ is the trivial character in \mathbb{K}^* , and $\nu_u(x) = \emptyset$ for $x \neq \mathbb{1}$.

Example. If $\mu \in \mathcal{P}^{\Phi}$ is given by

$$\mu = \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}^{(f)}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}^{(g)}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}^{(h)} \right), \quad \text{where } d(f) = 1, d(g) = d(h) = 2,$$

then $\|\boldsymbol{\mu}\| = 1 \cdot 4 + 2 \cdot 2 + 2 \cdot 5 = 18$, $n(\boldsymbol{\mu}) = 1 \cdot 2 + 2 \cdot 0 + 2 \cdot 1 = 4$,

$$\boldsymbol{\mu}_s = \left(\begin{array}{c} \square^{(f)} \\ \square \\ \square \\ \square \end{array}, \begin{array}{c} \square^{(g)} \\ \square \end{array}, \begin{array}{c} \square^{(h)} \\ \square \\ \square \\ \square \end{array} \right) \quad \text{and} \quad \boldsymbol{\mu}_u = \left(\begin{array}{cccccccc} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{array} \right)^{(t-1)}.$$

2.3. Levi subgroups and maximal tori. Let \mathcal{X} be either Φ or Θ as in Section 2.2.

For $\boldsymbol{\nu} \in \mathcal{P}_n^{\mathcal{X}}$, let

$$(2.10) \quad L_{\boldsymbol{\nu}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} L_{\boldsymbol{\nu}}(x), \quad \text{where} \quad \mathcal{X}_{\boldsymbol{\nu}} = \{x \in \mathcal{X} \mid \boldsymbol{\nu}(x) \neq \emptyset\},$$

and for $x \in \mathcal{X}_{\boldsymbol{\nu}}$,

$$(2.11) \quad L_{\boldsymbol{\nu}}(x) = \begin{cases} \text{U}(|\boldsymbol{\nu}(x)|, \mathbb{K}_{2|x|}) & \text{if } |x| \text{ is odd,} \\ \text{GL}(|\boldsymbol{\nu}(x)|, \mathbb{K}_{|x|}) & \text{if } |x| \text{ is even.} \end{cases}$$

Then $L_{\boldsymbol{\nu}}$ is a Levi subgroup of $U_n = \text{U}(n, \mathbb{K}_2)$ (though not uniquely determined by $\boldsymbol{\nu}$). The Weyl group

$$(2.12) \quad W_{\boldsymbol{\nu}} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} S_{|\boldsymbol{\nu}(x)|},$$

of $L_{\boldsymbol{\nu}}$ has conjugacy classes indexed by

$$(2.13) \quad \mathcal{P}_s^{\boldsymbol{\nu}} = \{\gamma \in \mathcal{P}^{\mathcal{X}} \mid \gamma_s = \boldsymbol{\nu}_s\},$$

and the size of the conjugacy class c_{γ} is

$$(2.14) \quad |c_{\gamma}| = \frac{|W_{\gamma}|}{z_{\gamma}}, \quad \text{where} \quad z_{\gamma} = \prod_{x \in \mathcal{X}} z_{\gamma(x)} \quad \text{and} \quad z_{\gamma} = \prod_{i=1}^{\ell(\gamma)} i^{m_i} m_i!,$$

for $\gamma = (1^{m_1} 2^{m_2} \dots) \in \mathcal{P}$.

For every $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_{\ell}) \in \mathcal{P}_n$ there exists a maximal torus (unique up to isomorphism) $T_{\boldsymbol{\nu}}$ of U_n such that

$$T_{\boldsymbol{\nu}} \cong \mathbb{M}_{\nu_1} \times \mathbb{M}_{\nu_2} \times \dots \times \mathbb{M}_{\nu_{\ell}}.$$

For every $\gamma \in \mathcal{P}_s^{\boldsymbol{\mu}}$, there exists a maximal torus (unique up to isomorphism) $T_{\gamma} \subseteq L_{\boldsymbol{\nu}}$ such that

$$(2.15) \quad T_{\gamma} = \bigoplus_{x \in \mathcal{X}_{\boldsymbol{\nu}}} T_{\gamma}(x), \quad \text{where} \quad T_{\gamma}(x) \cong \mathbb{M}_{|x|\gamma(x)_1} \times \dots \times \mathbb{M}_{|x|\gamma(x)_{\ell}}.$$

Note that as a maximal torus of U_n , the torus $T_{\gamma} \cong T_{\gamma_u(\mathbb{1})}$.

3. CONJUGACY CLASSES AND JORDAN DECOMPOSITION

Let $U_n = U(n, \mathbb{K}_2)$ as in (2.2).

Proposition 3.1. *The conjugacy classes c_μ of U_n are indexed by $\mu \in \mathcal{P}_n^\Phi$.*

Proof. For any $g \in U_n$, the monic minimal polynomial $f \in \mathbb{K}_2[t]$ of g satisfies $f(t) = \tilde{f}(t)$, and in fact every invariant factor of g satisfies this. Every polynomial of this kind may be factored into F -irreducibles. Wall [21] proved that an element $g \in \text{GL}(n, \mathbb{K}_2)$ is an element of U_n if and only if for every irreducible f , the polynomials f and \tilde{f} have the same multiplicities as elementary divisors of g .

Each $g \in U_n$ acts on the vector space \mathbb{K}_2^n and thus gives it a $\mathbb{K}_2[t]$ -module structure, where $t \cdot v = gv$ for each $v \in \mathbb{K}_2^n$. It is a result of Wall [21] that two elements of U_n are conjugate in U_n if they are conjugate in $G_n = \text{GL}(n, \mathbb{K}_2)$. So, two elements of U_n are conjugate if and only if they induce the same $\mathbb{K}_2[t]$ -module structure on \mathbb{K}_2^n . For a conjugacy class c of U_n , we call this $\mathbb{K}_2[t]$ -module V_c . Since the minimal polynomial f of $g \in U_n$ satisfies $f(t) = \tilde{f}(t)$ and annihilates V_c , we have that the conjugacy classes of U_n are parametrized by $\mathbb{K}_2[t]$ -modules whose annihilators are generated by such polynomials.

It follows that V_c is the direct sum of cyclic modules of the form $\mathbb{K}_2[t]/(h)^m$, where h is irreducible, and $(\tilde{h})^m$ appears in the direct sum whenever $(h)^m$ occurs. That is, when h^m and \tilde{h}^m are distinct, we may pair them together. In this way, to each $f \in \Phi$ we have assigned a partition $\mu(f) = (\mu(f)_1, \mu(f)_2, \dots)$ such that

$$V_c \cong \bigoplus_{f,i} \mathbb{K}_2[t]/(f)^{\mu(f)_i}.$$

Since $f \in \Phi$, then either f is irreducible, or $f = g\tilde{g}$ where g is irreducible. So V_c determines a Φ -partition μ of size n .

Conversely, every Φ -partition μ of size n determines a conjugacy class of U_n . \square

For $r \in \mathbb{Z}_{\geq 0}$, let

$$(3.1) \quad \psi_r(x) = \prod_{i=1}^r (1 - x^i).$$

Proposition 3.2 (Wall). *Let $g \in c_\mu$. The order a_μ of the centralizer g in U_n is*

$$a_\mu = (-1)^{\|\mu\|} \prod_{f \in \Phi} a_{\mu(f)}((-q)^{d(f)}), \text{ where } a_\mu(x) = x^{|\mu|+2n(\mu)} \prod_j \psi_{m_j}(x^{-1}),$$

for $\mu = (1^{m_1} 2^{m_2} 3^{m_3} \dots) \in \mathcal{P}$.

Proof. It follows from the results of Wall [21, Section 2.6], along with the description of F -irreducible polynomials given in Section 2 above, that

$$a_{\boldsymbol{\mu}} = \prod_{f \in \Phi} q^{d(2\sum_{i < j} im_i m_j + \sum_i (i-1)m_i^2)} \prod_j \left(q^{dm_j(m_j-1)/2} \prod_{i=1}^{m_j} (q^{di} - (-1)^{di}) \right).$$

where $d = d(f)$ and $\boldsymbol{\mu}(f) = (1^{m_1} 2^{m_2} \dots)$. From

$$\prod_{i=1}^{m_j} (q^{di} - (-1)^i) = q^{dm_j(m_j+1)/2} \prod_{i=1}^{m_j} (1 - (1/(-q)^d)^i),$$

we obtain

$$a_{\boldsymbol{\mu}} = \prod_{f \in \Phi} q^{d(2\sum_{i < j} im_i m_j + \sum_i im_i^2)} \prod_j \psi_{m_j}((-q)^{-d}).$$

Note that for any partition $\boldsymbol{\mu} = (1^{m_1} 2^{m_2} \dots) \in \mathcal{P}$, it follows from

$$\sum_{j \leq i+1} m_j = \mu'_{i+1} \quad \text{and} \quad m_i = \mu'_i - \mu'_{i+1},$$

that

$$2 \sum_{i < j} im_i m_j + \sum_i im_i^2 = |\boldsymbol{\mu}| + 2n(\boldsymbol{\mu}).$$

Thus

$$\begin{aligned} a_{\boldsymbol{\mu}} &= \prod_{f \in \Phi} (q^d)^{|\boldsymbol{\mu}(f)| + 2n(\boldsymbol{\mu}(f))} \prod_j \psi_{m_j}((-q)^{-d}) \\ &= \prod_{f \in \Phi} (-1)^{d|\boldsymbol{\mu}(f)|} ((-q)^d)^{|\boldsymbol{\mu}(f)| + 2n(\boldsymbol{\mu}(f))} \prod_j \psi_{m_j}((-q)^{-d}) \\ &= (-1)^{\|\boldsymbol{\mu}\|} \prod_{f \in \Phi} a_{\boldsymbol{\mu}(f)}((-q)^{d(f)}), \end{aligned}$$

as desired. \square

For $\boldsymbol{\mu} \in \mathcal{P}^\Phi$, let $L_{\boldsymbol{\mu}}$ be as in (2.10). Note that $|L_{\boldsymbol{\mu}}| = a_{\boldsymbol{\mu}_s}$.

Lemma 3.1. *Suppose $g \in c_{\boldsymbol{\mu}}$ with Jordan decomposition $g = su$. Then*

- (a) $s \in c_{\boldsymbol{\mu}_s}$ and $u \in c_{\boldsymbol{\mu}_u}$, where $\boldsymbol{\mu}_s$ and $\boldsymbol{\mu}_u$ are as in (2.8) and (2.9),
- (b) the centralizer $C_{U_n}(s)$ of s in U_n is isomorphic to $L_{\boldsymbol{\mu}}$.

Proof. (a) Suppose $f = t^d - a_{d-1}t^{d-1} - \dots - a_1t - a_0 \in \Phi$ is irreducible (so d is odd). Define

$$(3.2) \quad J(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{d-1} \end{pmatrix} \in \text{GL}(d, \mathbb{K}_2).$$

For $\nu = (\nu_1, \nu_2, \dots, \nu_\ell) \in \mathcal{P}$, let

$$J_\nu(f) = J_{\nu_1}(f) \oplus \dots \oplus J_{\nu_\ell}(f), \text{ where } J_m(f) = \begin{pmatrix} J(f) & \text{Id}_d & & 0 \\ 0 & J(f) & \ddots & \\ \vdots & \ddots & \ddots & \text{Id}_d \\ 0 & \dots & 0 & J(f) \end{pmatrix}$$

is an $md \times md$ matrix and Id_d is the $d \times d$ identity matrix. Note that

$$J_m(f) = J_{(1^m)}(f)u_{(m)}(f), \text{ where } u_{(m)}(f) = \begin{pmatrix} \text{Id}_d & J(f)^{-1} & & 0 \\ & \text{Id}_d & \ddots & \\ & & \ddots & J(f)^{-1} \\ 0 & & & \text{Id}_d \end{pmatrix}$$

is the Jordan decomposition of $J_m(f)$ in $\text{GL}(d, \mathbb{K}_2)$.

Similarly, if $f = h\tilde{h} \in \Phi$ (or $d(f)$ is even), then define

$$\begin{aligned} J(f) &= J(h) \oplus J(\tilde{h}) \\ J_m(f) &= J_m(h) \oplus J_m(\tilde{h}) \\ J_\nu(f) &= J_\nu(h) \oplus J_\nu(\tilde{h}), \end{aligned}$$

so

$$J_m(f) = (J_{(1^m)}(h) \oplus J_{(1^m)}(\tilde{h}))(u_{(m)}(h) \oplus u_{(m)}(\tilde{h}))$$

is the Jordan decomposition of $J_m(f)$. If $g \in c_\mu$, then g is conjugate to

$$J_\mu = \bigoplus_{f \in \Phi} J_{\mu(f)}(f)$$

in $\text{GL}(\|\mu\|, \mathbb{K}_2)$. Claim (a) follows from the uniqueness of the Jordan decomposition.

(b) In $\text{GL}(n, \mathbb{K}_2)$,

$$C_{\text{GL}(n, \mathbb{K}_2)}(J_{(1^m)}(f)) \cong \begin{cases} \text{GL}(m, \mathbb{K}_{2d(f)}) & \text{if } d(f) \text{ is odd,} \\ \text{GL}(m, \mathbb{K}_{d(f)}) \oplus \text{GL}(m, \mathbb{K}_{d(f)}) & \text{if } d(f) \text{ is even.} \end{cases}$$

If $d(f)$ is odd and $s = xJ_{(1^m)}(f)x^{-1}$ for some $x \in \text{GL}(n, \mathbb{K}_2)$, then

$$(x\text{GL}(m, \mathbb{K}_{2d(f)})x^{-1})^F = \text{GL}(m, \mathbb{K}_{2d(f)})^{x \circ F} \cong \text{GL}(m, \mathbb{K}_{2d(f)})^F = \text{U}(m, \mathbb{K}_{2d(f)}).$$

Suppose $d(f)$ is even so that $f = h\tilde{h}$, and let $y \in \text{GL}(n, \mathbb{K}_2)$ such that $\text{GL}(n, \mathbb{K})^F = y\text{GL}(n, \mathbb{K})^{F'}y^{-1}$ (see the comment after (2.2)). The element $J(f)$ is conjugate to

$$J(h) \oplus (F'(J(h))) \in \text{GL}(n, \mathbb{K}_2)^{F'},$$

whose centralizer in $\text{GL}(n, \mathbb{K}_2)^{F'}$ is

$$\{g \oplus F'(g) \mid g \in \text{GL}(m, \mathbb{K}_{d(f)})\} \cong \text{GL}(m, \mathbb{K}_{d(f)}). \quad \square$$

4. THE ENNOLA CONJECTURE

4.1. The characteristic map. Let $X = \{X_1, X_2, \dots\}$ be an infinite set of variables and let $\Lambda(X)$ be the graded \mathbb{C} -algebra of symmetric functions in the variables $\{X_1, X_2, \dots\}$. For $\nu = (\nu_1, \nu_2, \dots, \nu_\ell) \in \mathcal{P}$, the *power-sum symmetric function* $p_\nu(X)$ is

$$p_\nu(X) = p_{\nu_1}(X)p_{\nu_2}(X)\cdots p_{\nu_\ell}(X), \quad \text{where } p_m(X) = X_1^m + X_2^m + \cdots.$$

The irreducible characters ω^λ of S_n are indexed by $\lambda \in \mathcal{P}_n$. Let $\omega^\lambda(\nu)$ be the value of ω^λ on a permutation with cycle type ν . For $\lambda \in \mathcal{P}$, the *Schur function* $s_\lambda(X)$ is given by

$$(4.1) \quad s_\lambda(X) = \sum_{\nu \in \mathcal{P}_{|\lambda|}} \omega^\lambda(\nu) z_\nu^{-1} p_\nu(X), \quad \text{where } z_\nu = \prod_{i \geq 1} i^{m_i} m_i!$$

is the order of the centralizer in S_n of the conjugacy class corresponding to $\nu = (1^{m_1} 2^{m_2} \dots) \in \mathcal{P}$. Let $t \in \mathbb{C}$. For $\mu \in \mathcal{P}$, the *Hall-Littlewood symmetric function* $P_\mu(X; t)$ is given by

$$(4.2) \quad s_\lambda(X) = \sum_{\mu \in \mathcal{P}_{|\lambda|}} K_{\lambda\mu}(t) P_\mu(X; t),$$

where $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial (as in [18, III.6]). For $\nu, \mu \in \mathcal{P}_n$, the *classical Green function* $Q_\nu^\mu(t)$ is given by

$$(4.3) \quad p_\nu(X) = \sum_{\mu \in \mathcal{P}_{|\nu|}} Q_\nu^\mu(t^{-1}) t^{n(\mu)} P_\mu(X; t).$$

As a graded ring,

$$\begin{aligned} \Lambda(X) &= \mathbb{C}\text{-span}\{p_\nu(X) \mid \nu \in \mathcal{P}\} \\ &= \mathbb{C}\text{-span}\{s_\lambda(X) \mid \lambda \in \mathcal{P}\} \\ &= \mathbb{C}\text{-span}\{P_\mu(X; t) \mid \mu \in \mathcal{P}\}. \end{aligned}$$

For every $f \in \Phi$, fix a set of independent variables $X^{(f)} = \{X_1^{(f)}, X_2^{(f)}, \dots\}$, and for any symmetric function h , we let $h(f) = h(X^{(f)})$ denote the symmetric function in the variables $X^{(f)}$. Let

$$\Lambda = \mathbb{C}\text{-span}\{P_\mu \mid \mu \in \mathcal{P}^\Phi\}, \quad \text{where } P_\mu = (-q)^{-n(\mu)} \prod_{f \in \Phi} P_{\mu(f)}(f; (-q)^{d(f)}).$$

Then

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n, \quad \text{where } \Lambda_n = \mathbb{C}\text{-span}\{P_\mu \mid \|\mu\| = n\},$$

makes Λ a graded \mathbb{C} -algebra. Define a Hermitian inner product on Λ by

$$\langle P_\mu, P_\nu \rangle = a_\mu^{-1} \delta_{\mu\nu}.$$

For each $\varphi \in \Theta$ let $Y^{(\varphi)} = \{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \dots\}$ be an infinite variable set, and for a symmetric function h , let $h(\varphi) = h(Y^{(\varphi)})$. Relate symmetric

functions in the X variables to symmetric functions in the Y variables via the transform

$$(4.4) \quad p_n(\varphi) = (-1)^{n|\varphi|-1} \sum_{x \in \mathbb{M}_{n|\varphi|}} \xi(x) p_{n|\varphi|/d(f_x)}(f_x),$$

where $\varphi \in \Theta$, $\xi \in \varphi$, and $f_x \in \Phi$ satisfies $f_x(x) = 0$.

Then

$$(4.5) \quad \Lambda = \mathbb{C}\text{-span}\{s_\lambda \mid \lambda \in \mathcal{P}^\Theta\}, \quad \text{where} \quad s_\lambda = \prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(\varphi).$$

Let C_n denote the set of complex-valued class functions of the group U_n , and for $\|\mu\| = n$, let $\pi_\mu : U_n \rightarrow \mathbb{C}$ be given by

$$\pi_\mu(u) = \begin{cases} 1 & \text{if } u \in c_\mu, \\ 0 & \text{otherwise,} \end{cases} \quad \text{where } u \in U_n.$$

Then the π_μ form a \mathbb{C} -basis for C_n . By Proposition 3.2, the usual inner product on class functions of finite groups, $\langle \cdot, \cdot \rangle : C_n \times C_n \rightarrow \mathbb{C}$, satisfies

$$\langle \pi_\mu, \pi_\lambda \rangle = a_\mu^{-1} \delta_{\mu\lambda}.$$

For $\alpha_i \in C_{n_i}$, Ennola [5] defined a product $\alpha_1 \star \alpha_2 \in C_{n_1+n_2}$, which takes the following value on the conjugacy class c_λ :

$$\alpha_1 \star \alpha_2(c_\lambda) = \sum_{\|\mu_i\|=n_i} g_{\mu_1\mu_2}^\lambda \alpha_1(c_{\mu_1}) \alpha_2(c_{\mu_2}),$$

where $g_{\mu_1\mu_2}^\lambda$ is the product of Hall polynomials (see [18, Chapter II])

$$g_{\mu_1\mu_2}^\lambda = \prod_{f \in \Phi} g_{\mu_1(f)\mu_2(f)}^{\lambda(f)}((-q)^{d(f)}).$$

Extend the inner product to

$$C = \bigoplus_{n \geq 0} C_n,$$

by requiring the components C_n and C_m to be orthogonal for $n \neq m$. This gives C a graded \mathbb{C} -algebra structure. The *characteristic map* is

$$\begin{aligned} \text{ch} : C &\longrightarrow \Lambda \\ \pi_\mu &\longmapsto P_\mu, \quad \text{for } \mu \in \mathcal{P}^\Phi. \end{aligned}$$

Proposition 4.1. *Let multiplication in the character ring C of U_n be given by \star . Then the characteristic map $\text{ch} : C \rightarrow \Lambda$ is an isometric isomorphism of graded \mathbb{C} -algebras.*

Proof. It is clear that ch is a linear graded isometric isomorphism of vector spaces. It is a ring homomorphism because

$$\begin{aligned} \text{ch}(\pi_\mu \pi_\nu) &= \sum_{\lambda \in \mathcal{P}^\Phi} g_{\mu\nu}^\lambda \text{ch}(\pi_\lambda) \\ &= \sum_{\lambda \in \mathcal{P}^\Phi} g_{\mu\nu}^\lambda P_\lambda \\ &= P_\mu P_\nu \quad (\text{by [18, III.3.6]}) \\ &= \text{ch}(\pi_\mu) \text{ch}(\pi_\nu). \quad \square \end{aligned}$$

4.2. The Ennola conjecture I. Following the work of Green [8] on the general linear group, Ennola was able to obtain the following result. We follow the proof in Macdonald [18, IV.4] on the general linear group case.

Proposition 4.2 (Ennola). *The set $\{s_\lambda \mid \lambda \in \mathcal{P}^\Theta\}$ is an orthonormal basis for Λ .*

Proof. For $u \in \Lambda$, if

$$u = \sum_{\mu \in \mathcal{P}^\Phi} u_\mu P_\mu \quad \text{then let} \quad \bar{u} = \sum_{\mu \in \mathcal{P}^\Phi} \bar{u}_\mu P_\mu,$$

where $\bar{x} \in \mathbb{C}$ is the complex conjugate of $x \in \mathbb{C}$. It follows from [18, I.4.3] that

$$(4.6) \quad \log \left(\sum_{\lambda \in \mathcal{P}^\Theta} s_\lambda \otimes \bar{s}_\lambda \right) = \sum_{n \geq 1} \frac{1}{n} \sum_{\xi \in \mathbb{M}_n^*} p_{n/|\varphi_\xi|}(\varphi_\xi) \otimes \overline{p_{n/|\varphi_\xi|}(\varphi_\xi)}.$$

where $\varphi_\xi \in \Theta$ is the orbit such that $\xi \in \varphi_\xi$. Apply the transform (4.4) and orthogonality of characters to obtain

$$(4.7) \quad \sum_{\xi \in \mathbb{M}_n^*} p_{n/|\varphi_\xi|}(\varphi_\xi) \otimes \overline{p_{n/|\varphi_\xi|}(\varphi_\xi)} = |\mathbb{M}_n| \sum_{x \in \mathbb{M}_n} p_{n/d(f_x)}(f_x) \otimes p_{n/d(f_x)}(f_x).$$

From (4.6) and (4.7), we have

$$(4.8) \quad \begin{aligned} \log \left(\sum_{\lambda \in \mathcal{P}^\Theta} s_\lambda \otimes \bar{s}_\lambda \right) &= \sum_{n \geq 1} \frac{(q^n - (-1)^n)}{n} \sum_{x \in \mathbb{M}_n} p_{n/d(f_x)}(f_x) \otimes p_{n/d(f_x)}(f_x) \\ &= \sum_{n \geq 1} \frac{1}{n} \sum_{f \in \Phi} (q^{nd(f)} - (-1)^{nd(f)}) p_n(f) \otimes p_n(f). \end{aligned}$$

On the other hand, note that by [18, III.4.4] the $P_\nu(t)$ satisfy a Cauchy kernel formula

$$\sum_{\nu \in \mathcal{P}} t^{|\nu|+2n(\nu)} a_\nu(t^{-1}) P_\nu(x; t) P_\nu(y; t) = \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j}.$$

With substitutions $t \mapsto (-q)^{-d(f)}$ and $x_i \mapsto q^{d(f)}x_i$ we obtain

$$\begin{aligned} & \prod_{i,j} \frac{1 - (-1)^{d(f)}x_i y_j}{1 - q^{d(f)}x_i y_j} \\ &= \sum_{\nu \in \mathcal{P}} (-1)^{|\nu|d(f)} q^{-2n(\nu)d(f)} a_\nu ((-q)^{d(f)}) P_\nu(x; (-q)^{-d(f)}) P_\nu(y; (-q)^{-d(f)}). \end{aligned}$$

Make the further substitution $x_i \mapsto X_i^{(f)} \otimes 1$, $y_j \mapsto 1 \otimes X_j^{(f)}$ and take the product on both sides over all $f \in \Phi$ to get

$$\prod_{f \in \Phi} \prod_{i,j} \frac{1 - (-1)^{d(f)} X_i^{(f)} \otimes X_j^{(f)}}{1 - q^{d(f)} (X_i^{(f)} \otimes X_j^{(f)})} = \sum_{\mu \in \mathcal{P}^\Phi} a_\mu P_\mu \otimes P_\mu.$$

Take logarithms of both sides to obtain

$$\begin{aligned} & \log \left(\sum_{\mu \in \mathcal{P}^\Phi} a_\mu P_\mu \otimes P_\mu \right) \\ &= \sum_{\substack{f \in \Phi \\ i,j}} \left(\log(1 - (-1)^{d(f)} X_i^{(f)} \otimes X_j^{(f)}) - \log(1 - q^{d(f)} X_i^{(f)} \otimes X_j^{(f)}) \right) \\ &= \sum_{n \geq 1} \frac{1}{n} \sum_{f \in \Phi} \sum_{i,j} (q^{d(f)n} - (-1)^{d(f)n}) (X_i^{(f)} \otimes X_j^{(f)})^n \\ (4.9) \quad &= \sum_{n \geq 1} \frac{1}{n} \sum_{f \in \Phi} (q^{nd(f)} - (-1)^{nd(f)}) p_n(f) \otimes p_n(f). \end{aligned}$$

We may conclude from (4.8) and (4.9) that

$$(4.10) \quad \sum_{\lambda \in \mathcal{P}^\Theta} s_\lambda \otimes \bar{s}_\lambda = \sum_{\mu \in \mathcal{P}^\Phi} a_\mu P_\mu \otimes P_\mu.$$

Suppose

$$s_\lambda = \sum_{\mu \in \mathcal{P}^\Phi} u_{\lambda\mu} P_\mu = \sum_{\mu \in \mathcal{P}^\Phi} v_{\lambda\mu} a_\mu P_\mu.$$

Compare coefficients of $a_\mu P_\mu \otimes P_\mu$ on both sides of (4.10) to obtain

$$\sum_{\lambda \in \mathcal{P}^\Phi} u_{\lambda\mu} \bar{v}_{\lambda\nu} = \delta_{\mu\nu}.$$

It follows that $\langle s_\lambda, s_\nu \rangle = \delta_{\lambda\nu}$, as desired. \square

Now let $\chi^\lambda \in R$ be class functions so that $\chi^\lambda(1) > 0$ and $\text{ch}(\chi^\lambda) = \pm s_\lambda$. Ennola conjectured that $\{\chi^\lambda \mid \lambda \in \mathcal{P}_n^\Theta\}$ is the set of irreducible characters of U_n . He pointed out that if one could show that the product \star takes virtual characters to virtual characters, then the conjecture would follow. There is no known direct proof of this fact, however. Significant progress on Ennola's conjecture was only made after the work of Deligne and Lusztig [3].

4.3. Deligne-Lusztig Induction. Let $T_\nu \cong \mathbb{M}_{\nu_1} \times \cdots \times \mathbb{M}_{\nu_\ell}$ be a maximal torus of U_n . If $t \in T_\nu$, then t is conjugate to

$$J_{(1^{m_1})}(f_1) \oplus \cdots \oplus J_{(1^{m_\ell})}(f_\ell), \quad \text{where } f_i \in \Phi, m_i d(f_i) = \nu_i.$$

Define $\gamma_t \in \mathcal{P}^\Phi$ by

$$(4.11) \quad \gamma_t(f) \quad \text{has parts} \quad \{m_i \mid f_i = f\}.$$

Note that $(\gamma_t)_u(t-1) = \nu$, but in general $t \notin c_{\gamma_t}$.

Example. If $t \in T_{(4,4,2,2,1)}$ and t is conjugate to

$$J_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}}(t-1) \oplus J_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}(t^2+1) \oplus J_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}(t-1) \oplus J_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}(t^2+1) \oplus J_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}(t-1),$$

then

$$\gamma_t = \left(\begin{array}{c} \begin{array}{cccc} \square & \square & \square & \square \end{array}^{(t-1)} \\ \begin{array}{cc} \square & \square \\ \square & \square \end{array} \\ \begin{array}{c} \square \\ \square \end{array} \end{array}, \begin{array}{c} \begin{array}{cc} \square & \square \end{array}^{(t^2+1)} \\ \begin{array}{c} \square \\ \square \end{array} \end{array} \right).$$

For $\mu \in \mathcal{P}^\Phi$, let L_μ , $\gamma \in \mathcal{P}_s^\mu$ and T_γ be as in Section 2.3. Let θ be a character of T_ν . The *Deligne-Lusztig character* $R_\nu(\theta) = R_{T_\nu}^{U_n}(\theta)$ is the virtual character of U_n given by

$$(R_\nu(\theta))(g) = \sum_{\substack{t \in T_\nu \\ \gamma_t \in \mathcal{P}_s^\mu}} \theta(t) Q_{T_{\gamma_t}}^{L_\mu}(u),$$

where $g \in c_\mu$ has Jordan decomposition $g = su$ (thus, by Lemma 3.1 $C_{U_n}(s) \cong L_\mu$), and $Q_{T_{\gamma_t}}^{L_\mu}(u)$ is a Green function for the unitary group (see, for example, [2]).

Deligne and Lusztig proved

$$\begin{aligned} C_n &= \{\text{class functions of } U_n\} \\ &= \mathbb{C}\text{-span}\{R_\nu(\theta) \mid \nu \in \mathcal{P}_n, \theta \in \text{Hom}(T_\nu, \mathbb{C}^\times)\}, \end{aligned}$$

so we may define *Deligne-Lusztig induction* by

$$(4.12) \quad \begin{array}{ccc} R_{U_m \oplus U_n}^{U_{m+n}} : & C_m \otimes C_n & \longrightarrow C_{m+n} \\ & R_\alpha^{U_m}(\theta_\alpha) \otimes R_\beta^{U_n}(\theta_\beta) & \mapsto R_{T_\alpha \oplus T_\beta}^{U_{m+n}}(\theta_\alpha \otimes \theta_\beta), \end{array}$$

for $\alpha \in \mathcal{P}_m$, $\beta \in \mathcal{P}_n$, $\theta_\alpha \in \text{Hom}(T_\alpha, \mathbb{C})$, and $\theta_\beta \in \text{Hom}(T_\beta, \mathbb{C})$.

Let Λ and C be as in Section 4.1, except we now give C a graded \mathbb{C} -algebra structure using Deligne-Lusztig induction. That is, we define a multiplication \circ on C by

$$\chi \circ \eta = R_{U_m \oplus U_n}^{U_{m+n}}(\chi \otimes \eta), \quad \text{for } \chi \in C_m \text{ and } \eta \in C_n.$$

We recall the characteristic map defined in Section 4.2,

$$\begin{array}{ccc} \text{ch} : C & \longrightarrow & \Lambda \\ \pi_\mu & \mapsto & P_\mu \quad \text{for } \mu \in \mathcal{P}^\Phi. \end{array}$$

As noted in Proposition 4.1, it is immediate that ch is an isometric isomorphism of vector spaces, but it is not yet clear if ch is also a ring homomorphism when C has multiplication given by Deligne-Lusztig induction.

4.4. The Ennola conjecture II. To prove the Ennola conjecture we require two further ingredients:

- (1) Theorem 4.2 and Corollary 4.1 establish that this new characteristic map is also a ring isomorphism by using a key result by Kawanaka on Green functions to evaluate $\text{ch}(R_\nu(\theta))$.
- (2) Corollary 4.3 uses an explicit decomposition by Lusztig and Srinivasan of irreducible characters of U_n into Deligne-Lusztig characters to complete the reproof of the Ennola conjecture.

To compute $\text{ch}(R_\nu(\theta))$, we need to write the Green functions $Q_{T_\gamma}^{L_\mu}(u)$ as polynomials in q . These Green functions turn out to be those of the general linear group, except with q replaced by $-q$, which is the essence of Ennola's original idea. This fact was proven by Hotta and Springer [11] for the case that $p = \text{char}(\mathbb{F}_q)$ is large compared to n , and was finally proven in full generality by Kawanaka [14].

Theorem 4.1 (Hotta-Springer, Kawanaka). *The Green functions for the unitary group are given by $Q_{T_\gamma}^{L_\mu}(u) = Q_\gamma^\mu(-q)$, where*

$$Q_\gamma^\mu(-q) = \prod_{f \in \Phi_\mu} Q_{\gamma(f)}^{\mu(f)}((-q)^{d(f)}),$$

and $Q_\gamma^\mu(q)$ is the classical Green function as in (4.3).

For $\nu = (\nu_1, \nu_2, \dots, \nu_\ell) \in \mathcal{P}$ and $\theta = \theta_1 \otimes \theta_2 \otimes \dots \otimes \theta_\ell$ a character of T_ν , define

$$p_{\mu_{\nu\theta}} = \prod_{\varphi \in \Theta} p_{\mu_{\nu\theta}(\varphi)}(\varphi), \quad \text{where } \mu_{\nu\theta}(\varphi) = (\nu_i/|\varphi| \mid \theta_i \in \varphi).$$

Theorem 4.2. *Let $\nu = (\nu_1, \nu_2, \dots, \nu_\ell) \in \mathcal{P}$, $\theta = \theta_1 \otimes \theta_2 \otimes \dots \otimes \theta_\ell$ be a character of T_ν , and $\nu = \mu_{\nu\theta} \in \mathcal{P}^\Theta$. Then*

$$\text{ch}(R_\nu(\theta)) = (-1)^{|\nu| - \ell(\nu)} p_\nu.$$

Proof. By Theorem 4.1, and since $\text{ch}(\pi_\mu) = P_\mu$, it suffices to show that the coefficient of P_μ in the expansion of $(-1)^{|\nu| - \ell(\nu)} p_\nu$ is

$$\sum_{\substack{t \in T_\nu \\ \gamma_t \in \mathcal{P}_s^\mu}} \theta_\nu(t) Q_{\gamma_t}^\mu(-q).$$

Since

$$p_\nu = p_{\nu_1/|\varphi_1|}(\varphi_1) \cdots p_{\nu_\ell/|\varphi_\ell|}(\varphi_\ell), \quad \text{where } \theta_i \in \varphi_i,$$

the transform (4.4) implies

$$(-1)^{|\nu|-\ell(\nu)} p_\nu = \prod_{i=1}^{\ell} \sum_{t_i \in \mathfrak{k}_{\nu_i}^\times} \theta_i(t_i) p_{\nu_i/d_i}(f_i) = \sum_{t \in T_\nu} \theta(t) \left(\prod_{i=1}^{\ell} p_{\nu_i/d_i}(f_i) \right),$$

where $f_i \in \Phi$ satisfies $f_i(t_i) = 0$ and $d_i = d(f_i)$. By definition (4.11),

$$(4.13) \quad (-1)^{|\nu|-\ell(\nu)} p_\nu = \sum_{t \in T_\nu} \theta(t) \left(\prod_{f \in \Phi} p_{\gamma_t(f)}(f) \right).$$

Change the basis from power-sums to Hall-Littlewood polynomials (4.3), and we obtain

$$\begin{aligned} & (-1)^{|\nu|-\ell(\nu)} p_\nu \\ &= \sum_{t \in T_\nu} \theta(t) \left(\prod_{f \in \Phi} \sum_{\mu(f) \in \mathcal{P}_{|\gamma_t(f)|}} Q_{\gamma_t(f)}^{\mu(f)}((-q)^{d(f)}) (-q)^{-d(f)} P_{\mu(f)}(f; (-q)^{-d(f)}) \right) \\ &= \sum_{t \in T_\nu} \theta(t) \sum_{\substack{\mu \in \mathcal{P}^\Phi \\ \gamma_t \in \mathcal{P}_s^\mu}} Q_{\gamma_t}^\mu(-q) P_\mu \\ &= \sum_{\mu \in \mathcal{P}^\Phi} \left(\sum_{\substack{t \in T_\nu \\ \gamma_t \in \mathcal{P}_s^\mu}} \theta(t) Q_{\gamma_t}^\mu(-q) \right) P_\mu, \end{aligned}$$

as desired. \square

Corollary 4.1. *Let multiplication in the character ring C of U_n be given by \circ . Then the characteristic map $\text{ch} : C \rightarrow \Lambda$ is an isometric isomorphism of graded \mathbb{C} -algebras.*

Proof. Let $\alpha \in \mathcal{P}_m$, $\beta \in \mathcal{P}_n$, $\theta_\alpha \in \text{Hom}(T_\alpha, \mathbb{C}^\times)$, $\theta_\beta \in \text{Hom}(T_\beta, \mathbb{C}^\times)$, $\alpha = \mu_\alpha \theta_\alpha$, and $\beta = \mu_\beta \theta_\beta$. By (4.12) and Theorem 4.2,

$$\begin{aligned} \text{ch} \left(R_{U_m \oplus U_n}^{U_{m+n}} (R_\alpha^{U_m}(\theta_\alpha) \otimes R_\beta^{U_n}(\theta_\beta)) \right) &= \text{ch} (R_{T_\alpha \oplus T_\beta}^{U_{m+n}} (\theta_\alpha \otimes \theta_\beta)) \\ &= (-1)^{m+n-\ell(\alpha)-\ell(\beta)} p_\alpha p_\beta \\ &= \text{ch}(R_\alpha^{U_m}(\theta_\alpha)) \text{ch}(R_\beta^{U_n}(\theta_\beta)). \end{aligned}$$

Since the Deligne-Lusztig characters span C , this product extends linearly to C . \square

An immediate consequence is that the graded multiplication that Ennola originally defined on C is exactly Deligne-Lusztig induction, or

Corollary 4.2. *Let $\chi \in C_m$ and $\eta \in C_n$. Then*

$$\chi \circ \eta = \chi \star \eta.$$

We therefore having the advantage of taking either definition when convenience demands.

For $\lambda \in \mathcal{P}^\Theta$, let L_λ , W_λ , and T_γ , $\gamma \in \mathcal{P}_s^\lambda$, be as in Section 2.3.

Note that the combinatorics of γ almost specifies character θ_γ of T_γ in the sense that

$$\theta_\gamma(T_\gamma(\varphi)) = \theta_\varphi(T_\gamma(\varphi)), \quad \text{for some } \theta_\varphi \in \varphi.$$

In fact, we may define

$$(4.14) \quad R_\gamma = R_{T_\gamma}^{U_n}(\theta_\gamma) = \text{ch}^{-1}((-1)^{|\gamma| - \ell(\gamma)} p_\gamma),$$

where θ_γ is any choice of the θ_φ 's.

For every $\lambda \in \mathcal{P}^\Theta$ there exists a character ω^λ of W_λ defined by

$$\omega^\lambda(\gamma) = \prod_{\varphi \in \Theta} \omega^{\lambda(\varphi)}(\gamma(\varphi)),$$

where $\omega^{\lambda(\varphi)}$ is the value of ω^λ on the conjugacy class c_γ corresponding to $\gamma \in \mathcal{P}_s^\Theta$.

In [17], Lusztig and Srinivasan decomposed the irreducible characters of U_n as linear combinations of Deligne-Lusztig characters, as follows.

Theorem 4.3 (Lusztig-Srinivasan). *Let $\lambda \in \mathcal{P}_n^\Theta$. Then there exists $\tau'(\lambda) \in \mathbb{Z}_{\geq 0}$ such that the class function*

$$R(\lambda) = (-1)^{\tau'(\lambda) + \lfloor n/2 \rfloor + \sum_{\varphi \in \Theta} |\lambda(\varphi)| + |\varphi| \lfloor |\lambda(\varphi)|/2 \rfloor} \sum_{\gamma \in \mathcal{P}_s^\lambda} \frac{\omega^\lambda(\gamma)}{z_\gamma} R_\gamma$$

is an irreducible character of U_n (z_γ is as in (2.14)).

Remark. The sign

$$(-1)^{\lfloor n/2 \rfloor + \sum_{\varphi \in \Theta} |\lambda(\varphi)| + |\varphi| \lfloor |\lambda(\varphi)|/2 \rfloor} = (-1)^{\mathbb{F}_q\text{-rank of } U_n + \mathbb{F}_q\text{-rank of } L_\lambda},$$

and Theorem 5.1 will show that

$$\tau'(\lambda) = n(\lambda') + \|\lambda\| - \sum_{\varphi \in \Theta} |\varphi| \lfloor |\lambda(\varphi)|/2 \rfloor = n(\lambda') + \sum_{\varphi \in \Theta} |\varphi| \lfloor |\lambda(\varphi)|/2 \rfloor.$$

Corollary 4.3 (Ennola Conjecture). *For $\lambda \in \mathcal{P}^\Theta$, there exists $\tau(\lambda) \in \mathbb{Z}_{\geq 0}$ such that*

$$\left\{ \text{ch}^{-1} \left((-1)^{\tau(\lambda)} s_\lambda \right) \mid \lambda \in \mathcal{P}_n^\Theta \right\}$$

is the set of irreducible characters of U_n .

Proof. By Theorem 4.3 and Theorem 4.2,

$$\begin{aligned} \text{ch}(R(\lambda)) &= (-1)^{\tau'(\lambda) + \lfloor n/2 \rfloor + \sum_{\varphi \in \Theta} |\lambda(\varphi)| + |\varphi| \lfloor |\lambda(\varphi)|/2 \rfloor} \sum_{\gamma \in \mathcal{P}_s^\lambda} \frac{\omega^\lambda(\gamma)}{z_\gamma} (-1)^{n - \ell(\gamma)} p_\gamma \\ &= (-1)^{\tau'(\lambda) + \lfloor n/2 \rfloor + n + \sum_{\varphi \in \Theta} |\varphi| \lfloor |\lambda(\varphi)|/2 \rfloor} \sum_{\gamma \in \mathcal{P}_s^\lambda} \frac{\omega^\lambda(\gamma)}{z_\gamma} (-1)^{\sum_{\varphi \in \Theta} |\lambda(\varphi)| - \ell(\gamma)} p_\gamma \end{aligned}$$

Note that the sign character ω^{λ_s} of W_{λ} acts by

$$\omega^{\lambda_s}(\gamma) = (-1)^{\sum_{\varphi \in \Theta} |\gamma(\varphi)| - \ell(\gamma)},$$

and that $\omega^{\lambda} \otimes \omega^{\lambda_s} = \omega^{\lambda'}$, so since $\gamma \in \mathcal{P}_s^{\lambda}$,

$$\begin{aligned} \text{ch}(R(\lambda)) &= (-1)^{\tau'(\lambda) + [n/2] + n + \sum_{\varphi \in \Theta} |\varphi| \lceil |\lambda(\varphi)|/2 \rceil} \sum_{\gamma \in \mathcal{P}_s^{\lambda}} \frac{(\omega^{\lambda} \otimes \omega^{\lambda_s})(\gamma)}{z_{\gamma}} p_{\gamma} \\ &= (-1)^{\tau'(\lambda) + [n/2] + n + \sum_{\varphi \in \Theta} |\varphi| \lceil |\lambda(\varphi)|/2 \rceil} \sum_{\gamma \in \mathcal{P}_s^{\lambda}} \frac{\omega^{\lambda'}(\gamma)}{z_{\gamma}} p_{\gamma}, \end{aligned}$$

and by applying (4.1) to a product over Θ ,

$$= (-1)^{\tau'(\lambda) + [n/2] + n + \sum_{\varphi \in \Theta} |\varphi| \lceil |\lambda(\varphi)|/2 \rceil} s_{\lambda'}. \quad \square$$

Remark. There are at least two natural ways to index the irreducible characters of U_n by Θ -partitions: Theorem 4.3 gives a natural indexing by Θ -partitions, but Corollary 4.3 indicates that the conjugate choice is equally natural. Since we like to think of Schur functions as irreducible characters, we have chosen the latter indexing. However, several references, including Ennola [5], Ohmori [20], and Henderson [10], make use of the former.

5. CHARACTERS DEGREES

Let $\lambda \in \mathcal{P}^{\Theta}$, and suppose $\square \in \lambda$ is in position (i, j) in $\lambda(\varphi)$ for some $\varphi \in \Theta$. The *hook length* $\mathbf{h}(\square)$ of \square is

$$\mathbf{h}(\square) = |\varphi| h(\square), \quad \text{where } h(\square) = \lambda(\varphi)_i - \lambda(\varphi)'_j - i - j + 1,$$

is the usual hook length for partitions.

Example. For $|\theta| = 1$, $|\varphi| = 3$ and

$$\lambda = \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}^{(\theta)}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}^{(\varphi)} \right), \text{ the hook lengths are } \left(\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 & \square \\ \hline \end{array}^{(\theta)}, \begin{array}{|c|c|} \hline 12 & 3 \\ \hline 6 & \square \\ \hline 3 & \square \\ \hline \end{array}^{(\varphi)} \right).$$

Example 2 in [18, I.1] implies that

$$(5.1) \quad \sum_{\square \in \lambda} \mathbf{h}(\square) = \|\lambda\| + n(\lambda) + n(\lambda').$$

For $\lambda \in \mathcal{P}^{\Theta}$, let

$$\eta^{\lambda} = \text{ch}^{-1}(s_{\lambda}).$$

Theorem 5.1. *Let $\lambda \in \mathcal{P}^{\Theta}$ and let 1 be the identity in $U_{\|\lambda\|}$. Then*

$$\eta^{\lambda}(1) = (-1)^{\tau(\lambda)} q^{n(\lambda')} \frac{\prod_{1 \leq i \leq \|\lambda\|} (q^i - (-1)^i)}{\prod_{\square \in \lambda} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})},$$

where $\tau(\lambda) = \|\lambda\|(\|\lambda\| + 3)/2 + n(\lambda) \equiv \lfloor \|\lambda\|/2 \rfloor + n(\lambda) \pmod{2}$. So for each λ , we have $\chi^\lambda = (-1)^{\tau(\lambda)}\eta^\lambda$.

Proof. We follow the computations in [18, IV.6]. Let η_μ^λ be the value of η^λ on the conjugacy class c_μ . Then

$$s_\lambda = \sum_{\mu \in \mathcal{P}_{\|\lambda\|}^\Phi} \eta_\mu^\lambda P_\mu,$$

implies $\eta_\mu^\lambda = \langle s_\lambda, a_\mu P_\mu \rangle$. Since the s_λ are orthonormal,

$$(5.2) \quad a_\mu P_\mu = \sum_{\lambda} \eta_\mu^\lambda \bar{s}_\lambda \quad \text{where} \quad \bar{s}_\lambda = \sum_{\mu \in \mathcal{P}_{\|\lambda\|}^\Phi} \bar{\eta}_\mu^\lambda P_\mu$$

is obtained by taking the complex conjugates of the coefficients of the P_μ .

If c_{μ_1} is the conjugacy class corresponding to the identity element 1 of U_m , then $\mu_1(t-1) = (1^m)$ and $\mu_1(f) = 0$ for $f \neq t-1$. From the definition of a_μ , and the fact that $P_{(1^m)}(x; t) = e_m(x)$ (see [18, III.8]), we have

$$\begin{aligned} a_{\mu_1} P_{\mu_1} &= (-1)^m (-q)^{m+m(m-1)} \psi_m(-q^{-1}) (-q)^{-m(m-1)/2} P_{(1^m)}(X^{(t-1)}; (-q)^{-1}) \\ &= (-1)^m (-q)^{m(m+1)/2} \psi_m((-q)^{-1}) P_{(1^m)}(X^{(t-1)}; (-q)^{-1}) \\ &= \psi_m(-q) e_m(t-1). \end{aligned}$$

Therefore, by (5.2),

$$(5.3) \quad \psi_m(-q) e_m(t-1) = \sum_{\|\lambda\|=m} \eta^\lambda(1) \bar{s}_\lambda.$$

Let $\delta : \Lambda \rightarrow \mathbb{C}$ be the \mathbb{C} -algebra homomorphism defined by

$$\delta(p_m(f)) = \begin{cases} (-1)^{m-1}/(q^m - (-1)^m), & \text{if } f = t-1, \\ 0, & \text{if } f \neq t-1. \end{cases}$$

Apply $\delta \otimes 1$ to both sides of Equation (4.8) to obtain

$$\log \left(\sum_{\lambda \in \mathcal{P}^\Theta} \delta(s_\lambda) \bar{s}_\lambda \right) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} p_m(t-1) = \log \prod_i (1 + X_i^{(t-1)}).$$

By exponentiating and expanding the product to e_m 's, we have

$$\sum_{m \geq 0} e_m(t-1) = \sum_{\lambda} \delta(s_\lambda) \bar{s}_\lambda,$$

which gives

$$e_m(t-1) = \sum_{\|\lambda\|=m} \delta(s_\lambda) \bar{s}_\lambda.$$

Compare coefficients with (5.3) to deduce

$$(5.4) \quad \eta^\lambda(1) = \psi_m(-q) \delta(s_\lambda), \quad \text{where} \quad \delta(s_\lambda) = \prod_{\varphi \in \Theta} \delta(s_{\lambda(\varphi)}(\varphi)).$$

By the definitions of δ and $p_m(\varphi)$, $\varphi \in \Theta$,

$$\delta(p_m(\varphi)) = (-1)^{|\varphi|m} ((-q)^{|\varphi|m} - 1)^{-1} = (-1)^{|\varphi|m} \sum_{i \geq 1} (-q)^{-i|\varphi|m},$$

so

$$\delta(p_m(Y_1^{(\varphi)}, Y_2^{(\varphi)}, \dots)) = (-1)^{|\varphi|m} p_m((-q)^{-|\varphi|}, (-q)^{-2|\varphi|}, \dots).$$

That is, applying δ to a homogeneous symmetric function $f(Y^{(\varphi)})$ of degree m replaces each $Y_i^{(\varphi)}$ by $(-q)^{-i|\varphi|}$ and multiplies by $(-1)^{|\varphi|m}$. In particular, for $\lambda \in \mathcal{P}$, $\varphi \in \Theta$,

$$(5.5) \quad \delta(s_\lambda(\varphi)) = (-1)^{|\lambda||\varphi|} s_\lambda((-q)^{-|\varphi|}, (-q)^{-2|\varphi|}, \dots).$$

Example 2 in [18, I.3] implies

$$(5.6) \quad \delta(s_\lambda(\varphi)) = (-1)^{|\varphi||\lambda|} (-q)^{-|\varphi|(|\lambda|+n(\lambda))} \prod_{\square \in \lambda} (1 - (-q)^{-|\varphi|h(\square)})^{-1}.$$

Combine (5.4) and (5.6) to get

$$\begin{aligned} \delta(s_\lambda) &= (-1)^{|\lambda|} (-q)^{-|\lambda|-n(\lambda)} \prod_{\square \in \lambda} (1 - (-q)^{-\mathbf{h}(\square)})^{-1} \\ &= (-1)^{|\lambda|} (-q)^{-|\lambda|-n(\lambda)+\sum_{\square \in \lambda} \mathbf{h}(\square)} \prod_{\square \in \lambda} ((-q)^{\mathbf{h}(\square)} - 1)^{-1} \\ &= (-1)^{|\lambda|} (-q)^{n(\lambda')} \prod_{\square \in \lambda} ((-q)^{\mathbf{h}(\square)} - 1)^{-1}, \quad (\text{by (5.1)}) \\ &= (-1)^{|\lambda|+n(\lambda')+ \sum_{\square \in \lambda} \mathbf{h}(\square)} q^{n(\lambda')} \prod_{\square \in \lambda} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})^{-1}. \end{aligned}$$

Since

$$|\lambda| + n(\lambda') + \sum_{\square \in \lambda} \mathbf{h}(\square) = 2|\lambda| + 2n(\lambda') + n(\lambda) \equiv n(\lambda) \pmod{2},$$

we have

$$\delta(s_\lambda) = (-1)^{n(\lambda)} q^{n(\lambda')} \prod_{\square \in \lambda} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})^{-1}.$$

Since

$$\psi_m(-q) = \prod_{i=1}^m (1 - (-q)^i) = (-1)^{m+m(m+1)/2} \prod_{i=1}^m (q^i - (-1)^i),$$

we have

$$\eta^\lambda(1) = (-1)^{\tau(\lambda)} q^{n(\lambda')} \prod_{1 \leq i \leq |\lambda|} (q^i - (-1)^i) \prod_{\square \in \lambda} (q^{\mathbf{h}(\square)} - (-1)^{\mathbf{h}(\square)})^{-1}. \quad \square$$

By the Littlewood-Richardson rule, for any $\mu, \nu \in \mathcal{P}$, we have

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda,$$

where $c_{\mu\nu}^\lambda$ is the number of tableaux T of shape $\lambda - \mu$ and weight ν such that the word $w(T)$ is a lattice permutation [18, I.9]. So we have

$$(5.7) \quad s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda \quad \text{where} \quad c_{\mu\nu}^\lambda = \prod_{\varphi \in \Theta} c_{\mu(\varphi)\nu(\varphi)}^{\lambda(\varphi)}.$$

Corollary 5.1. *Let $\mu, \nu \in \mathcal{P}^\Theta$. Then $\chi^\mu \circ \chi^\nu$ is a character if and only if every $\lambda \in \mathcal{P}^\Theta$ such that $c_{\mu\nu}^\lambda > 0$ satisfies*

$$n(\mu) + n(\nu) \equiv n(\lambda) + \|\mu\| \|\nu\| \pmod{2}$$

Proof. Since

$$\text{ch}(\chi^\mu \circ \chi^\nu) = (-1)^{\tau(\mu) + \tau(\nu)} \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda.$$

the class function $\chi^\mu \circ \chi^\nu$ is a character if and only if every λ such that $c_{\mu\nu}^\lambda > 0$ satisfies

$$\tau(\mu) + \tau(\nu) \equiv \tau(\lambda) \pmod{2}.$$

For each λ such that $c_{\mu\nu}^\lambda > 0$, we have, for every $\varphi \in \Theta$, that there is a tableau of shape $\lambda(\varphi) - \mu(\varphi)$ and weight $\nu(\varphi)$. In particular, we have $|\mu(\varphi)| + |\nu(\varphi)| = |\lambda(\varphi)|$ for every $\varphi \in \Theta$, so that $\|\mu\| + \|\nu\| = \|\lambda\|$. The result follows from the definition of τ . \square

It follows that the operation \circ does not always take a pair of characters to a character. For example, let φ_1 be the orbit of the trivial character, and define $\mu = \nu = (\square^{(\varphi_1)})$. Then one λ such that $c_{\mu\nu}^\lambda > 0$ is

$$\lambda = \left(\begin{array}{c} \square \\ \square \end{array} \right)^{(\varphi_1)}.$$

Then we have $n(\mu) + n(\nu) = 0$ while $n(\lambda) + \|\mu\| \|\nu\| = 1$, and so $\chi^\mu \circ \chi^\nu$ is not a character by Corollary 5.1.

5.1. A character degree sum. Let d_r denote the number of F -orbits in Θ of size r . Since the F -action preserves \mathbb{M}_m^* and $|\mathbb{M}_m^*| = q^m - (-1)^m$, we have

$$(5.8) \quad q^m - (-1)^m = \sum_{r|m} r d_r, \quad \text{for } m \in \mathbb{Z}_{>0}.$$

Theorem 5.2. *The sum of the degrees of the complex irreducible characters of U_m is given by*

$$\sum_{\|\lambda\|=m} \chi^\lambda(1) = (q+1)q^2(q^3+1)q^4(q^5+1)\cdots\left(q^m + \frac{1 - (-1)^m}{2}\right).$$

Proof. Following a similar approach to [18, IV.6, Example 5], we consider the coefficient of t^m in the series

$$S = \sum_{\lambda \in \mathcal{P}^\Theta} (-1)^{n(\lambda) + \|\lambda\|} \delta(s_\lambda) t^{\|\lambda\|},$$

where $n(\lambda)$ is as in (2.7). Note that $S = S_o S_e$, where

$$S_o = \prod_{\substack{|\varphi| \\ \text{odd}}} \sum_{\lambda} (-1)^{n(\lambda)} \delta(s_{\lambda}(\varphi)) (-t)^{|\lambda||\varphi|} \text{ and } S_e = \prod_{\substack{|\varphi| \\ \text{even}}} \sum_{\lambda} \delta(s_{\lambda}(\varphi)) t^{|\lambda||\varphi|}.$$

Combine the following identity from Example 6 in [18, I.5],

$$\sum_{\lambda} (-1)^{n(\lambda)} s_{\lambda} = \prod_i (1 - x_i)^{-1} \prod_{i < j} (1 + x_i x_j)^{-1}$$

with (5.5) to obtain

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} (-1)^{n(\lambda)} (-1)^{|\varphi||\lambda|} \delta(s_{\lambda}(\varphi)) t^{|\lambda||\varphi|} \\ = \prod_{i \geq 1} (1 - (t(-q)^{-i})^{|\varphi|})^{-1} \prod_{1 \leq i < j} (1 + (t^2(-q)^{-i-j})^{|\varphi|})^{-1}. \end{aligned}$$

Taking the logarithm,

$$\begin{aligned} \log S_o &= \sum_{m \text{ odd}} d_m \left(\sum_{i \geq 1} \sum_{r \geq 1} \frac{(t(-q)^{-i})^{mr}}{r} + \sum_{1 \leq i < j} \sum_{r \geq 1} \frac{(- (t^2(-q)^{-i-j})^m)^r}{r} \right) \\ &= \sum_{m \text{ odd}} d_m \sum_{r \geq 1} \left(\frac{t^{mr}}{r} \sum_{i \geq 1} (-q)^{-imr} + (-1)^r \sum_{1 \leq i < j} t^{2mr} (-q)^{-(i+j)mr} \right) \\ &= \sum_{m \text{ odd}} d_m \sum_{r \geq 1} \frac{t^{mr}}{r((-q)^{mr} - 1)} \left(1 + (-1)^r \sum_{i \geq 1} t^{mr} (-q)^{-2imr} \right) \\ &= \sum_{m \text{ odd}} d_m \sum_{r \geq 1} \frac{t^{mr}}{r(q^{mr} - (-1)^{mr})} \left((-1)^{mr} + \sum_{i \geq 1} t^{mr} q^{-2imr} \right), \end{aligned}$$

where d_m is as in (5.8). Similarly for $|\varphi|$ even, by Example 4 in [18, I.5],

$$\sum_{\lambda} s_{\lambda} = \prod_i (1 - x_i)^{-1} \prod_{i < j} (1 - x_i x_j)^{-1}$$

so

$$\sum_{\lambda \in \mathcal{P}} \delta(s_{\lambda}(\varphi)) t^{|\lambda||\varphi|} = \prod_{i \geq 1} (1 - (tq^{-i})^{|\varphi|})^{-1} \prod_{1 \leq i < j} (1 - (t^2 q^{-i-j})^{|\varphi|})^{-1}.$$

Taking logarithms,

$$\begin{aligned} \log S_e &= \sum_{m \text{ even}} d_m \left(\sum_{i \geq 1} \sum_{r \geq 1} \frac{(tq^{-i})^{mr}}{r} + \sum_{1 \leq i < j} \sum_{r \geq 1} \frac{(t^2 q^{-i-j})^{mr}}{r} \right) \\ &= \sum_{m \text{ even}} d_m \sum_{r \geq 1} \frac{t^{mr}}{r(q^{mr} - (-1)^{mr})} \left((-1)^{mr} + \sum_{i \geq 1} t^{mr} q^{-2imr} \right). \end{aligned}$$

Set $N = mr$, and by (5.8),

$$\begin{aligned} \log S &= \log S_o + \log S_e \\ &= \sum_{N \geq 1} \left(\frac{(-t)^N}{N} + \sum_{i \geq 1} \frac{t^{2N} q^{-2iN}}{N} \right) \\ &= \log(1+t)^{-1} + \sum_{i \geq 1} \log(1 - t^2 q^{-2i})^{-1} \end{aligned}$$

By exponentiating,

$$S = \frac{1}{1+t} \prod_{i \geq 1} \frac{1}{1 - t^2 q^{-2i}} = (1-t) \prod_{i \geq 0} \frac{1}{1 - t^2 q^{-2i}} = (1-t) \sum_{l \geq 0} \frac{t^{2l}}{\psi_l(q^{-2})},$$

where the last equality comes from Example 4 in [18, I.2]. Multiply the coefficient of t^m by $(-1)^m |\psi_m(-q)| = (-1)^m \prod_{i=1}^m (q^i - (-1)^i)$ to obtain

$$\sum_{\|\lambda\|=m} \chi^\lambda(1) = (q+1)q^2(q^3+1)q^4(q^5+1) \cdots \left(q^m + \frac{(1 - (-1)^m)}{2} \right). \quad \square$$

Write $f_{U_m}(q) = \sum_{\|\lambda\|=m} \chi^\lambda(1)$. The polynomial $f_{G_m}(q)$ expressing the sum of the degrees of the complex irreducible characters of $G_m = \mathrm{GL}(m, \mathbb{F}_q)$, was computed in [6] for odd q and in [16] and Example 6 of [18, IV.6] for general q . From these results we see that

$$f_{U_m}(q) = (-1)^{m(m+1)/2} f_{G_m}(-q),$$

another example of Ennola duality.

Gow [6] and Klyachko [16] proved that the sum of the degrees of the complex irreducible characters of G_n is equal to the number of symmetric matrices in G_n . Gow accomplished this by considering the split extension of G_n by the transpose-inverse automorphism,

$$(5.9) \quad G_n^+ = \langle G_n, \kappa \mid \kappa^2 = 1, \kappa g \kappa = {}^t g^{-1} \text{ for every } g \in G_n \rangle,$$

and showed that every complex irreducible representation of G_n^+ could be realized over the real numbers. Instead, we show directly that the sum of the degrees of the complex irreducible characters of U_n is equal to the number of symmetric matrices in U_n , and make conclusions about the reality properties of the characters of U_n^+ , where

$$(5.10) \quad U_n^+ = \langle U_n, \kappa \mid \kappa^2 = 1, \kappa u \kappa = {}^t u^{-1} \text{ for every } u \in U_n \rangle.$$

Corollary 5.2. *The sum of the degrees of the complex irreducible characters of $U(n, \mathbb{F}_{q^2})$ is equal to the number of symmetric matrices in $U(n, \mathbb{F}_{q^2})$.*

Proof. Define S_{U_n} to be

$$S_{U_n} = \{\text{symmetric matrices in } U_n\}.$$

Then U_n acts on the set S_{U_n} by the action

$$u \cdot s = {}^t u s u, \quad s \in S_{U_n}, u \in U_n$$

To find the number of elements in S_{U_n} , we find the stabilizers of the orbits under this action. Consider the same action of $G_n = \text{GL}(n, \mathbb{F}_q)$ on its subset of symmetric matrices. That is, let

$$S_{G_n} = \{\text{symmetric matrices in } G_n\},$$

and let G_n act on the set S_{G_n} by the action

$$g \cdot s = {}^t g s g, \quad s \in S_{G_n}, g \in G_n.$$

Gow [7] proved, using the Lang-Steinberg theorem, that there is a one-to-one correspondence between conjugacy classes of G_n^+ (5.9) and U_n^+ (5.10) of elements of the form $g\kappa$ and $u\kappa$, for $g \in G_n$ and $u \in U_n$. Moreover, Gow proved that this correspondence preserves orders of the elements, and the corresponding centralizers are isomorphic. The conjugacy classes of order 2 elements of this form correspond to the orbits of symmetric matrices in G_n and U_n , and their centralizers to the stabilizers under the action described above. So to find the stabilizers of the orbits in the case for U_n , it is enough to do this for G_n .

First let q be odd. Then the orbits of S_{G_n} correspond exactly to equivalence classes of quadratic forms, of which there are exactly 2 (see, for example, [9, Chapter 9] for a complete discussion). The stabilizers of these orbits are the orthogonal groups corresponding to the quadratic forms. When n is odd, these two orthogonal groups are isomorphic, with order

$$|O_1| = |O_2| = 2q^{(n-1)^2/4} \prod_{i=1}^{(n-1)/2} (q^{2i} - 1).$$

When n is even, the orthogonal groups fixing the two classes of quadratic forms are not isomorphic, and their orders are

$$|O_1| = 2q^{(n^2-2n)/4} (q^{n/2} - 1) \prod_{i=1}^{(n-2)/2} (q^{2i} - 1), \text{ and}$$

$$|O_2| = 2q^{(n^2-2n)/4} (q^{n/2} + 1) \prod_{i=1}^{(n-2)/2} (q^{2i} - 1).$$

From Gow's result, the orders of the stabilizers of orbits is the same for U_n . Using the fact that

$$|U_n| = q^{n(n-1)/2} \prod_{i=1}^n (q^i - (-1)^i),$$

we calculate that the number of elements in S_{U_n} is

$$\begin{aligned} |S_{U_n}| &= \frac{|U_n|}{|O_1|} + \frac{|U_n|}{|O_2|} \\ &= (q+1)q^2(q^3+1)q^4(q^5+1) \cdots \left(q^n + \frac{(1-(-1)^n)}{2} \right), \end{aligned}$$

and for this case, the result now follows from Theorem 5.2.

When q is even, the orbits of S_{G_n} are not necessarily in one-to-one correspondence with inequivalent quadratic forms. When n is odd and q is even, however, there is exactly one orbit, and its stabilizer is the unique orthogonal group, O (see [9, Chapter 14]), with order

$$|O| = q^{(n-1)^2/4} \prod_{i=1}^{(n-1)/2} (q^{2i} - 1).$$

So the size of S_{U_n} is again found to be

$$|S_{U_n}| = \frac{|U_n|}{|O|} = (q+1)q^2(q^3+1)q^4(q^5+1)\cdots q^{n-1}(q^n+1).$$

When q and n are both even, there are two orbits, however these do not correspond to classes of quadratic forms. One orbit consists of symmetric matrices which have at least one nonzero entry on the diagonal, the order of whose stabilizer, $\text{Stab}(1)$, is computed in [19]. The order is

$$|\text{Stab}(1)| = q^{n^2/4} \prod_{i=1}^{(n-2)/2} (q^{2i} - 1)$$

We note that in [19], the author calls $\text{Stab}(1)$ the orthogonal group, but this is different from what is commonly defined as the orthogonal group for characteristic 2 in the current literature.

The other orbit, consisting of symmetric matrices with zero diagonal, corresponds to the unique class of alternating form, with stabilizer the symplectic group, $Sp = \text{Sp}(n, \mathbb{F}_q)$, with order

$$|Sp| = q^{n^2/4} \prod_{i=1}^{n/2} (q^{2i} - 1).$$

As in the previous cases, we calculate the size of S_{U_n} , which in this case is

$$\begin{aligned} |S_{U_n}| &= \frac{|U_n|}{|\text{Stab}(1)|} + \frac{|U_n|}{|Sp|} \\ &= (q+1)q^2(q^3+1)q^4(q^5+1)\cdots(q^{n-1}+1)q^n. \quad \square \end{aligned}$$

The connection between Corollary 5.2 and the characters of U_n^+ is most easily understood using twisted Frobenius-Schur indicators, which we now briefly discuss. More details may be found in [15] and [1].

Let G be a finite group, ι an order 2 automorphism of G , and (π, V) a complex irreducible representation of G with character χ . Suppose that ${}^{\iota}\pi \cong \hat{\pi}$, where ${}^{\iota}\pi(g) = \pi(\iota g)$, and where $\hat{\pi}$ is the contragredient of π , or the irreducible representation with character $\bar{\chi}$. This isomorphism gives rise to a nondegenerate bilinear form B_{ι} satisfying

$$B_{\iota}(\pi(g)v, {}^{\iota}\pi(g)w) = B_{\iota}(v, w) \text{ for all } g \in G \text{ and } v, w \in V.$$

This bilinear form is unique up to scalar by Schur's lemma, from which it follows that for every $v, w \in V$, we have

$$B_\iota(v, w) = \varepsilon_\iota(\pi)B_\iota(w, v),$$

where $\varepsilon_\iota(\pi) = \pm 1$. If ${}^\iota\pi \not\cong \hat{\pi}$, then define $\varepsilon_\iota(\pi) = 0$, and we also write $\varepsilon_\iota(\chi) = \varepsilon_\iota(\pi)$. Kawanaka and Matsuyama [15] obtained the following formula for this indicator:

$$(5.11) \quad \varepsilon_\iota(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g {}^\iota g).$$

This formula generalizes that of the classical Frobenius-Schur indicator, which is the case that ι is trivial, in which case we write $\varepsilon_\iota(\pi) = \varepsilon(\pi)$. Frobenius and Schur proved that $\varepsilon(\pi) = 1$ when π is a real representation, $\varepsilon(\pi) = -1$ when π is not a real representation but χ is real-valued, and $\varepsilon(\pi) = 0$ when χ is not real-valued. The orthogonality relations and equation (5.11) imply the following generalization of the Frobenius-Schur involution formula (for a proof, see [1, Proposition 1]):

$$(5.12) \quad \sum_{\chi \in \text{Irr}(G)} \varepsilon_\iota(\chi)\chi(1) = |\{g \in G \mid {}^\iota g^{-1} = g\}|.$$

For a character χ of a finite group H which is a subgroup of G , we say that χ *extends* to G if there is a character χ' of G such that χ' restricted to H is χ .

Corollary 5.3. *Let U_n^+ be the split extension of U_n by the transpose-inverse involution, as in (5.10). Let χ be a complex irreducible character of U_n . Then we have the following:*

- (1) *If $\varepsilon(\chi) = 0$, then χ induces to an irreducible character $\psi = \text{Ind}_{U_n^+}^{U_n^+}(\chi)$ of U_n^+ such that $\varepsilon(\psi) = 1$.*
- (2) *If $\varepsilon(\chi) = 1$, then χ extends to an irreducible character χ' of U_n^+ such that $\varepsilon(\chi') = 1$.*
- (3) *If $\varepsilon(\chi) = -1$, then χ extends to an irreducible character χ' of U_n^+ such that $\varepsilon(\chi') = 0$.*

Proof. If ${}^\iota g = {}^\iota g^{-1}$, then Corollary 5.2 and (5.12) imply that $\varepsilon_\iota(\chi) = 1$ for every irreducible χ of U_n . First let χ of U_n be an irreducible character such that $\varepsilon(\chi) = 0$. It follows from Frobenius reciprocity that $\psi = \text{Ind}_{U_n^+}^{U_n^+}(\chi)$ is irreducible. Applying the formula for an induced character and (5.11), we find that $\varepsilon(\psi) = \varepsilon_\iota(\chi) = 1$.

Now let χ be an irreducible character of U_n such that $\varepsilon(\chi) = \pm 1$. Then $\psi = \text{Ind}_{U_n^+}^{U_n^+}(\chi)$ is reducible, and is the sum of two extensions of χ to U_n^+ , $\psi = \chi' + \chi''$. Again, using the formula for an induced character and Equation (5.11), we calculate that

$$\begin{aligned} \varepsilon(\chi') + \varepsilon(\chi'') &= \varepsilon(\chi) + \varepsilon_\iota(\chi) \\ &= \varepsilon(\chi) + 1. \end{aligned}$$

If $\varepsilon(\chi) = 1$, then we must have $\varepsilon(\chi') = \varepsilon(\chi'') = 1$. If $\varepsilon(\chi) = -1$, then χ cannot have an extension with a Frobenius-Schur indicator equal to 1, and so the only choice is that $\varepsilon(\chi') = \varepsilon(\chi'') = 0$ \square

5.2. A second character degree sum. A Θ -partition λ is *even* if every part of $\lambda(\varphi)$ is even for every $\varphi \in \Theta$.

Theorem 5.3. *The sum of the degrees of the complex irreducible characters of U_{2m} corresponding to λ such that λ' is even is given by*

$$\sum_{\substack{||\lambda||=2m \\ \lambda' \text{ even}}} \chi^\lambda(1) = (q+1)q^2(q^3+1) \cdots q^{2m-2}(q^{2m-1}+1).$$

Proof. To calculate this sum, we find the coefficient of t^{2m} in the series

$$T = \sum_{\lambda' \text{ even}} (-1)^{n(\lambda)+||\lambda||} \delta(s_\lambda) t^{||\lambda||} = T_o T_e,$$

where

$$T_o = \prod_{\substack{|\varphi| \\ \text{odd}}} \sum_{\lambda' \text{ even}} (-1)^{n(\lambda)} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|} \text{ and } T_e = \prod_{\substack{|\varphi| \\ \text{even}}} \sum_{\lambda' \text{ even}} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|}.$$

It follows from the computation in Example 6 of [18, I.5] that

$$\sum_{\lambda' \text{ even}} (-1)^{n(\lambda)} s_\lambda = \prod_{i < j} (1 + x_i x_j)^{-1},$$

and applying this yields

$$\sum_{\lambda' \text{ even}} (-1)^{n(\lambda)} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|} = \prod_{1 \leq i < j} \left(1 + (t^2(-q)^{-i-j})^{|\varphi|}\right)^{-1}.$$

From Example 5(b) in [18, I.5], we have the identity

$$\sum_{\lambda' \text{ even}} s_\lambda = \prod_{i < j} (1 - x_i x_j)^{-1},$$

from which it follows, for $|\varphi|$ even, that

$$\sum_{\lambda' \text{ even}} \delta(s_\lambda(\varphi)) t^{|\lambda||\varphi|} = \prod_{1 \leq i < j} \left(1 - (t^2 q^{-i-j})^{|\varphi|}\right)^{-1}.$$

Taking logarithms gives

$$\begin{aligned} \log T_o &= \sum_{m \text{ odd}} d_m \left(\sum_{1 \leq i < j} \sum_{r \geq 1} \frac{(- (t^2(-q)^{-i-j})^m)^r}{r} \right) \\ &= \sum_{m \text{ odd}} d_m \sum_{r \geq 1} \frac{t^{mr}}{r(q^{mr} - (-1)^{mr})} \left(\sum_{i \geq 1} t^{mr} q^{-2imr} \right), \end{aligned}$$

where d_m is as in (5.8), and

$$\begin{aligned} \log T_e &= \sum_{m \text{ even}} d_m \left(\sum_{1 \leq i < j} \sum_{r \geq 1} \frac{(t^2 q^{-i-j})^{mr}}{r} \right) \\ &= \sum_{m \text{ even}} d_m \sum_{r \geq 1} \frac{t^{mr}}{r(q^{mr} - (-1)^{mr})} \left(\sum_{i \geq 1} t^{mr} q^{-2imr} \right). \end{aligned}$$

Adding the two terms, making the change $N = mr$, and applying (5.8), we obtain

$$\log T = \sum_{N \geq 1} \sum_{i \geq 1} \frac{t^{2N} q^{-2iN}}{N},$$

and after exponentiating,

$$T = \prod_{i \geq 1} \frac{1}{1 - t^2 q^{-2i}} = (1 - t^2) \prod_{i \geq 0} \frac{1}{1 - t^2 q^{-2i}} = (1 - t^2) \sum_{l \geq 0} \frac{t^{2l}}{\psi_l(q^{-2})}.$$

Multiplying the coefficient of t^{2m} by $|\psi_{2m}(-q)| = \prod_{i=1}^{2m} (q^i - (-1)^i)$, we obtain

$$\sum_{\substack{\|\lambda\|=2m \\ \lambda' \text{ even}}} \chi^\lambda(1) = (q+1)q^2(q^3+1) \cdots q^{2m-2}(q^{2m-1}+1). \quad \square$$

Write $g_{U_m}(q) = \sum_{\|\lambda\|=2m, \lambda' \text{ even}} \chi^\lambda(1)$, and let $g_{G_m}(q)$ denote the corresponding sum for G_m . The polynomial $g_{G_m}(q)$ was calculated in Example 7 of [18, IV.6], and similar to the previous example, we see that we have

$$g_{U_m}(q) = (-1)^m g_{G_m}(-q).$$

In the case that q is odd, Proposition 5.3 follows from the following stronger result obtained by Henderson [10]. Let $Sp_{2n} = \text{Sp}(2n, \mathbb{F}_q)$ be the symplectic group over the finite field \mathbb{F}_q .

Theorem 5.4 (Henderson). *Let q be odd. The decomposition of $\text{Ind}_{Sp_{2n}}^{U_{2n}}(\mathbf{1})$ into irreducibles is given by*

$$\text{Ind}_{Sp_{2n}}^{U_{2n}}(\mathbf{1}) = \sum_{\substack{\|\lambda\|=2n \\ \lambda' \text{ even}}} \chi^\lambda.$$

The fact that Proposition 5.3 holds for all q suggests that Theorem 5.4 should as well.

6. A DELIGNE-LUSZTIG MODEL

A *model* of a finite group G is a representation ρ , which is a direct sum of representations induced from one-dimensional representations of subgroups of G , such that every irreducible representation of G appears as a component with multiplicity 1 in the decomposition of ρ .

Klyachko [16] and Inglis and Saxl [12] obtained a model for $\mathrm{GL}(n, \mathbb{F}_q)$, where the induced representations can be written as a Harish-Chandra product of Gelfand-Graev characters and the permutation character of the finite symplectic group.

In this section we show that the same result is true for the finite unitary group, except the Harish-Chandra product is replaced by Deligne-Lusztig induction. The result is therefore not a model for $\mathrm{U}(n, \mathbb{F}_q)$ in the finite group character induction sense, but rather from the Deligne-Lusztig point of view.

6.1. The Gelfand-Graev character. Let $U'_n = \mathrm{GL}(n, \mathbb{K})^{F'}$ as in (2.3), and let

$$B_{<} = \{u \in U'_n \mid u \text{ unipotent and uppertriangular}\} \subseteq U'_n.$$

Fix a nontrivial character $\psi : \mathbb{K}_2^+ \rightarrow \mathbb{C}^\times$ of the additive group of the field \mathbb{K}_2 such that the restriction to the subgroup $\{x \in \mathbb{K}_2 \mid x^q + x = 0\}$ is also nontrivial. The map $\psi_{(n)} : B_{<} \rightarrow \mathbb{C}$ given by

$$\psi_{(n)}(u) = \psi(u_{12} + \cdots + u_{[n/2]-1, [n/2]} + u_{[n/2], [n/2]+1}), \text{ for } u = (u_{ij}) \in B_{<},$$

is a linear character of $B_{<}$. Then

$$\Gamma'_{(n)} = \mathrm{Ind}_{B_{<}}^{U'_n}(\psi_{(n)})$$

is the *Gelfand-Graev character* of U'_n . Let $\Gamma_{(n)}$ be the corresponding Gelfand-Graev character of $U_n = \mathrm{GL}(n, \mathbb{K})^F$. For $\lambda \in \mathcal{P}^\Theta$, define

$$\mathrm{ht}(\lambda) = \max\{\ell(\lambda(\varphi)) \mid \varphi \in \Theta\}.$$

The following also appears in Section 5.2 of [20].

Theorem 6.1. *The decomposition of $\Gamma_{(m)}$ into irreducibles is given by*

$$\Gamma_{(m)} = \sum_{\substack{\lambda \in \mathcal{P}_m^\Theta \\ \mathrm{ht}(\lambda)=1}} \chi^\lambda.$$

Proof. By [3, Theorem 10.7],

$$\Gamma_{(m)} = (-1)^{\lfloor m/2 \rfloor} \sum_{\gamma \in \mathcal{P}_m^\Theta} \frac{(-1)^{m-\ell(\gamma)}}{z_\gamma} R_\gamma.$$

Therefore, Theorem 4.2 implies

$$\begin{aligned} \mathrm{ch}(\Gamma_{(m)}) &= (-1)^{\lfloor m/2 \rfloor} \sum_{\gamma \in \mathcal{P}_m^\Theta} \frac{1}{z_\gamma} p_\gamma \\ &= (-1)^{\lfloor m/2 \rfloor} \sum_{\gamma \in \mathcal{P}_m^\Theta} \frac{\omega^{\gamma'_s}(\gamma)}{z_\gamma} p_\gamma \quad (\omega^{\gamma'_s} \text{ is the trivial character of } W_\gamma) \\ &= (-1)^{\lfloor m/2 \rfloor} \sum_{\substack{\lambda \in \mathcal{P}_m^\Theta \\ \mathrm{ht}(\lambda)=1}} s_\lambda \quad (\text{by (4.1)}). \end{aligned}$$

Since $\lfloor m/2 \rfloor + n(\boldsymbol{\lambda}) = \lfloor m/2 \rfloor$ for all $\boldsymbol{\lambda}$ with $\text{ht}(\boldsymbol{\lambda}) = 1$, Theorem 5.1 implies

$$\Gamma_{(m)} = \text{ch}^{-1}(\text{ch}(\Gamma_{(m)})) = \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{P}_m^\Theta \\ \text{ht}(\boldsymbol{\lambda})=1}} \chi^\lambda. \quad \square$$

6.2. The model. For a partition λ , let $o(\lambda)$ denote the number of odd parts of λ , and for $\boldsymbol{\lambda} \in \mathcal{P}^\Theta$, let $o(\boldsymbol{\lambda}) = \sum_{\varphi \in \Theta} |\varphi| o(\boldsymbol{\lambda}(\varphi))$.

Theorem 6.2. *Let q be odd. For each r such that $0 \leq 2r \leq m$,*

$$\Gamma_{m-2r} \circ \text{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1}) = \sum_{o(\boldsymbol{\lambda}')=m-2r} \chi^\lambda.$$

Furthermore,

$$\sum_{0 \leq 2r \leq m} \Gamma_{m-2r} \circ \text{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1}) = \sum_{\|\boldsymbol{\lambda}\|=m} \chi^\lambda$$

Proof. Suppose $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}^\Theta$, such that $\text{ht}(\boldsymbol{\mu}) = 1$ and $\boldsymbol{\nu}'$ is even. From (5.7), Corollary 4.3, and Pieri's formula [18, I.5.16],

$$(6.1) \quad \chi^\boldsymbol{\mu} \circ \chi^\boldsymbol{\nu} = (-1)^{\tau(\boldsymbol{\mu})+\tau(\boldsymbol{\nu})} \sum_{\boldsymbol{\lambda}} \chi^\lambda,$$

where the sum is taken over all $\boldsymbol{\lambda}$ such that for every $\varphi \in \Theta$, $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$ -strip.

We now use Corollary 5.1 to show that $\chi^\boldsymbol{\mu} \circ \chi^\boldsymbol{\nu}$ is a character. As $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$ -strip, the part $\boldsymbol{\lambda}(\varphi)_i'$ is either $\boldsymbol{\nu}(\varphi)_i'$ or $\boldsymbol{\nu}(\varphi)_i' + 1$ for every $i = 1, 2, \dots, \ell(\boldsymbol{\lambda}(\varphi))$. By assumption, $\boldsymbol{\nu}'$ is even, so $\boldsymbol{\nu}(\varphi)_i'$ is even for every $\varphi \in \Theta$, and so

$$\binom{\boldsymbol{\nu}(\varphi)_i' + 1}{2} = \boldsymbol{\nu}(\varphi)_i' + \binom{\boldsymbol{\nu}(\varphi)_i'}{2} \equiv \binom{\boldsymbol{\nu}(\varphi)_i'}{2} \pmod{2}.$$

Thus, $n(\boldsymbol{\lambda}(\varphi)) = \sum_i \binom{\boldsymbol{\lambda}(\varphi)_i'}{2} \equiv n(\boldsymbol{\nu}(\varphi)) \pmod{2}$. The assumption $\text{ht}(\boldsymbol{\mu}) = 1$ implies $n(\boldsymbol{\mu}(\varphi)) = 0$, and since $\|\boldsymbol{\nu}\|$ is even,

$$n(\boldsymbol{\mu}) + n(\boldsymbol{\nu}) \equiv n(\boldsymbol{\lambda}) + \|\boldsymbol{\mu}\| \|\boldsymbol{\nu}\| \pmod{2}.$$

By Corollary 5.1, $\chi^\boldsymbol{\mu} \circ \chi^\boldsymbol{\nu}$ is a character.

Use the decompositions of Theorem 5.4 and Theorem 6.1 in the product (6.1) to observe that the irreducible characters χ^λ in the decomposition of $\Gamma_{m-2r} \circ \text{Ind}_{Sp_{2r}}^{U_{2r}}(\mathbf{1})$ are indexed by $\boldsymbol{\lambda} \in \mathcal{P}_m^\Theta$ such that for every φ , $\boldsymbol{\lambda}(\varphi) - \boldsymbol{\nu}(\varphi)$ is a horizontal $|\boldsymbol{\mu}(\varphi)|$ -strip, where $\|\boldsymbol{\mu}\| = m - 2r$, for some $\boldsymbol{\nu}(\varphi)$ such that $\boldsymbol{\nu}(\varphi)'$ is even. Then the number of odd parts of $\boldsymbol{\lambda}(\varphi)'$ is exactly $|\boldsymbol{\mu}(\varphi)|$, and so the $\boldsymbol{\lambda}$ in the decomposition must satisfy $\sum_{\varphi \in \Theta} |\varphi| o(\boldsymbol{\lambda}(\varphi)') = \|\boldsymbol{\mu}\| = m - 2r$. \square

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