Math 4140: Homework 8

Due March 11, 2009

1. Find the determinants of all the Cartan matrices for the irreducible root systems.

2. Fix a field $\mathbb{F}$. For an $n \times n$ matrix $A \in M_n(\mathbb{F})$, define

$$\exp(A) = \mathrm{Id}_n + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots.$$

Let e_{ij} be the $n \times n$ matrix with 1 in the (i, j) and zeroes elsewhere. That is,

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}.$$

(a) For $t \in \mathbb{F}$ and $1 \leq i, j \leq n$, find

$$\exp(te_{ij}).$$

How does the characteristic of the field matter for this computation?

(b) Show that for $1 \leq i \neq j \leq n$, as groups

$$\langle \exp(te_{ij}) \mid t \in \mathbb{F} \rangle$$

is isomorphic to the additive group of $\mathbb{F}$ (the set $\mathbb{F}$ under addition).

(c) For $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, compare

$$e_{ij}e_{kl} - e_{kl}e_{ij} \qquad \text{and} \qquad \exp(e_{ij})\exp(e_{kl})\exp(e_{ij})^{-1}\exp(e_{kl})^{-1}.$$

Describe how one determines the other and vice-versa.

The exponential map gives a connection between the matrix ring $M_n(\mathbb{F})$ and the group $\mathrm{GL}_n(\mathbb{F})$. This is a fundamental connection in Lie theory.

3. For a root system $R \subseteq V$, the corresponding *weight lattice* is the set

$$\Lambda = \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z}, \alpha \in R\}.$$

(a) Show that Λ is an additive group.

(b) Show that the Weyl group W of R fixes Λ.

(c) For $R(A_2)$, draw in the weights on the A_2-graph paper.