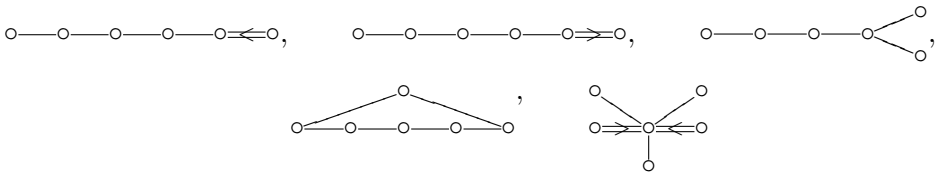
Math 4140: Homework 7

Due March 4, 2009

1. For the following graphs,

   (a) Find the corresponding Cartan matrix,

   (b) Find a set B of basis vectors in $\mathbb{R}^6$ such that the graph is the Dynkin diagram of B.



2. Suppose R is a root system in V. Let

$$R^\vee = \{\alpha^\vee \mid \alpha \in R\}.$$

   (a) Show that $R^\vee$ is a root system of V.

   (b) Show that its Cartan matrix is the transpose of the Cartan matrix of R.

   (c) Show that $R(B_n)^\vee = R(C_n)$, and $R(D_n)^\vee = R(D_n)$ (see Homework 6).

   (d) Explain why R and $R^\vee$ are not always isomorphic?

   We typically call $R^\vee$ the *dual root system* to R.

3. Let

$$C_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

   Show that

   (a) $\det(C_n) = 2\det(C_{n-1}) - \det(C_{n-2})$ for $n \geq 3$,

   (b) Find a formula for $\det(C_n)$ (a non-recursive formula).

   Hint: Look up the Laplace expansion for the determinant of a matrix.