Math 4140: Homework 2

Due January 28, 2009

1. Let R be a ring. An element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}_{\geq 1}$.

   (a) Show that if $a, b \in R$ are nilpotent elements of a commutative ring, then $a + b$ is nilpotent.

   (b) Find an example that shows that the converse of (a) is not true.

   Hint: Think about nilpotent elements in $M_2(\mathbb{Z})$.

   (c) Show that R has no nonzero nilpotent elements if and only if the equation $x^2 = 0$ has only one solution in R.

2. (a) Find an example of a ring R with a multiplicative identity 1, and a subring $U \subseteq R$ with a different multiplicative identity $1'$.

   Hint: Think about subrings of $M_2(\mathbb{Z})$, or subrings of $\mathbb{Z}/6\mathbb{Z}$.

   (b) Show that (a) cannot happen if R is a field.

3. Let F be a field with multiplicative identity 1. The *characteristic* of F is the smallest integer $n \in \mathbb{Z}_{\geq 1}$ such that $n \cdot 1 = 0$. If no such n exists, then the characteristic of F is 0.

   (a) Show that if $s, t \in F$, then $st = 0$ implies $s = 0$ and/or $t = 0$,

   (b) Use (a) to show that if the characteristic is nonzero, then the characteristic is a prime number.

   Hint: Factor $(pq) \cdot 1$.

   (c) Show that if p is the characteristic of F, then $pt = 0$ for all $t \in F$.